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ON THE MEAN ERGODIC THEOREM FOR CESÀRO BOUNDED OPERATORS

BY

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This paper is dedicated to the memory of Anzelm Iwanik

Abstract. For a Cesàro bounded operator in a Hilbert space or a reflexive Banach space the mean ergodic theorem does not hold in general. We give an additional geometrical assumption which is sufficient to imply the validity of that theorem. Our result yields the mean ergodic theorem for positive Cesàro bounded operators in L^p $(1 < p < \infty)$. We do not use the tauberian theorem of Hardy and Littlewood, which was the main tool of previous authors. Some new examples, interesting for summability theory, are described: we build an example of a mean ergodic operator T in a Hilbert space such that $\|T^n\|/n$ does not converge to 0, and whose adjoint operator is not mean ergodic (its Cesàro averages converge only weakly).

1. Introduction. A bounded linear operator T in a Banach space $\mathbb{B}$ is *power bounded* when $\sup_n \|T^n\| < \infty$. It is said to be *Cesàro bounded* when $\sup_n n^{-1}\|\sum_{i=0}^{n-1} T^i\| < \infty$. If $\mathbb{B}$ is reflexive and T is Cesàro bounded, then $n^{-1}\sum_{i=0}^{n-1} T^i x$ converges strongly for every $x \in \mathbb{B}$ for which we have strong-$\lim_n T^n x/n = 0$; then one says that the *mean ergodic theorem holds* for x. Therefore, if T is power bounded, the theorem holds for every $x \in \mathbb{B}$ (the best reference for the ergodic theorems is Krengel's book [K]; see also [DS], Chap. VIII,5).

This form of the mean ergodic theorem goes back to the thirties. The basic step was von Neumann's theorem for a unitary operator in a Hilbert space. An example showing that power boundedness is not necessary appeared in [Hi]. Only in 1983 was it observed that the condition $\lim_n T^n x/n = 0$ is redundant if T is a *positive* Cesàro bounded operator in an L^2 space or, more generally, in a reflexive Banach lattice ([B], [E]). The L^p spaces, for $1 \le p \le \infty$, are classical examples of Banach lattices, and T *positive* means that the cone of positive elements is stable under T. In the present paper we shall give a geometrical condition on T, without any order relation, sufficient to imply the mean ergodic theorem when T is

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Cesàro bounded. This condition is obviously satisfied by a positive operator on an L^p space. Our approach does not depend on the tauberian theorem of Hardy and Littlewood which was the main argument in [B] or [E]. It can be extended to some systems of averages besides the usual Cesàro averages.

In Section 2 we consider the Hilbert space case, our approach being then simpler. In Section 3 we consider reflexive Banach spaces. In Section 4 some examples are discussed; in particular it is shown that for a Cesàro bounded operator in a Hilbert space, weak convergence of the Cesàro averages may hold without strong convergence.

2. The Hilbert space case. Here is our main result.

Theorem 1. *Let $\mathbb{H}$ be a real Hilbert space and T a Cesàro bounded linear operator acting in $\mathbb{H}$. If $x \in \mathbb{H}$ satisfies $\langle T^n x, T^m x\rangle \geq 0$ for any integers n and m, then $(n+1)^{-1}\sum_{i=0}^n T^i x$ converges strongly as $n \to \infty$.*

Remarks. 1. By $\langle x, y\rangle$ we denote the scalar product of $\mathbb{H}$.

2. The limit, when it exists, is a fixed point of T because, by taking differences, the convergence of $(n+1)^{-1}\sum_{i=0}^n T^i x$ implies $T^n x/n \to 0$.

3. From the mean ergodic theorem it is clear that all we need to prove is the strong convergence to 0 of $T^n x/n$.

4. When $\mathbb{H}$ is an L^2 space and T is positive, that is, when $f \geq 0 \Rightarrow Tf \geq 0$, then, obviously, each nonnegative $f \in L^2$ satisfies the assumption of Theorem 1. Since each function of L^2 is the difference of two nonnegative functions, Theorem 1 implies that the mean ergodic theorem holds in L^2 for a positive Cesàro bounded operator T.

Before giving the proof of Theorem 1 we need to recall a few elementary facts about the Cesàro averages of order $\alpha > -1$, which will be useful tools. Following the classical authors ([H], [Z]), we put

$$A_n^\alpha = \frac{(n+\alpha)(n+\alpha-1)\dots(\alpha+1)}{n(n-1)\dots 2}, \quad \text{i.e.} \quad \sum_{n=0}^{\infty} A_n^\alpha z^n = (1-z)^{-\alpha-1}$$

and $S_n^\alpha = \sum_{k=0}^n A_{n-k}^{\alpha-1} T^k$. The nth *Cesàro mean* of order α of the powers of T is defined by $M_n^\alpha = S_n^\alpha / A_n^\alpha$. For $\alpha = 1$ we find $M_n^1 = (n+1)^{-1}\sum_{i=0}^n T^i$, the standard Cesàro averages. We recall that the convergence of the Cesàro-α means implies the convergence of the Cesàro-β means for every $\beta > \alpha$. For $\alpha \geq 0$ all the coefficients of the Cesàro-α means are nonnegative; thus the Cesàro-α means are weighted averages in the usual sense (see [Z], pp. 76–78).

The following lemma is a simple "abelian" property of the Cesàro means which is valid in any normed space.

Lemma 1. *If $\sup_n \|M_n^\alpha\| = K < \infty$ with $\alpha \geq 0$, then $\sup_n \|M_n^\beta\| \leq K$ and $\lim_n \|M_n^\beta (I-T)\| = 0$ for every $\beta > \alpha$.*

Proof. We use the elementary identity $S_n^\beta = \sum_{k=0}^n A_{n-k}^{\beta-\alpha-1} S_k^\alpha$ (it is an application of the summation by parts formula; [Z], pp. 76–78). The assumption $\|S_n^\alpha\| \le K A_n^\alpha$ yields

$$\|S_n^\beta\| \le K \sum_{k=0}^n A_{n-k}^{\beta-\alpha-1} A_k^\alpha = K A_n^\beta.$$

To get the second assertion we write

$$S_n^\beta T - S_n^\beta = S_{n+1}^\beta - S_n^\beta - A_{n+1}^{\beta-1} = S_{n+1}^{\beta-1} - A_{n+1}^{\beta-1}.$$

If $\beta - \alpha \ge 1$, we get

$$\|S_n^\beta T - S_n^\beta\| \le \|S_{n+1}^{\beta-1}\| + A_{n+1}^{\beta-1} \le (1+K) A_{n+1}^{\beta-1}$$

and we are done since $A_n^\alpha \sim C n^\alpha$ as $n \to \infty$.

If $\beta - \alpha < 1$, we write $S_{n+1}^{\beta-1} = \sum_{k=0}^{n+1} A_{n+1-k}^{\beta-\alpha-2} S_k^\alpha$. The coefficients $A_{n+1-k}^{\beta-\alpha-2}$ are negative for $k \ne n+1$. Therefore

$$\begin{aligned}\|S_n^\beta T - S_n^\beta\| &\le \|S_{n+1}^\alpha\| - \sum_{k=0}^n A_{n+1-k}^{\beta-\alpha-2} \|S_k^\alpha\| + A_{n+1}^{\beta-1} \\ &\le K A_{n+1}^\alpha - K(A_{n+1}^{\beta-1} - A_{n+1}^\alpha) + A_{n+1}^{\beta-1}\end{aligned}$$

and the same estimate $A_n^\alpha \sim C n^\alpha$ proves the desired result.

Remark. An obvious consequence of Lemma 1 is that the sequence of means M_n^β defines an *ergodic net* for T (according to the terminology of [K], p. 75), when $\beta > \alpha$. Since this notion is important in our reasoning we recall its definition:

"Given a continuous linear operator T in a Banach space $\mathbb{B}$, a sequence $(M_n)_{n\ge 0}$ of operators which are convex combinations of T^k $(k \ge 0)$ is an *ergodic net* for T when $\sup_n \|M_n\| < \infty$ and $\lim_n M_n(I-T)x = 0$ for every $x \in \mathbb{B}$."

An ergodic net is also called an *almost invariant system of means*.

Proof of Theorem 1. First of all we shall apply the abstract form of the mean ergodic theorem to the Cesàro-β means with $\beta > 1$ ([K], p. 76). For the convenience of the reader we recall this statement:

"If T is a continuous linear operator in a Banach space $\mathbb{B}$ and admits an ergodic net $(M_n)_{n\ge 0}$, then, for any $x, y \in \mathbb{B}$, the following conditions are equivalent:

(i) $Ty = y$ and y belongs to the closed convex hull of the orbit of x under T,

(ii) $y = \lim_n M_n x$,

(iii) y is a weak cluster point of the sequence $M_n x$."

According to Lemma 1 the assumption that T is Cesàro bounded implies that the Cesàro-β means are an *ergodic net* for T, for each $\beta > 1$. The unit ball of the Hilbert space $\mathbb{H}$ being weakly compact, $M_n^\beta u$ has a weak cluster point for every $u \in \mathbb{H}$. Therefore the closed convex hull of the T-orbit of $u \in \mathbb{H}$ contains a unique T-fixed point $v \in \mathbb{H}$ and $\lim_n M_n^\beta u = v$ for each $\beta > 1$; this holds for any $u \in \mathbb{H}$.

Consider now $x \in \mathbb{H}$ for which the assumption holds: $\langle T^n x, T^m x\rangle \geq 0$ for every n and m.

Let y be the unique T-fixed point in the closed convex hull of the orbit of x. The main part of the following proof is to show that any weak cluster point z of the Cesàro averages $(n+1)^{-1}\sum_{i=0}^n T^i x$ must be y, in order to prove first the weak convergence. The difference with the usual argument of the ergodic theorem is that the invariance of the weak cluster point z requires a specific proof, since we do not assume even weak-almost invariance of the Cesàro averages (i.e., we do not assume weak-$\lim_n (n+1)^{-1}\sum_{i=0}^n T^i(T-I) = 0$).

For $\beta > 1$ we have $\lim_n \langle M_n^\beta x, T^m x\rangle = \langle y, T^m x\rangle$. Since

$$\frac{A_n^\beta}{A_{n-k}^{\beta-1}} \geq \frac{\beta+n}{\beta}$$

for every $0 \leq k \leq n$, using the assumption on x, we get

$$\left\langle \frac{1}{n+1}\sum_{i=0}^n T^i x, T^m x\right\rangle \geq \frac{\beta+n}{\beta(1+n)}\langle M_n^\beta x, T^m x\rangle,$$

thus

$$\liminf_n \left\langle \frac{1}{n+1}\sum_{i=0}^n T^i x, T^m x\right\rangle \geq \frac{1}{\beta}\langle y, T^m x\rangle$$

and then, letting $\beta \to 1$,

$$\liminf_n \left\langle \frac{1}{n+1}\sum_{i=0}^n T^i x, T^m x\right\rangle \geq \langle y, T^m x\rangle$$

for every integer m. Now, let z be a weak cluster point of the sequence $(n+1)^{-1}\sum_{i=0}^n T^i x$, existing by the boundedness of the sequence in $\mathbb{H}$. The preceding inequality yields $\langle z, T^m x\rangle \geq \langle y, T^m x\rangle$ for every m.

On the other hand we have $\sum_{i=0}^n T^i(T-I) = T^{n+1} - I$, thus

$$\liminf_n \left\langle \frac{1}{n+1}\sum_{i=0}^n T^i(T-I)x, T^m x\right\rangle = \liminf_n \frac{1}{n+1}\langle (T^{n+1}-I)x, T^m x\rangle \geq 0$$

by the hypothesis on x. Taking a weakly convergent subsequence we get $\langle Tz - z, T^m x\rangle \geq 0$ for every m. By the same method we find $\langle T^{k+1}z - T^k z, T^m x\rangle \geq 0$ and, by addition, we get $\langle T^k z - z, T^m x\rangle \geq 0$ for every m

and k. Since z belongs to the closed convex hull of the orbit of x, by the abstract mean ergodic theorem recalled above, the Cesàro-β means $M_n^\beta z$ converge strongly to y, for $\beta > 1$. However $M_n^\beta z - z$ is a convex combination of the sequence $T^k z - z$. Therefore $\langle y - z, T^m x\rangle \geq 0$ for every m.

Combining the two inequalities of the preceding two paragraphs, we get $\langle y - z, T^m x\rangle = 0$ for every m. However, $y - z$ is a weak limit point of linear combinations of the $T^i x$, hence $\langle y - z, y - z\rangle = 0$ and $y = z$. We just proved that any weak cluster point of the sequence $(n+1)^{-1}\sum_{i=0}^n T^i x$ must be y. Hence the sequence weakly converges to y.

It remains to prove the strong convergence. By taking differences the sequence $T^n x/n$ converges weakly to 0. On the other hand we have

$$M_{2n}^2 x = \frac{2}{2n(2n+1)}\sum_{i=0}^{2n}(2n - i + 1)T^i x;$$

for $i = n$ we find, in this sum, the term $(n+1)T^n x/(n(2n+1))$, thus $2M_{2n}^2 x - T^n x/n$ is a linear combination of $T^k x$, $k \geq 0$, with *nonnegative* coefficients. Hence

$$\left\langle 2M_{2n}^2 x - \frac{T^n x}{n}, T^m x\right\rangle \geq 0 \quad \text{for every } m.$$

Then we write

$$\left\|\frac{T^n x}{n}\right\|^2 = -\left\langle \frac{T^n x}{n}, 2M_{2n}^2 x - \frac{T^n x}{n}\right\rangle + \left\langle \frac{T^n x}{n}, 2M_{2n}^2 x\right\rangle.$$

Since $T^n x/n$ converges weakly to 0 and $M_{2n}^2 x$ converges strongly to y, the last term converges to 0; the first term of the sum being nonpositive, we get $\lim_n \|T^n x/n\| = 0$. A last appeal to the mean ergodic theorem yields the desired result: the Cesàro averages of $T^n x$ converge strongly to y, the unique fixed point belonging to the closed convex hull of the orbit of x.

Remarks and comments. 1. In [B] or [E1, E2] the ergodic theorem for Cesàro bounded and positive operators in L^2 depended on the tauberian theorem of Hardy and Littlewood which says that, for nonnegative sequences, the convergence of the Abel means implies (in fact is equivalent to) the convergence of the Cesàro means. This theorem, with the elegant proof of Karamata ([H] or [Z]) is rightly considered a pearl of real analysis (another proof was given by Feller [F]). The above proof does not use this deep result. Nevertheless it gives more, as will be shown in the second remark.

2. For power bounded operators, the abstract mean ergodic theorem shows the convergence of the Cesàro-α means for any $\alpha > 0$ or the Abel means as well, because each of these systems of means defines an ergodic net. If convergence holds for one ergodic net, it holds for all. Therefore it is natural to ask whether Theorem 1 remains true if the standard Cesàro means

are replaced by the Cesàro-α means, with $\alpha > 0$. The answer is affirmative. The proof is exactly the same after the obvious change of the values of the coefficients of the means. However the proof given in [E2] does not allow this generalisation: if the Cesàro-α means are bounded, with $0 < \alpha < 1$, it is easy to deduce that the Abel means are bounded too and then convergent; but the tauberian theorem of Hardy and Littlewood, for nonnegative sequences, asserts that the Abel convergence implies only the Cesàro-1 convergence and not the stronger Cesàro-α convergence for $0 < \alpha < 1$.

3. The Banach space case. In this section we consider a Cesàro bounded operator T leaving stable a convex cone $\mathcal{F}$ in a Banach space $\mathbb{B}$. A subset $\mathcal{F}$ is *stable* under T when $T\mathcal{F} \subseteq \mathcal{F}$. This situation extends the setting of Theorem 1 and gives a better insight into its proof.

THEOREM 2. *Let $\mathbb{B}$ be a reflexive Banach space and T a Cesàro bounded linear operator acting in $\mathbb{B}$. Let $\mathcal{F}$ be a closed convex cone which is stable under T, and such that $\mathcal{F} \cap (-\mathcal{F}) \subseteq \mathrm{Ker}(I - T)$. If $x \in \mathcal{F}$ then the sequence $(n+1)^{-1} \sum_{i=0}^{n} T^i x$ converges weakly as $n \to \infty$.*

REMARKS. 1. Recall that a strongly closed convex set is necessarily weakly closed.

2. These assumptions are satisfied when $\mathbb{B} = L^p$, with $1 < p < \infty$, and T is a positive Cesàro bounded operator, with $\mathcal{F} = L^p_+$ the cone of nonnegative functions.

3. They are also satisfied in the setting of Theorem 1 if $\mathcal{F}$ is taken as the closed convex cone generated by the orbit of x. Under the assumption $\langle T^n x, T^m x\rangle \geq 0$ for all integers n and m, we then have $\langle u, v\rangle \geq 0$ for all u, $v \in \mathcal{F}$, thus $\mathcal{F} \cap (-\mathcal{F}) = \{0\}$.

4. The weak limit of $(n+1)^{-1} \sum_{i=0}^{n} T^i x$ is, of course, the unique T-fixed point y belonging to the closed convex hull of the orbit of x, whose existence follows from the mean ergodic theorem applied to the Cesàro-β means, with $\beta > 1$, which define an ergodic net (Lemma 1; beginning of the proof of Theorem 1). In Section 4 an illustration of Theorem 2 is given.

Proof of Theorem 2. The proof is quite similar to the first part of the proof of Theorem 1. By reflexivity the bounded sequence $(n+1)^{-1} \sum_{i=0}^{n} T^i x$ must have a weak cluster point $z \in \mathcal{F}$. Using the convexity and closedness of the cone $\mathcal{F}$ and applying the same inequalities as in the proof of Theorem 1, we get, on the one hand, $z - y \in \mathcal{F}$, and, on the other hand, $y - z \in \mathcal{F}$ (as above, y is the fixed point in the closed convex hull of the orbit of x). By the hypothesis $\mathcal{F} \cap (-\mathcal{F}) \subseteq \mathrm{Ker}(I - T)$, we get $Tz = z$. Since the fixed point in the closed convex hull of an orbit is unique, $z = y$ and the weak convergence of $(n+1)^{-1} \sum_{i=0}^{n} T^i x$ follows.

Corollary 1. *Let T be a positive Cesàro bounded operator on an L^p space with $1 < p < \infty$. Then the Cesàro averages $(n+1)^{-1}\sum_{i=0}^{n} T^i f$ converge strongly as $n \to \infty$ for every $f \in L^p$.*

Proof. Let f be a nonnegative element. Theorem 2, applied to the cone $\mathcal{F} = L^p_+$, yields the weak convergence. Hence $T^n f/n \to 0$ weakly. On the other hand we have $2M_{2n}^2 f - T^n f/n \geq 0$, and the sequence $M_{2n}^2 f$ converges strongly, as we saw in the proof of Theorem 1. In an L^p space it is well known that, given two sequences g_n and h_n with $g_n \geq h_n \geq 0$, if weak-$\lim_n h_n = 0$ and the strong limit of g_n exists then h_n converges strongly to 0, because of uniform integrability. Therefore $T^n f/n \to 0$ strongly, and our corollary follows from the mean ergodic theorem.

The preceding result was proved in [E2] with the help of the tauberian theorem of Hardy and Littlewood.

In an abstract Hilbert space it is possible to deduce the strong convergence from the weak convergence when the "angle at the summit" of the cone $\mathcal{F}$ is not more than $\pi/2$, that is, when $\langle u, v\rangle \geq 0$ for all $u, v \in \mathcal{F}$; this was the situation of Theorem 1 with the closed convex cone generated by the orbit of x. Otherwise this deduction is impossible. An example will be given in the next section.

4. Examples. We first recall some previously known facts.

Even in a finite-dimensional space the Cesàro averages of a Cesàro bounded operator need not converge: a very simple example, due to Assani, is given in [E1]; it is defined by the 2×2 matrix $\left(\begin{smallmatrix} -1 & 2 \\ 0 & -1 \end{smallmatrix}\right)$.

In a finite-dimensional space, a linear operator, that is, a matrix, which is ergodic, is necessarily power bounded: see [S, Chap. 1, §3] (the proof relies on the fact that a Jordan block of an ergodic matrix, corresponding to an eigenvalue of unit modulus, is necessarily diagonal). A matrix with nonnegative elements which is Cesàro bounded is ergodic (i.e. its Cesàro averages converge); this fact may be seen as a byproduct of our Theorem 1. Thus nonnegative Cesàro bounded matrices are necessarily power bounded.

In [DL] there is an example of a positive, Cesàro bounded but not power bounded operator T in an L^1 space. Theorem 2.1 of [DL] asserts then that $\lim_n \|T^n f\|_1/n = 0$. But the proof shows even more: $\|T^n\| \leq K(n+1)/\ln n$ where K is the uniform bound of the Cesàro averages of T, thus $\lim_n \|T^n\|_1/n = 0$. In that example T is also a contraction in the L^∞ norm; by convexity we then get $\lim_n \|T^n\|_p/n = 0$ in every L^p norm, $1 \leq p \leq \infty$; it is shown in [E2] that T is not power bounded in L^p $(1 < p < \infty)$.

We now describe a class of new examples which are of interest in summability theory.

For any real sequence $a = (a_n)_{n\ge 1}$ we put

$$v_p(a) = \left[\sum_{n=1}^{\infty}\left|\frac{1}{n}\sum_{i=1}^{n} a_i\right|^p\right]^{1/p} \qquad \text{for } 1 \le p < \infty,$$
$$v_\infty(a) = \sup_{n\ge 1}\left|\frac{1}{n}\sum_{i=1}^{n} a_i\right|.$$

For $1 \le p \le \infty$, we denote by V_p the space of real sequences a such that $v_p(a) < \infty$. The space V_p is a Banach space for the norm v_p (we keep the standard notation $\|u\|_p$ for the norm in the classical l_p space).

In fact, each V_p space is the bijective and isometric image of the space $l_p = \{(u_n)_{n\ge1} : \|u\|_p = (\sum_{n=1}^{\infty}|u_n|^p)^{1/p} < \infty\}$ through the linear map $S : l_p \to V_p$ defined by

$$S(u) = a \quad \text{with} \quad a_n = nu_n - (n-1)u_{n-1} \quad \text{for } n \ge 2, \quad a_1 = u_1$$

(or equivalently $n^{-1}\sum_{i=1}^{n} a_i = u_n$).

We consider the shift operator T defined by $(Ta)_n = a_{n+1}$ for every $n \ge 1$; we shall explore its ergodic properties in different V_p spaces.

PROPOSITION 1. *In each V_p space $(1 < p \le \infty)$, the shift operator T is bounded and leaves stable the closed convex cone of nonnegative elements. In V_1 it is not bounded.*

P r o o f. Using the convexity of the real function x^p for $1 < p$, we find

$$\left|\frac{1}{n}\sum_{i=1}^{n} a_{i+1}\right|^p \le 2^{p-1}\left(\left|\frac{1}{n}\sum_{i=1}^{n+1} a_i\right|^p + \left|\frac{a_1}{n}\right|^p\right),$$

thus

$$v_p(Ta) \le \left(2^{2p-1} + 2^{p-1}\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^p\right)^{1/p} v_p(a) \le Cv_p(a).$$

If $\lim_n \sum_{i=2}^{n} a_i = -a_1 \ne 0$ and $a \in V_1$ then $Ta \notin V_1$. The other assertions are obvious.

PROPOSITION 2. *In V_∞ the shift operator T is Cesàro bounded: the operator norm satisfies*

$$v_\infty\left(\frac{1}{k+1}\sum_{j=0}^{k} T^j\right) \le 4,$$

but $v_\infty(T^k) \ge k$ for every k.

P r o o f. Let us introduce the expression

$$w_\infty(a) = \sup_n \frac{1}{n}\sup_{i\le n}\left|\sum_{j=i}^{n} a_j\right|.$$

It is easy to check $w_\infty(a) \le 2v_\infty(a)$. Then we have

$$v_\infty\left(\frac{1}{k}\sum_{j=0}^{k-1} T^j a\right) = \sup_n \frac{1}{nk}\Big|\sum_{i=1}^{n}\sum_{j=0}^{k-1} a_{i+j}\Big|,$$

and

$$\frac{1}{nk}\Big|\sum_{i=1}^{n}\sum_{j=0}^{k-1} a_{i+j}\Big| \le \frac{1}{nk}\sum_{i=1}^{n}(i+k-1)w_\infty(a) = \frac{1}{k}\left((k-1)+\frac{n+1}{2}\right)w_\infty(a),$$

which is less than $2w_\infty(a)$ when $n \le k$; since n and k play a symmetric role the desired inequality is proved.

Finally, put $\delta_i^k = 1$ if $k = i$ and $\delta_i^k = 0$ if $k \ne i$ for $k,\, i \ge 1$ (Kronecker symbol). Then the sequence δ^k is in V_∞; observe that $v_\infty(\delta^k) = 1/k$ and $v_\infty(T^{k-1}\delta^k) = 1$.

PROPOSITION 3. *Let A be the set of real sequences $(a_n)_{n\ge1}$ which converge in the Cesàro sense. Then A is a closed subspace of V_∞. On A the restriction of the shift operator T is mean ergodic, that is, for every $a \in A$, the sequence $(k+1)^{-1}\sum_{i=0}^{k} T^i a$ converges in v_∞-norm. In other words, if $\lim_n n^{-1}\sum_{i=1}^{n} a_i = l$ then*

$$\lim_k \frac{1}{k+1}\sup_n \frac{1}{n}\Big|\sum_{i=0}^{k}\sum_{j=1}^{n}(a_{i+j}-l)\Big| = 0.$$

P r o o f. By the isometry S introduced above, the subspace A corresponds to the space of convergent sequences which is a closed subspace of l_∞. By Proposition 2, T is Cesàro bounded on the Banach space A with the norm v_∞. We leave to the interested reader the exercise to check that:

(i) for $a \in A$, $\lim_k v_\infty(T^k a)/k = 0$.
(ii) for $a \in A$, the sequence $(n+1)^{-1}\sum_{i=0}^{n} T^i a$ converges weakly in A.

Then the proposition follows from the mean ergodic theorem.

PROPOSITION 4. *The space V_2 is a Hilbert space with the scalar product*

$$\langle a,b\rangle_{V_2} = \sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} b_i\right),$$

and norm v_2. The shift operator T is Cesàro bounded in V_2, that is,

$$\sup \frac{1}{k+1} v_2\Big(\sum_{i=0}^{k} T^i\Big) < \infty,$$

but $v_2(T^k) \geq k$ *for every* k; *moreover* T *is mean ergodic in* V_2:

$$\lim_k \frac{1}{k+1} v_2\Big(\sum_{i=0}^{k} T^i a\Big) = 0 \quad \textit{for every } a \in V_2.$$

Proof. The Hilbert space structure of V_2 is deduced from the structure of l_2 through the isometry S defined above.

In V_2 the sequence $e^k = k(\delta^k - \delta^{k+1})$, for $k \geq 1$, is an orthonormal Hilbert basis (δ_i^k is the Kronecker symbol as above). It is the image under S of the canonical basis δ^k of the space l_2.

We have $v_2(T^k e^{k+1}) = k+1$ and $T^k e^l = 0$ if $k \geq l+2$. Therefore the proposition will be proved as soon as the Cesàro boundedness of T is checked.

To check that T is Cesàro bounded in V_2 we need rather involved computations. We put

$$s^{j,m} = \sum_{l=0}^{m} T^l e^j = \begin{cases} j(\delta^{j-m} - \delta^{j+1}) & \text{if } m \leq j-1, \\ -j\delta^{j+1} & \text{if } m \geq j. \end{cases}$$

We compute

$$\langle s^{i,m}, s^{j,m}\rangle_{V_2} = \sum_{n=1}^{\infty} \Big(\frac{1}{n}\sum_{\iota=1}^{n} s_\iota^{i,m}\Big)\Big(\frac{1}{n}\sum_{\iota=1}^{n} s_\iota^{j,m}\Big) \quad \text{for } 1 \leq i \leq j.$$

We find

$$\langle s^{i,m}, s^{j,m}\rangle_{V_2} = \begin{cases} 0 & \text{for } i \geq m+1 \text{ and } j \geq m+i+1, \\ ij\sum_{n=j-m}^{i}(1/n^2) & \text{for } i \geq m+1 \text{ and } j < m+i+1, \\ -ij\sum_{n=\max((i+1),(j-m))}^{j+1}(1/n^2) & \text{for } i \leq m \text{ and } j \geq m+1, \\ ij\sum_{n=j+1}^{\infty}(1/n^2) & \text{for } i \leq m \text{ and } j \leq m. \end{cases}$$

Let $a \in V_2$ have the expansion $a = \sum_{i=1}^{\infty} x_i e^i$ with respect to the Hilbert basis (e^i) where $x \in l_2$, that is, $\sum_{i=1}^{\infty} x_i^2 = \|x\|_2^2 = v_2(a) < \infty$. Then

$$\begin{aligned} v_2\Big(\sum_{l=0}^{m} T^l a\Big)^2 &= \sum_{i=1}^{\infty} x_i^2 v_2(s^{i,m})^2 + 2\sum_{1\leq i<j} x_i x_j \langle s^{i,m}, s^{j,m}\rangle_{V_2} \\ &= \sum_{i=1}^{m} x_i^2 \sum_{n=i+1}^{\infty} \frac{i^2}{n^2} + \sum_{i=m+1}^{\infty} x_i^2 \sum_{n=i-m}^{i} \frac{i^2}{n^2} \end{aligned}$$

$$+\,2\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}x_ix_j\sum_{n=j+1}^{\infty}\frac{ij}{n^2}+2\sum_{i=1}^{m}\sum_{j=m+1}^{m+i}x_ix_j\sum_{n=i+1}^{j+1}\frac{-ij}{n^2}$$

$$+\,2\sum_{i=1}^{m}\sum_{j=m+i+1}^{\infty}x_ix_j\sum_{n=j-m}^{j+1}\frac{-ij}{n^2}+2\sum_{i=m+1}^{\infty}\sum_{j=i+1}^{m+i}x_ix_j\sum_{n=j-m}^{i}\frac{ij}{n^2}.$$

We have to show the existence of an absolute constant C, independent of m and x, such that $v_2(\sum_{l=0}^{m}T^la)^2\le C(m+1)^2\|x\|_2^2$. We shall prove it for each of the six expressions appearing above; the letter C may denote different values along the computations, but always represents an absolute constant.

To begin with, we replace each term by its absolute value, and we estimate

$$\sum_{n=t+1}^{u}\frac{1}{n^2}\le\frac{u-t}{tu}.$$

The estimation of the first two expressions is straightforward.

The next two expressions are easily majorized using the Cauchy–Schwarz inequality, and estimates like $\sum_{i=t}^{u}i\le C(u^2-t^2)$.

The fifth expression is less than

$$2\sum_{i=1}^{m}\sum_{j=m+i+1}^{\infty}x_ix_j\frac{ij(m+1)}{(j-m)(j+1)}\le 2\|x\|_2(m+1)\sum_{i=1}^{m}\sqrt{i}\,x_i\le C(m+1)^2\|x\|_2^2,$$

by using the Cauchy–Schwarz inequality twice.

The last expression requires a more delicate treatment. After interchanging the sums on i and j, the last expression is majorized by

$$(m+1)C\bigg[\sum_{j=2m+1}^{\infty}\frac{j|x_j|}{j-m}\sum_{i=j-m}^{j-1}|x_i|+\sum_{j=2+m}^{2m}\frac{j|x_j|}{j-m}\sum_{i=1+m}^{j-1}|x_i|\bigg]$$
$$\le 2(m+1)^2C\bigg[\sum_{j=2m+1}^{\infty}\frac{|x_j|}{m+1}\sum_{i=j-m}^{j-1}|x_i|+\sum_{j=2+m}^{2m}\frac{|x_j|}{j-m}\sum_{i=1+m}^{j-1}|x_i|\bigg].$$

By convexity the sequence $y_j=(m+1)^{-1}\sum_{i=j-m}^{j-1}|x_i|$ belongs to l_2 and $\|y\|_2\le\|x\|_2$. The sequence $z_j=(j-m)^{-1}\sum_{i=1+m}^{j-1}|x_i|$ also belongs to l_2 and $\|z\|_2\le 2\|x\|_2$ by a classical inequality of Hardy (see [DS], p. 522, ex. 22; [Z], p. 20); this inequality means that $a\in l_2\Rightarrow a\in V_2$ and $v_2(a)\le 2\|a\|_2$. Hence the Cauchy–Schwarz inequality yields the desired result.

Remarks. 1. Proposition 4 gives an example of a mean ergodic operator T in a Hilbert space such that $\|T^n\|/n$ does not converge to 0.

2. For a real sequence a_n, Proposition 4 means that

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)^2 < \infty \Rightarrow \lim_m \sum_{n=1}^{\infty}\left(\frac{1}{nm}\sum_{i=1}^{n}\sum_{j=0}^{m-1} a_{i+j}\right)^2 = 0.$$

3. By convexity, the shift operator T is also Cesàro bounded in V_p for $2 \le p \le \infty$. For $1 < p < 2$ this property is unknown to the author. It would be desirable to give a more direct proof of this property in V_2 or V_p.

4. Consider now the dual operator T^* of the shift operator T in the Hilbert space V_2. It is also Cesàro bounded, and by duality we get

$$\text{weak-}\lim_k \frac{1}{k+1}\sum_{i=0}^{k} T^{*i}a = 0 \quad \text{for every } a \in V_2.$$

The strong convergence does not hold, as the following easy computations show: for $a \in V_2$ we have $\langle e^k, a\rangle_{V_2} = k^{-1}(a_1 + \ldots + a_k)$; we also have $\langle T^{*i}e^1, a\rangle_{V_2} = a_{i+1}$, thus $k^{-1}\sum_{i=0}^{k-1} T^{*i}e^1 = e^k$, where $(e^k)_{k\ge1}$ is the orthonormal Hilbert basis of the space V_2, $e^k = k(\delta^k - \delta^{k+1})$, introduced above.

5. The weak convergence of the Cesàro averages of T^* can be deduced from our Theorem 2. In V_2 the set of nonnegative sequences is a closed convex cone V_2^+. The dual cone is $\mathcal{F} = \{b \in V_2 : \langle b, a\rangle_{V_2} \ge 0 \text{ for every } a \in V_2^+\}$. It is obvious that V_2^+ is stable under T; then $\mathcal{F}$ is a closed convex cone which is stable under T^*. It is easy to check that $\mathcal{F} \cap (-\mathcal{F}) = 0$, and $\mathcal{F} - \mathcal{F} = V_2$. Therefore Theorem 2 applies; this is an example where the strong convergence does not hold. The geometric idea is that the cone V_2^+ being "thin" in the space V_2, the dual cone $\mathcal{F}$ is "wide" in V_2.

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