COLLOQUIUM MATHEMATICUM

VOL. 84/85 2000 PART 2

A $\mathbb{Z}^d$ GENERALIZATION OF THE DAVENPORT–ERDŐS CONSTRUCTION OF NORMAL NUMBERS

BY

MORDECHAY B. LEVIN (RAMAT-GAN) AND
MEIR SMORODINSKY (TEL-AVIV)

Dedicated to the memory of the late Anzelm Iwanik,
a friend and fellow mathematician

Abstract. We extend the Davenport and Erdős construction of normal numbers to the $\mathbb{Z}^d$ case.

1. Introduction. A number $\alpha \in (0,1)$ is said to be *normal* to the base b if in the b-ary expansion of α, $\alpha = .d_1d_2\ldots$ ($d_i \in \{0,1,\ldots,b-1\}$, $i = 1,2,\ldots$), each fixed finite block of digits of length k appears with an asymptotic frequency of b^{-k} along the sequence $(d_i)_{i\ge1}$. Normal numbers were introduced by Borel [B]. Champernowne [C] gave an explicit construction of such a number, namely

$$\theta = .123456789101112\ldots$$

obtained by successively concatenating all the natural numbers written to base 10.

Let $\varphi(x) = \alpha x^r + \alpha_1 x^{r-1} + \ldots + \alpha_{r-1}x + \alpha_r$ ($\alpha > 0$, $r \ge 1$) be a polynomial with integer coefficients such that $\varphi(n) \ge 0$ ($n = 1,2,\ldots$). Davenport and Erdős [DE] generalized Champernowne's construction and proved that the number

$$.\varphi(1)\varphi(2)\ldots\varphi(n)\ldots$$

obtained by successively concatenating the b-expansions of the numbers $\varphi(n)$ ($n = 1,2,\ldots$) is also normal. We refer the reader to other generalizations of Champernowne's construction which appear in [AKS] and [SW].

In [LeSm] we extend Champernowne's construction to $\mathbb{Z}^d$, $d > 1$, arrays of random variables, which we shall call $\mathbb{Z}^d$-*processes*. We shall deal with *stationary* $\mathbb{Z}^d$-*processes*, that is, processes with distribution invariant

2000 *Mathematics Subject Classification*: Primary 11K16, 28D15.
Work supported in part by the Israel Science Foundation Grant No. 366-172.

under the $\mathbb{Z}^d$ action. We shall call a specific realization of a $\mathbb{Z}^d$-process a "configuration".

In this note we generalize the Davenport and Erdős construction to the $\mathbb{Z}^d$ case. For the sake of clarity, we carry out the proof only for the case $d = 2$. The generalization for general $d > 2$ is easy and straightforward. We begin with a very simple generalization (see also [Ci] and [KT]).

We denote by $\mathbb{N}$ the set of non-negative integers. Let $d, b \geq 2$ be two integers, $\mathbb{N}^d = \{(n_1, \dots, n_d) \mid n_i \in \mathbb{N},\ i = 1, \dots, d\}$, $\Delta_b = \{0, 1, \dots, b-1\}$, $\Omega = \Delta_b^{\mathbb{N}^d}$.

We shall call $\omega \in \Omega$ a *configuration* (*lattice configuration*). A configuration is thus a function $\omega : \mathbb{N}^d \to \Delta_b$.

Given a subset F of $\mathbb{N}^d$, ω_F will be the restriction of the function ω to F. Let $\mathbf{N} \in \mathbb{N}^d$, $\mathbf{N} = (N_1, \dots, N_d)$. We denote a *rectangular block* by

$$F_{\mathbf{N}} = \{(f_1, \dots, f_d) \in \mathbb{N}^d \mid 0 \leq f_i < N_i,\ i = 1, \dots, d\},$$

$\mathbf{h} = (h_1, \dots, h_d)$, $h_i \geq 1$, $i = 1, \dots, d$; $G = G_{\mathbf{h}}$ is a fixed block of digits $G = (g_{\mathbf{i}})_{\mathbf{i} \in F_{\mathbf{h}}}$, $g_{\mathbf{i}} \in \Delta_b$, $\chi_{\omega,G}(\mathbf{f})$ is the characteristic function of the block of digits G shifted by the vector $\mathbf{f}$ in the configuration ω:

$$\chi_{\omega,G}(\mathbf{f}) = \begin{cases} 1 & \text{if } \omega(\mathbf{f} + \mathbf{i}) = g_{\mathbf{i}},\ \forall \mathbf{i} \in F_{\mathbf{h}}, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Definition. $\omega \in \Omega$ is said to be *rectangular normal* if for any $\mathbf{h} \in \mathbb{N}^d$ and block $G_{\mathbf{h}}$,

$$\#\{\mathbf{f} \in F_{\mathbf{N}} \mid \chi_{\omega,G}(\mathbf{f}) = 1\} - b^{-h_1 \dots h_d} N_1 \dots N_d = o(N_1 \dots N_d) \tag{2}$$

as $\max(N_1, \dots, N_d) \to \infty$.

As remarked, in what follows we shall consider the case $d = 2$.

Construction. The map

$$L(f_1, f_2) = \begin{cases} f_1^2 + f_2 & \text{if } f_2 < f_1, \\ f_2^2 + 2f_2 - f_1 & \text{if } f_2 \geq f_1, \end{cases} \tag{3}$$

is a bijection between $\mathbb{N}$ and $\mathbb{N}^2$, inducing a total order on $\mathbb{N}^2$ from the usual one on $\mathbb{N}$. Let $I_n = [\alpha^{-1/(2r)} b^{2n^2/r}]$, $n = 1, 2, \dots$ We define the configuration ω_n on $F_{(2nI_n, 2nI_n)}$ as the concatenation of I_n^2 $2n \times 2n$ blocks of digits with the lower left corner $(2nx, 2ny)$, $0 \leq x, y < I_n$. To each of these blocks we assign the number $\varphi(L(x, y))$. Next we use the b-expansion of $\varphi(L(x, y))$ according to the order L to obtain the digits of the $2n \times 2n$ block considered. It is easy to obtain an analytic expression for the digits of the configuration ω_n:

$$\omega_n(2nx + s, 2ny + t) = a_{L(s,t)}(u), \tag{4}$$

where

$$u = u(x, y) = \varphi(L(x, y)), \tag{5}$$

s, t, x, y are integers, $0 \le x, y < I_n, 0 \le s, t < 2n$, and

$$n = \sum_{i \ge 0} a_i(n) b^i \tag{6}$$

is the b-expansion of the integer n.

Next we define inductively a sequence of increasing configurations ω_n on $F_{(2nI_n, 2nI_n)}$. Put $\omega'_1 = \omega_1$, $\omega'_{n+1}(\mathbf{f}) = \omega'_n(\mathbf{f})$ for $\mathbf{f} \in F_{(2nI_n, 2nI_n)}$ and $\omega'_{n+1}(\mathbf{f}) = \omega_{n+1}(\mathbf{f})$ otherwise. Put

$$\omega_\infty = \lim\ \omega'_n, \qquad (\omega_\infty)_{F_{(2nI_n, 2nI_n)}} = \omega'_n, \qquad n = 1, 2, \ldots \tag{7}$$

THEOREM. ω_∞ *is rectangular normal.*

The proof of the Theorem is given in Section 3.

2. Auxiliary notation and results. Let $(u_x)_{x \ge 0}$ be an arbitrary sequence in $[0, 1)$. The quantity

$$D(N) = D((u_x)_{x=0}^{N-1}) = \sup_{\gamma \in (0,1]} \left| \frac{1}{N} \#\{0 \le n \le N - 1 \mid u_x \in [0, \gamma)\} - \gamma \right| \tag{8}$$

is called the *discrepancy* of $(u_x)_{x=0}^{N-1}$. The sequence $(u_x)_{x \ge 0}$ is said to be *uniformly distributed* in $[0, 1)$ if $D(N) \to 0$.

To estimate the discrepancy we use the Erdős–Turán inequality (see, for example, [DrTi], p. 15)

$$ND(N) \le \frac{3}{2} \left(\frac{2N}{H + 1} + \sum_{0 < |m| \le H} \frac{|\sum_{x=0}^{N-1} e(mu_x)|}{\overline{m}} \right), \tag{9}$$

where $e(y) = e^{2\pi i y}$, $\overline{m} = \max(1, |m|)$ and $H \ge 1$ is arbitrary.

We shall use the following Weyl inequality (see, for example, [DrTi], p. 15):

$$\left| \sum_{x=1}^{L} e(\psi(x)) \right| \le C(\theta) L^{1+\theta} (q^{-1} + L^{-1} + qL^{-k})^{2^{1-k}}, \tag{10}$$

where $\psi(x) = \beta x^k + \beta_1 x^{k-1} + \ldots + \beta_{k-1} x + \beta_k$, $|\beta - p/q| < 1/q^2$, $(p, q) = 1$ and $\theta > 0$ is arbitrary.

3. Proof of the Theorem. Consider the configuration ω_n, where n satisfies the following inequality:

$$2(n - 1) I_{n-1} \le \max(N_1, N_2) < 2nI_n.$$

Let $h_1, h_2 \ge 1$ be integers and

$$d_{i_1, i_2} \in \{0, 1, \ldots, b - 1\}, \qquad 0 \le i_1 < h_1,\ 0 \le i_2 < h_2.$$

We consider the block of digits $G = (d_{i_1, i_2})_{0 \le i_1 < h_1,\, 0 \le i_2 < h_2}$, the configuration ω_n, and the block of digits $\omega_0 = (\omega_n(i, j))_{0 \le i < N_1,\, 0 \le j < N_2}$.

To compute the number of appearances of the block G in the configuration ω_0, we introduce the following notation (see (1)):

$$V_{n,G}(L_1, M_1; L_2, M_2) = \bigcup_{(i,j)\in[L_1,L_1+M_1)\times[L_2,L_2+M_2)} \{(i,j) \mid \chi_{\omega_n,G}(i,j) = 1\}, \tag{11}$$

$$V_{n,G}(N_1, N_2) = V_{n,G}(0, N_1; 0, N_2). \tag{12}$$

Let

$$N_1 = 2nN_{11} + N_{12}, \qquad N_2 = 2nN_{21} + N_{22}, \quad \text{with } N_{12}, N_{22} \in [0, 2n). \tag{13}$$

Next, we fix $s, t \in [0, 2n)$, and compute the number of appearances of G in the configuration $\omega_0 = (\omega_n(i,j))_{0\le i<N_1,\,0\le j<N_2}$ such that the shift of the block G by the vector (i,j) satisfies $i \equiv s \pmod{2n}$, $j \equiv t \pmod{2n}$. Set

$$A_{s,t,G}(M_1, M_2) = \bigcup_{(i,j)\in[0,2nM_1)\times[0,2nM_2)} \{(i,j) \mid \chi_{\omega_n,G}(i,j) = 1 \text{ and } i \equiv s,\ j \equiv t \pmod{2n}\}. \tag{14}$$

Let $\varepsilon > 0$ be arbitrary. To complete the proof of the Theorem it is sufficient to prove that for all $s, t \in [\varepsilon n, 2n(1-\varepsilon))$,

$$|\#A_{s,t,G}(M_1, M_2) - b^{-h_1h_2} M_1M_2| < \varepsilon M_1M_2.$$

Observe that

$$V_{n,G}(N_1, N_2) = V_{n,G}(2nN_{11}, 2nN_{21}) \cup V_{n,G}(0, 2nN_1; 2nN_{21}, N_{22}) \cup V_{n,G}(2nN_{11}, N_{12}; 0, N_2) \tag{15}$$

and

$$V_{n,G}(2nN_{11}, 2nN_{21}) = \bigcup_{0\le s<2n} \bigcup_{0\le t<2n} A_{s,t,G}(N_{11}, N_{21}), \tag{16}$$

$$V_{n,G}(0, 2nN_{11}; 2nN_{21}, N_{22}) = \bigcup_{0\le s<2n} \bigcup_{0\le t<N_{22}} (A_{s,t,G}(N_{11}, N_{21}+1) \setminus A_{s,t,G}(N_{11}, N_{21})). \tag{17}$$

Now let

$$v(i_1, i_2) = v(s, t, i_1, i_2) = L(s + i_1, t + i_2). \tag{18}$$

Everywhere below $0 \le s, t < 2n - h_1h_2$.

Using (4)–(6) we see that the condition

$$\omega_n(2nx + s + i_1, 2ny + t + i_2) = d_{i_1,i_2}, \qquad \forall (i_1, i_2) \in [0, h_1) \times [0, h_2), \tag{19}$$

is equivalent to the statement

$$a_{v(i_1,i_2)}(u(x,y)) = d_{i_1,i_2}, \qquad \forall (i_1, i_2) \in [0, h_1) \times [0, h_2). \tag{20}$$

From (14), (1) and (19), (20) we obtain

$$\begin{aligned}(21)\quad & A_{s,t,G}(M_1,M_2)\\ &= \{(2nx+s,2ny+t)\in[0,2nM_1)\times[0,2nM_2) \mid \\ &\qquad a_{v(i_1,i_2)}(u(x,y)) = d_{i_1,i_2},\ \forall (i_1,i_2)\in[0,h_1)\times[0,h_2)\}.\end{aligned}$$

Let $k_1,\ldots,k_h$ ($h=h_1h_2$) be an increasing sequence of integers from the set

$$(22)\quad \{v(s,t,i_1,i_2)+1 \mid i_1=0,1,\ldots,h_1-1,\ i_2=0,1,\ldots,h_2-1\},$$

and $\mu(i_1,i_2)\in[1,h]$ ($(i_1,i_2)\in[0,h_1)\times[0,h_2)$) be a sequence of integers so that

$$(23)\quad \mu(i_1,i_2)>\mu(j_1,j_2)\Leftrightarrow v(s,t,i_1,i_2)>v(s,t,j_1,j_2),$$

where $i_\nu,j_\nu\in[0,h_\nu)$, $\nu=1,2$. It is evident that

$$(24)\quad k_{\mu(i_1,i_2)} = v(s,t,i_1,i_2)+1,\quad i_1=0,1,\ldots,h_1-1,\ i_2=0,1,\ldots,h_2-1.$$

Now put

$$(25)\quad d_{\mu(i_1,i_2)} = d_{i_1,i_2},\quad i_1=0,1,\ldots,h_1-1,\ i_2=0,1,\ldots,h_2-1.$$

From (21)–(25) we see that

$$\begin{aligned}(26)\quad A_{s,t,G}(M_1,M_2) &= \{(2nx+s,2ny+t)\in[0,2nM_1)\times[0,2nM_2) \mid \\ &\qquad a_{k_i-1}(u(x,y)) = d_i,\ \forall i\in[1,h_1h_2]\}.\end{aligned}$$

LEMMA 1. *Let* $M_1,M_2\in[0,I_n)$, $I_n=[\alpha^{-1/(2r)}b^{2n^2/r}]$, $s,t\in[0,2n-15h]$, $h=h_1h_2$. *Then*

$$\begin{aligned}(27)\quad & \#A_{s,t,G}(M_1,M_2)\\ &= \sum_{x_2=0}^{b^{k_2-k_1-1}-1}\cdots\sum_{x_h=0}^{b^{k_h-k_{h-1}-1}-1} B_{st}(M_1,M_2,d(x_2,\ldots,x_h)),\end{aligned}$$

where

$$\begin{aligned}(28)\quad B_{st}(M_1,M_2,d) = \#\{&(x,y)\in[0,M_1)\times[0,M_2) \mid \\ &\{u(x,y)b^{-k_h}\}\in[d/b^{k_h-k_1+1},(d+1)/b^{k_h-k_1+1})\},\end{aligned}$$

and

$$\begin{aligned}(29)\quad d &= d(x_2,\ldots,x_h)\\ &= d_1 + x_2 b + d_2 b^{k_2-k_1} + \ldots + x_h b^{k_{h-1}-k_1+1} + d_h b^{k_h-k_1}.\end{aligned}$$

Proof. From (6), we see that the condition $a_{k_i-1}(u(x,y)) = d_i$, $\forall i\in[1,h]$, is equivalent to the statement

$$u(x,y) = x_1 + d_1 b^{k_1-1} + x_2 b^{k_1} + d_2 b^{k_2-1} + \ldots + x_h b^{k_{h-1}} + d_h b^{k_h-1} + x_{h+1} b^{k_h},$$

with $x_i\in[0,b^{k_i-k_{i-1}-1})$, $k_0=0$, $i=1,2,\ldots,h$, $x_{h+1}\ge 0$. Using (26) and (29) we get

$$(30)\quad A_{s,t,G}(M_1,M_2) = \bigcup_{x_2=0}^{b^{k_2-k_1-1}-1} \cdots \bigcup_{x_h=0}^{b^{k_h-k_{h-1}-1}-1} \{(2nx+s,2ny+t)\in[0,2nM_1)\times[0,2nM_2) \mid u(x,y)=x_1+d(x_2,\ldots,x_h)b^{k_1-1}+x_{h+1}b^{k_h}\}$$

for arbitrary integers $x_1 \in [0,b^{k_1-1})$, $x_{h+1}\ge 0$. Bearing in mind that the condition

$$u(x,y)=x_1+db^{k_1-1}+x_{h+1}b^{k_h}$$

is equivalent to the condition

$$\{u(x,y)b^{-k_h}\}\in\left[\frac{d}{b^{k_h-k_1+1}},\frac{d+1}{b^{k_h-k_1+1}}\right)$$

we deduce from (30) and (28) that

$$A_{s,t,G}(M_1,M_2) = \bigcup_{x_2=0}^{b^{k_2-k_1-1}-1} \cdots \bigcup_{x_h=0}^{b^{k_h-k_{h-1}-1}-1} \{(2nx+s,2ny+t)\in[0,2nM_1)\times[0,2nM_2) \mid \{u(x,y)b^{-k_h}\}\in[d/b^{k_h-k_1+1},(d+1)/b^{k_h-k_1+1})\}. \ \blacksquare$$

LEMMA 2. *Let* $1\le M_2\le M_1\in[b^{\xi 2n^2/r},I_n)$, $I_n=[\alpha^{-1/(2r)}b^{2n^2/r}]$, $\xi=(1-\varepsilon)^2+\varepsilon\in(0,1)$, $s,t\in[\varepsilon n,2n(1-\varepsilon)]$, $h=h_1h_2$, $n\ge 4/\varepsilon^2$, $\varepsilon\in(0,1/(4r))$ *and* $0<|m|\le H=b^{k_h-k_1+s+t}$. *Then*

$$(31)\quad S(m)=\sum_{y=0}^{M_2-1}\sum_{x=0}^{M_1-1}e(mu(x,y)b^{-k_h})=O(M_1M_2H^{-1}b^{-n^2\varepsilon^2 2^{-2r-2}}).$$

Proof. By (22), (18) and the condition of the lemma, we get

$$(32)\quad k_1=\max(s^2+t,t^2+t-s),\qquad k_1-s-t>\varepsilon^2n^2/2,$$

$$(33)\quad 0\le k_h-k_1\le 2sh_1+2th_2+2h_1^2+2h_2^2\le 8nh+4h^2,\qquad H=O(b^{16nh}).$$

Let

$$(34)\quad M_0\in[b^{\xi_1 2n^2/r},I_n],\qquad \xi_1=(1-\varepsilon)^2+\varepsilon^2,$$

and

$$\sigma(y)=\sum_{x=0}^{M_0-1}e(m\varphi(x^2+y)b^{-k_h}).$$

Applying Weyl's inequality (10) with $\theta=\varepsilon^2r2^{-2r-2}$, $L=M_0$, $k=2r$, $\beta=\alpha m b^{-k_h}$, $q=b^{k_h}/d$ and $d=\gcd(b^{k_h},\alpha m)$, where $\alpha>0$ is an integer, we obtain

(35) $$|\sigma(y)| \le C(\varepsilon^2 r 2^{-2r-2}) M_0^{1+\varepsilon^2 r 2^{-2r-2}} (b^{-k_h} d + M_0^{-1} + b^{k_h} d^{-1} M_0^{-2r})^{2^{-2r+1}}.$$

Using the assumption of the lemma, (34), (18), (22) and (32), (33) we get

(36) $$b^{-k_h} d \le b^{-k_h}\alpha|m| \le \alpha b^{-k_h} H = \alpha b^{-k_1+s+t} = O(b^{-k_1/2}) = O(b^{-\varepsilon^2 n^2/2}),$$

(37) $$M_0^{-1} \le b^{-2((1-\varepsilon)^2+\varepsilon^2)n^2/r} < b^{-n^2/r},$$

(38) $$b^{k_h} d^{-1} M_0^{-2r} \le b^{(k_h)_{\max}} (M_0)_{\min}^{-2r} \le b^{4n^2(1-\varepsilon)^2+2n-2r(((1-\varepsilon)^2+\varepsilon^2)2n^2/r)} = b^{-4n^2\varepsilon^2+2n} = O(b^{-2n^2\varepsilon^2}).$$

Now from (33)–(38) we have

$$M_0^{-1}\sigma(y) = O(M_0^{\varepsilon^2 r 2^{-2r-2}} b^{-\varepsilon^2 n^2 2^{-2r}}) = O(b^{-\varepsilon^2 n^2 2^{-2r-1}}),$$

and

(39) $$H M_0^{-1}\sigma(y) = O(b^{-\varepsilon^2 n^2 2^{-2r-2}}).$$

Putting

(40) $$\sigma_1 = \sum_{x=0}^{M_2^2-1} e(m\varphi(x) b^{-k_h}),$$

(41) $$\sigma_2 = \sum_{y=0}^{M_2-1} \sum_{x=0}^{M_1-1} e(m\varphi(x^2+y) b^{-k_h}),$$

(42) $$\sigma_3 = \sum_{x,y=0}^{M_2-1} e(m\varphi(x^2+y) b^{-k_h}),$$

and using (5) and (31), we obtain

(43) $$S(m) - \sigma_1 = \sum_{y=0}^{M_2-1} \sum_{x=M_2}^{M_1-1} e(mu(x,y) b^{-k_h}) = \sigma_2 - \sigma_3.$$

If $M_2 < b^{\xi_1 2n^2/r}$, we apply (39) with $M_0 = M_1$ for σ_2, and the trivial estimate for σ_1 and σ_3:

(44) $$H M_1^{-1} M_2^{-1} S(m) = O(b^{-\varepsilon^2 n^2 2^{-2r-2}} + (H M_1^{-1} M_2^{-1}) M_2^2) = O(b^{-\varepsilon^2 n^2 2^{-2r-2}} + b^{16nh+(\xi_1-\xi)2n^2/r}) = O(b^{-\varepsilon^2 n^2 2^{-2r-2}}).$$

Now let $M_2 \ge b^{\xi_1 2n^2/r}$. We apply (39) with $M_0 = M_2$ for σ_2 and for σ_3:

(45) $$H M_1^{-1} M_2^{-1}(|\sigma_2| + |\sigma_3|) = O(b^{-\varepsilon^2 n^2 2^{-2r-2}}).$$

To estimate the sum σ_1 we apply Weyl's inequality with $\theta = \varepsilon^2 r 2^{-2r-3}$, $L = M_2^2$, $k = r$, $\beta = \alpha m b^{-k_h}$, $q = b^{k_h}/d$, $d = \gcd(b^{k_h}, \alpha m)$, and repeat the calculations (35)–(39):

$$HM_1^{-1}M_2^{-1}|\sigma_1| = O(b^{-\varepsilon^2 n^2 2^{-2r-2}}). \tag{46}$$

By (44)–(46) the assertion of the lemma follows. ∎

Lemma 3. *Under the assumptions of Lemma* 2,

$$D = D((\{u(x,y)b^{-k_h}\})_{x=0,\ y=0}^{M_1-1,\ M_2-1}) = O(b^{k_1-k_h-s-t}). \tag{47}$$

Proof. We apply Lemma 2, (31), (33) and Erdős–Turán's inequality with $H = b^{k_h-k_1+s+t}$ to get

$$\begin{aligned} D &= O\biggl(H^{-1} + (M_1M_2)^{-1} \sum_{0<|m|\le H} \frac{|S(m)|}{\overline{m}}\biggr) \\ &= O\biggl(H^{-1}\biggl(1 + \frac{1}{s+t+1} \sum_{0<|m|\le H} \frac{1}{\overline{m}}\biggr)\biggr) \\ &= O(H^{-1}(1 + (s+t+1)^{-1}\log H)) \\ &= O(H^{-1}(1 + (s+t+1)^{-1}(k_h - k_1 + s + t))) = O(H^{-1}). \ \blacksquare \end{aligned}$$

Using the definition of discrepancy (8), we get:

Corollary 1. *Under the assumptions of Lemma* 2,

$$B_{st}(M_1, M_2, d) = M_1M_2b^{k_1-k_h-1}(1 + O(b^{-s-t})), \tag{48}$$

where $B_{st}(M_1, M_2, d)$ *is defined in* (28).

Corollary 2. *Under the assumptions of Lemma* 2,

$$\#A_{s,t,G}(M_1, M_2) = b^{-h}M_1M_2 + O(M_1M_2b^{-s-t}). \tag{49}$$

Proof. This follows from (28), Lemma 1 and Corollary 1. ∎

Lemma 4. *Under the assumptions of Lemma* 2, *let* $1 \le N_2 \le N_1 \in [2nb^{\varepsilon 2n^2/r}, 2nI_n)$. *Then*

$$\#V_{n,G}(N_1, N_2) - b^{-h}4n^2N_1N_2 = 200\varepsilon_0 N_1N_2 + O(N_1N_2/n), \qquad |\varepsilon_0| \le \varepsilon.$$

Proof. Using (16) we have

$$\begin{aligned} &V_{n,G}(2nN_{11}, 2nN_{21}) \\ &\quad = \bigcup_{\varepsilon n\le s,t<2n(1-\varepsilon)} \ \bigcup_{\min(s,t)<\varepsilon n} \ \bigcup_{2n(1-\varepsilon)\le \max(s,t)<2n} A_{s,t,G}(N_{11}, N_{21}). \end{aligned} \tag{50}$$

We apply (49) to the first union and the trivial estimates to the other unions:

$$\begin{aligned} &\#V_{n,G}(2nN_{11}, 2nN_{21}) \\ &\quad = \sum_{\varepsilon n\le s,t<2n(1-\varepsilon)} (b^{-h}N_{11}N_{21} + O(N_{11}N_{21}b^{-s-t})) + 16\varepsilon_1 n^2 N_{11}N_{21} \end{aligned} \tag{51}$$

$$= b^{-h}4n^2N_{11}N_{21} + 32\varepsilon_2 n^2 N_{11}N_{21} + O(N_{11}N_{21}),$$
$$N_{21} \ge 1, \quad |\varepsilon_i| < \varepsilon,\ i = 1, 2.$$

Similarly, from (17) and (49) we obtain

$$(52)\qquad \#V_{n,G}(0, 2nN_{11}; 2nN_{21}, N_{22}) = \sum_{0\le s<2n}\sum_{0\le t<N_{22}} (b^{-h}N_{11} + O(N_{11}b^{-s-t})) + 16\varepsilon_3 nN_{11}N_{22}$$
$$= b^{-h}2nN_{11}N_{22} + 32\varepsilon_4 nN_{11}N_{22} + O(N_{11}N_{22})$$

5 with $|\varepsilon_i| < \varepsilon$, $i = 3, 4$. We get a trivial estimate from (11)–(13):

$$\#V_{n,G}(2nN_{11}, N_{12}; 0, N_2) \le N_2N_{12} \le 2nN_2 < N_1N_2/n.$$

Now the assertion of the lemma follows from (13), (15), and (51)–(52). ∎

Similar notation is introduced for the configuration ω (instead of ω_n):

$$(53)\qquad V_G(P_1, P_2) = \{(v_1, v_2) \in [0, P_1) \times [0, P_2) \mid \omega(v_1 + i_1, v_2 + i_2) = d_{i_1,i_2},\ \forall (i_1, i_2) \in [0, h_1) \times [0, h_2)\}.$$

We prove the Theorem for the case $N_1 \ge N_2$. The other case is similar.

End of the proof of the Theorem. Let $1 \le N_2 \le N_1$, $N_1 \ge 4b^8$. Then there exists $n \ge 3$ so that

$$(54)\qquad N_1 \in [2(n-1)I_{n-1} - h, 2nI_n - h).$$

Now let

$$(55)\qquad N_1' = 2(n-1)I_{n-1} - h, \qquad N_2' = \min(N_2, N_1').$$

From (53) and the definition of the configurations ω, ω_n we get

$$(56)\qquad \#V_G(N_1, N_2) = \#V_{n,G}(N_1, N_2) - \#V_{n,G}(N_1', N_2') + \#V_G(N_1', N_2') + 2\varepsilon_1 hN_2' + 2\varepsilon_2 N_1 \min(h, N_2 - N_2')$$

with $|\varepsilon_i| \le 1$, $i = 1, 2$. It is easy to see that if $N_2 \le n$, then $N_2 = N_2'$, otherwise $h \le hN_2/n$ and

$$(57)\qquad \#V_G(N_1, N_2) - \#V_{n,G}(N_1, N_2) = \#V_G(N_1', N_2') - \#V_{n,G}(N_1', N_2') + 4\varepsilon_3 hN_1N_2/n \quad \text{with } |\varepsilon_3| \le 1.$$

Analogously

$$(58)\qquad \#V_G(N_1', N_2') - \#V_{n,G}(N_1', N_2') = \#V_G(N_1'', N_2'') - \#V_{n-1,G}(N_1'', N_2'') + 4\varepsilon_4 hN_1N_2/n,$$

where

$$N_1'' = 2(n-2)I_{n-2} - h, \qquad N_2'' = \min(N_2, N_1''), \qquad |\varepsilon_4| \le 1. \tag{59}$$

It is evident that

$$\#V_G(N_1'', N_2'') + \#V_{n,G}(N_1'', N_2'') \le 2N_1''N_2'' < 2N_1N_2/n. \tag{60}$$

From (56)–(60) we obtain

$$\begin{aligned}\#V_G(N_1, N_2) = {} & \#V_{n,G}(N_1, N_2) - \#V_{n,G}(N_1', N_2') \\ & + \#V_{n-1,G}(N_1', N_2') + O(N_1N_2/n).\end{aligned}$$

It is easy to verify that

$$b^{\xi 2n^2/r} = o(I_{n-1}),$$

where $\xi = (1-\varepsilon)^2 + \varepsilon \in (0,1)$, and $I_n = [\alpha^{-1/(2r)} b^{2n^2/r}]$. Hence $N_1 \in [2nb^{\xi 2n^2/r}, 2nI_n)$ and we can apply Lemma 4:

$$\begin{aligned}&\#V_G(N_1, N_2) \\ &\quad = b^{-h}N_1N_2 - b^{-h}N_1'N_2' + 400\varepsilon_5 N_1N_2 + b^{-h}N_1'N_2' + O(N_1N_2/n) \\ &\quad = b^{-h}N_1N_2 + 400\varepsilon_5 N_1N_2 + O(N_1N_2/n) \qquad \text{with } |\varepsilon_5| \le \varepsilon. \ \blacksquare\end{aligned}$$

Now from (1), (2), and (53) we obtain the assertion of the Theorem. ■

Acknowledgments. We are grateful to the referee for his corrections and suggestions.

REFERENCES

[AKS] R. Adler, M. Keane and M. Smorodinsky, *A construction of a normal number for the continued fraction transformation*, J. Number Theory 13 (1981), 95–105.

[B] E. Borel, *Les probabilités dénombrables et leurs applications arithmétiques*, Rend. Circ. Mat. Palermo 27 (1909), 247–271.

[C] D. J. Champernowne, *The construction of decimals normal in the scale ten*, J. London Math. Soc. 8 (1933), 254–260.

[Ci] J. Cigler, *Asymptotische Verteilung reeller Zahlen mod 1*, Monatsh. Math. 64 (1960), 201–225.

[DE] H. Davenport and P. Erdős, *Note on normal numbers*, Canad. J. Math. 4 (1953), 58–63.

[DrTi] M. Drmota and R. Tichy, *Sequences*, *Discrepancies and Applications*, Lecture Notes in Math. 1651, Springer, 1997.

[KT] P. Kirschenhofer and R. F. Tichy, *On uniform distribution of double sequences*, Manuscripta Math. 35 (1981), 195–207.

[LeSm] M. B. Levin and M. Smorodinsky, *On explicit construction of normal lattices*, preprint.

[SW] M. Smorodinsky and B. Weiss, *Normal sequences for Markov shifts and intrinsically ergodic subshifts*, Israel J. Math. 59 (1987), 225–233.

Department of Mathematics and Computer Science
Bar-Ilan University
52900 Ramat-Gan, Israel
E-mail: mlevin@macs.biu.ac.il

School of Mathematical Sciences
Tel Aviv University
69978 Tel-Aviv, Israel
E-mail: meir@math.tau.ac.il

Received 27 August 1999;
revised 17 February 2000 (3825)