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COMPLETE POSITIVITY OF ENTROPY AND NON-BERNOULLICITY FOR TRANSFORMATION GROUPS

BY

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Dedicated to the memory of Anzelm Iwanik

Abstract. The existence of non-Bernoullian actions with completely positive entropy is proved for a class of countable amenable groups which includes, in particular, a class of Abelian groups and groups with non-trivial finite subgroups. For this purpose, we apply a reverse version of the Rudolph–Weiss theorem.

1. Introduction. The study of group actions with completely positive entropy (c.p.e.) is an important trend in the contemporary entropy theory of dynamical systems. The primary results were obtained by V. Rokhlin and Ya. Sinai [7] who introduced the notion of c.p.e. for $\mathbb{Z}$-actions. Later it was transferred to $\mathbb{Z}^d$ by J. Conze [1], and then B. Kamiński started a more refined investigation of this phenomenon using the idea of perfect partitions [4]. A remarkable progress was made in the recent work by D. Rudolph and B. Weiss [8, Theorem 2.3], where it was demonstrated (surprisingly, with the use of the orbit theory of dynamical systems) that complete positivity for an amenable transformation group implies a rather strong mixing property (to be called the Rudolph–Weiss property below). In this context, it is desirable to prove the existence of non-Bernoullian c.p.e. actions of amenable groups. This is the goal of this work.

We present a construction of c.p.e. non-Bernoullian actions for a class of countable amenable groups. These actions are good to verify the Rudolph-Weiss property. This should certainly imply c.p.e. via reversing the result of [8]. Thus we need to show that the Rudolph–Weiss property is not only necessary, but also sufficient for c.p.e. We demonstrate this in Section 2 in the utmost generality, as advised by the referee.

The c.p.e. non-Bernoullian actions are produced for countable Abelian groups with infinite order elements (see Section 3, Corollary 7 and Remark 2). The case of nilpotent groups is considered in [3].

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2. The Rudolph–Weiss property and completely positive entropy. The following definition is due to D. Rudolph and B. Weiss. Let G be a countable amenable group and $K \subset G$ a finite set. A finite set $S \subset G$ is said to be *K-spread* if for all $\gamma_1 \neq \gamma_2 \in S$ one has $\gamma_1\gamma_2^{-1} \notin K$.

Recall also that the action of G is said to have a *completely positive entropy* (c.p.e.) if for any finite partition ξ, the mean entropy $h(\xi, G)$ is positive.

THEOREM 1. *Let G be a countable amenable group*, (X,μ) *a Lebesgue free G-space*, *and ξ a finite partition of X. Suppose that for any $\varepsilon > 0$ there exists a finite subset $K \subseteq G$ such that for any finite set $S \subset G$ which is K-spread*,

$$\left|\frac{1}{\#S} H\Big(\bigvee_{g\in S} g\xi\Big) - H(\xi)\right| < \varepsilon.$$

Then $h(\xi, G) > 0$.

We need the following

LEMMA 2. *The statement of Theorem* 1 *is valid in the case of a countable Abelian G.*

Proof. We write our Abelian group G additively, and let a be the action symbol. Let $T = \{g \in G : ng = e \text{ for some } n \in \mathbb{Z}\}$ be the torsion group of G. Fix an increasing sequence of finitely generated subgroups G_n with $\bigcup_n G_n = G$. Let also $\pi : G \to G/T$ and $\pi_n : G \to G/(G_n \cap T)$ stand for the natural projections, so $\pi_n(G_n) \cong \mathbb{Z}^{d(n)} \subset \pi(G)$. Choose a sequence of rectangles $Q_n \subset \pi_n(G_n)$ centered at 0 of G_n in such a way that $\bigcup_i \pi_i^{-1}(Q_i) = G$, and the following property of pavement is valid: for $N > n$ there exists a finite subset $S_{n,N} \subset Q_N$ $(\subset \pi(G))$ such that Q_N splits into a disjoint union $Q_N = \bigcup_{g\in Q_n}(g + S_{n,N})$. One can easily observe that, under our assumptions, Q_n is an increasing sequence of sets in $\pi(G)$, and $\pi_n^{-1}(Q_n)$ form a Følner sequence of sets in G.

With ξ being the given finite partition, choose some positive $\varepsilon < H(\xi)/2$. By our assumptions, one can find a finite set $K \subset G$ such that for any finite $S \subset G$ which is K-spread,

$$\left|\frac{1}{\#S} H\Big(\bigvee_{g\in S} a(g)\xi\Big) - H(\xi)\right| < \varepsilon,$$

and hence

$$\frac{1}{\#S} H\Big(\bigvee_{g\in S} a(g)\xi\Big) > H(\xi)/2.$$

Choose n so that $K \subset \pi_n^{-1}(Q_n)$ $(\subset G_n)$. Let $S'_{n,N}$ be a subset of $\pi_N^{-1}(S_{n,N})$ whose intersection with each $(G_n \cap T)$-coset is at most one point. Clearly, $S'_{n,N}$ is $\pi_n^{-1}(Q_n)$-spread (hence K-spread) for any $N > n$, and $\#\pi_N^{-1}(Q_N) = \#\pi_n^{-1}(Q_n) \cdot \#S'_{n,N}$. Now we have

$$\begin{aligned}
&\frac{1}{\#\pi_N^{-1}(Q_N)} H\Big(\bigvee_{g\in\pi_N^{-1}(Q_N)} a(g)\xi\Big)\\
&\quad > \frac{1}{\#\pi_n^{-1}(Q_n)\cdot\#S'_{n,N}} H\Big(\bigvee_{g\in\pi_n^{-1}(Q_n)}\ \bigvee_{s\in S'_{n,N}} a(g+s)\xi\Big)\\
&\quad > \frac{1}{\#\pi_n^{-1}(Q_n)\cdot\#S'_{n,N}} H\Big(\bigvee_{s\in S'_{n,N}} a(s)\xi\Big) > \frac{1}{2\#\pi_n^{-1}(Q_n)} H(\xi).
\end{aligned}$$

It remains to let N tend to infinity to obtain

$$h(\xi, G) \ge \frac{1}{2\#\pi_n^{-1}(Q_n)} H(\xi) > 0. \ \blacksquare$$

REMARK 1. One can use the same argument to obtain a relativized version of Lemma 2, with entropy being replaced by the conditional entropy with respect to a G-invariant measurable partition.

Proof of Theorem 1. This is essentially the argument used in [8] for proving the converse result, so we omit some details. To begin with, we quote from [8] the following

DEFINITION. Fix some enumeration $G = \{\gamma_1, \gamma_2, \ldots\}$. Suppose $S(x) = \{s_1(x), \ldots, s_k(x)\}$ is a Borel choice of k-element subsets of G. We say $S(x)$ is N-*quasi-spread* if for all x outside a subset of measure less than $1/N$, there is a subset $S'(x) \subseteq S(x)$ with $\#S'(x)/\#S(x) > 1 - 1/N$ and for all distinct $s, s' \in S'(x)$ we have

$$s^{-1}s' \notin \{\gamma_1, \ldots, \gamma_N\}.$$

Let (Y, ν) be a G-space, and form the product G-space $X \times Y$, with the product G-action $g(x, y) = (gx, gy)$. Let T be an ergodic automorphism on Y whose orbits are just the orbits of the G-action. This certainly means the existence of Borel maps $V_i : Y \to G$, $i \in \mathbb{Z}$, with $T^i y = V_i(y)y$. So, one has a $\mathbb{Z}$-action on $X \times Y$ generated by the automorphism $\overline{T}$ as follows: $\overline{T}^i(x, y) = (V_i(y)x, T^i y)$.

Let ξ be a given partition of X and $\varepsilon > 0$. By our assumptions, there is a finite subset $K \subset G$ such that for any finite K-spread subset $S \subset G$, one has

$$\left|\frac{1}{\#S}H\Big(\bigvee_{\gamma\in S}\gamma\xi\Big)-H(\xi)\right|<\frac{\varepsilon}{3}.$$

Now form a partition $\overline{\xi}=\xi\times\{\text{the trivial partition of } Y\}$ of $X\times Y$. Obviously, for $S\subset G$ being K-spread, one has

$$\left|\frac{1}{\#S}H\Big(\bigvee_{\gamma\in S}\gamma\overline{\xi}\Big)-H(\overline{\xi})\right|<\frac{\varepsilon}{3}$$

with respect to the above action of G on $X\times Y$.

For a subset $Q=\{q_1,\dots,q_{\#Q}\}\subset\mathbb{Z}$ there is a Borel function $V:Y\to G^{\#Q}$, $V(y)=(V_{q_1}(y),\dots,V_{q_{\#Q}}(y))$, such that $V_{q_i}(y)y=T^{q_i}y$, so V is uniform [8]. Let $M>3H(\xi)/\varepsilon$ be an integer such that $K\subset\{\gamma_1,\dots,\gamma_M\}$. By [8, Theorem 2.11] we can choose Q to be N-spread (that is, for $a,b\in Q$, $a\neq b$ implies $|a-b|>N$) with N being so large that V is M-quasi-spread. This implies that there is a Borel subset $B\subset Y$ with $\nu(B)<1-1/M$ such that for $y\in B$ there is a subset $I(y)\subset Q$ with $\#I(y)>(1-1/M)\#Q$ such that $\{V_i(y):i\in I(y)\}$ is K-spread, and so

$$\left|H(\xi)-\frac{1}{\#I(y)}H\Big(\bigvee_{i\in I(y)}V_i(y)\xi\Big)\right|<\frac{\varepsilon}{3}$$

for $y\in B$. Let $H(P\,|\,Y)$ be the conditional entropy of a partition P with respect to the σ-algebra of Y-measurable Borel subsets of $X\times Y$. Then

$$\begin{aligned}
&\left|\frac{1}{\#Q}H\Big(\bigvee_{i\in Q}\overline{T}^i\overline{\xi}\,\Big|\,Y\Big)-H(\overline{\xi}\,|\,Y)\right|\\
&\quad=H(\overline{\xi}\,|\,Y)-\frac{1}{\#Q}\int_Y H\Big(\Big(\bigvee_{i\in Q}\overline{T}^i\overline{\xi}\Big)\cap(X\times\{y\})\Big)\,d\nu(y)\\
&\quad=H(\overline{\xi})-\frac{1}{\#Q}\int_Y H\Big(\bigvee_{i\in Q}V_i(y)\xi\Big)\,d\nu(y)\\
&\quad=\int_Y\bigg(H(\xi)-\frac{1}{\#Q}H\Big(\bigvee_{i\in Q}V_i(y)\xi\Big)\bigg)\,d\nu(y)\\
&\quad\le\int_B\bigg(H(\xi)-\frac{1-1/M}{\#I(y)}H\Big(\bigvee_{i\in I(y)}V_i(y)\xi\Big)\bigg)d\nu(y)+\int_{Y\setminus B}H(\xi)\,d\nu(y)\\
&\quad\le\int_B\bigg(H(\xi)-\frac{1}{\#I(y)}H\Big(\bigvee_{i\in I(y)}V_i(y)\xi\Big)\bigg)\,d\nu(y)+\frac{1}{M}H(\xi)+H(\xi)\nu(Y\setminus B)\\
&\quad<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.
\end{aligned}$$

This means that a relativized version of the Rudolph–Weiss property (with respect to Y) is valid for the partition $\overline{\xi}$ and the $\mathbb{Z}$-action by powers of $\overline{T}$ on $X \times Y$.

So one can apply the relativized version of Lemma 2 (see Remark 1) to conclude that $h(\overline{\xi}, \mathbb{Z} \,|\, Y) > 0$. Again, going back to the orbit equivalent G-action on $X \times Y$, one can now apply [8, Theorem 2.6] to get

$$h(\overline{\xi}, G \,|\, Y) = h(\overline{\xi}, \mathbb{Z} \,|\, Y) > 0,$$

and so

$$h(\xi, G) = h(\overline{\xi}, G \,|\, Y) > 0. \ \blacksquare$$

Corollary 3. *A free action of a countable amenable group G on a Lebesgue space (X, μ) has c.p.e. if and only if for any finite partition ξ and any $\varepsilon > 0$ there exists a finite subset $K \subseteq G$ such that for any finite set $S \subset G$ which is K-spread*,

$$\left| \frac{1}{\#S} H\Big(\bigvee_{g \in S} g\xi \Big) - H(\xi) \right| < \varepsilon.$$

Proof. The "only if" part is due to D. Rudolph and B. Weiss [8]. The "if" part is an obvious consequence of Theorem 1. $\blacksquare$

3. Non-Bernoullian actions with completely positive entropy. To produce c.p.e. non-Bernoullian actions for a class of amenable groups, we need the following lemma on the entropy of finite index subgroups; it is well known in some special cases (cf. [1]). Let G be a countable amenable group and G_p be a subgroup of G of index p. Consider the space $G_p \backslash G$ of left G_p-cosets. Let $\delta_p \subset G$ be a "fundamental domain" (section) for this homogeneous space which contains the identity of G. Given a partition α, we denote by α^p the partition $\alpha_{\delta_p} = \bigvee_{g \in \delta_p} g \cdot \alpha$.

Lemma 4. *For a finite index subgroup G_p with index p*,

$$h(\alpha^p, G_p) = p h(\alpha, G).$$

Proof. Let F_n, $n \in \mathbb{N}$, be a right Følner sequence of subsets in G_p (recall that we consider right group actions). One has

$$\begin{aligned} h(\alpha^p, G_p) &= \lim_{n \to \infty} \frac{1}{\#F_n} H\Big(\bigvee_{h \in F_n} h\delta_p \alpha \Big) \\ &= \lim_{n \to \infty} \frac{1}{\#F_n} H\Big(\bigvee_{\delta \in \delta_p} \bigvee_{h \in F_n} h\delta\alpha \Big) = \lim_{n \to \infty} \frac{p}{\#(F_n \delta_p)} H\Big(\bigvee_{g \in F_n \delta_p} g\alpha \Big). \end{aligned}$$

To prove that the latter limit is $ph(\alpha, G)$, it suffices to verify that the $F_n \delta_p$ form a right Følner sequence in G. Note that, since $F_n \subset G_p$, $F_n h$ for different $h \in \delta_p$ are in different G_p-cosets, and hence disjoint. An arbitrary

$g \in G$ generates a one-to-one map of the right homogeneous space $G_p \backslash G$, and so there exist a bijection a_g of δ_p and a map $\gamma_g : \delta_p \to G_p$ such that $hg = \gamma_g(h) a_g(h)$ for $h \in \delta_p$. Hence

$$\frac{\#(F_n \delta_p g \triangle F_n \delta_p)}{\#(F_n \delta_p)} = \sum_{h \in \delta_p} \frac{\#(F_n \gamma_g(a_{g^{-1}}(h))h \triangle F_n h)}{\#(F_n \delta_p)} \underset{n\to\infty}{\longrightarrow} 0,$$

which was to be proved. ∎

The following lemma demonstrates that a non-Bernoullian c.p.e. action of a subgroup can be used to produce an action of the entire Abelian group with these properties via a sort of inducing procedure.

LEMMA 5. *Let G be a countable Abelian group, and N a subgroup of G with the quotient group G/N being finitely generated. Suppose we are given a c.p.e. non-Bernoullian N-space (X, μ). Then G also admits a c.p.e. non-Bernoullian action.*

Proof. Denote by $\pi : G \to G/N$ the natural projection and by $s : G/N \to G$ a section with $s(N) = 0$.

Form the product space $Y = X^{G/N}$ with the associated product measure ν and introduce an action of G on Y by

$$(gy)_\gamma = (s(\gamma) + g - s(\gamma g)) y_{\gamma + \pi(g)}, \quad y \in Y,\ \gamma \in G/N,\ g \in G,$$

with the given action of N on each direct factor of Y. An easy verification shows that this action is well defined (in particular, $s(\gamma) + g - s(\gamma g) \in N$). To see that this action is non-Bernoulli, we need the following simple proposition, valid for any countable amenable group G and its subgroup N.

PROPOSITION 6. *The restriction of a Bernoullian action of G to a subgroup N is also Bernoullian.*

Proof. Let ζ be a measurable generating partition for the G-action such that the family of partitions $\{g\zeta \mid g \in G\}$ is independent. Let $B \subset G$ be a set which meets each left N-coset Ng in exactly one point. Form the measurable partition $\eta = \bigvee_{g \in B} g\zeta$. Evidently, it is generating for the N-action and its shifts by the elements of N are independent, which proves our statement. ∎

We return to our construction. It follows from the definition of our G-action that its restriction to N is given by

$$(hy)_\gamma = g y_\gamma, \quad y \in Y,\ y_\gamma \in X,\ \gamma \in G/N,\ h \in N.$$

Observe that this action splits into a direct product of actions on direct factors of Y. Hence it has the original N-action as a factor, and thus is non-Bernoullian [6]. Now an application of Proposition 6 shows that the entire G-action is non-Bernoullian.

To prove that the constructed action of G has c.p.e., we first observe that G/N has the form $G/N = F \times \mathbb{Z}^m$, with F being a finite group and m a non-negative integer. Denote by $\tau : G/N \to \mathbb{Z}^m$ the natural projection.

It was mentioned above that the restriction of the G-action on Y to N splits into the direct product of N-actions on direct factors of Y. Since each of those is just the given N-action on X, the N-action on Y has c.p.e. [2]. That is, given a finite partition ξ of Y, the mean entropy $h(\xi, N)$ is positive. Choose a finite subset $Q \subset G/N$ such that for some finite partition $\overline{\eta}$ of X^Q and the corresponding partition η on Y one has $d(\xi, \eta) = H(\xi \,|\, \eta) + H(\eta \,|\, \xi) < \frac{1}{2}h(\xi, N)$. Consider a rectangle centered at 0 in $\mathbb{Z}^m$ which contains $\tau(Q)$, and denote by $\overline{Q}$ the direct product of F and $\tau(Q)$. Clearly $\overline{Q}$ is a fundamental domain for a finite index subgroup D in G/N. Note that D is $\overline{Q}$-spread and contains F.

Now consider the finite index subgroup $\pi^{-1}(D)$ of G. Let F_n and R_n be Følner sequences of sets in N and D, respectively. If the above sequences are chosen properly, $s(R_n)+F_n$ is a Følner sequence of sets in $\pi^{-1}(D)$. Assuming this to be true, we use the independence of the partitions $\eta_{s(\gamma)+F_n}$ for distinct $\gamma \in D$ to get

$$\begin{aligned}\frac{1}{\#(s(R_n)+F_n)} H(\eta_{s(R_n)+F_n}) &= \frac{1}{\#R_n \cdot \#F_n} \sum_{\gamma \in R_n} H(\eta_{s(\gamma)+F_n}) \\ &= \frac{1}{\#F_n} H(\eta_{F_n}) \underset{n\to\infty}{\longrightarrow} h(\eta, N),\end{aligned}$$

that is, $h(\eta, \pi^{-1}(D)) = h(\eta, N)$. Now we apply the relation $|h(\xi, G) - h(\eta, G)| < d(\xi, \eta)$ to get

$$\begin{aligned}h(\xi, \pi^{-1}(D)) &\ge h(\eta, \pi^{-1}(D)) - d(\xi, \eta) > h(\eta, N) - \tfrac{1}{2}h(\xi, N) \\ &\ge h(\xi, N) - d(\xi, \eta) - \tfrac{1}{2}h(\xi, N) \\ &> h(\xi, N) - \tfrac{1}{2}h(\xi, N) - \tfrac{1}{2}h(\xi, N) = 0,\end{aligned}$$

and hence by Lemma 4,

$$h(\xi, G) = \frac{1}{\#\overline{Q}} h(\xi_{\overline{Q}}, \pi^{-1}(D)) \ge \frac{1}{\#\overline{Q}} h(\xi, \pi^{-1}(D)) > 0,$$

which proves c.p.e. for the G-action. ■

COROLLARY 7. *Any countable Abelian group G containing an element of infinite order has a non-Bernoullian c.p.e. action.*

Proof. For G finitely generated, it suffices, due to Lemma 5, to prove that some subgroup of G admits an action with the required properties. Under our assumptions this subgroup is generated by an infinite order element, and the required action of this subgroup comes from [5].

In the general case, represent G as a union of an increasing sequence of finitely generated subgroups, $G = \bigcup_{i=0}^{\infty} G_i$, with G_0 being a cyclic group which has a c.p.e. non-Bernoullian action (X, μ) as discussed above. Form the G-space $Y = X^{G/G_0}$ exactly as in the proof of Lemma 5, and use the same argument to demonstrate that this G-space is non-Bernoullian.

Observe that if one sets $Y_n = X^{G_n/G_0}$, then for each $n \in \mathbb{N}$, $Y = Y_n^{G/G_n}$ splits into the direct product of G_n-spaces. Since G_n is finitely generated and the G_n-space Y_n has the same structure as in the proof of Lemma 5, one can demonstrate that the G_n-action has c.p.e. on Y_n and hence on Y.

Suppose we are given a finite partition ξ on Y and $\varepsilon > 0$. Now choose n as above so large that there is a partition η_0 of Y_n such that for the corresponding partition η of Y one has $d(\xi, \eta) < \varepsilon/3$. By Corollary 3, one can find a finite subset $K \subset G_n$ such that for any K-spread finite subset $S \subset G_n$,

$$\left| \frac{1}{\#S} H(\eta_S) - H(\eta) \right| < \frac{\varepsilon}{3}.$$

Now let $P \subset G$ be a K-spread finite subset, which without loss of generality can be assumed to contain 0. Split P into a disjoint finite union $P = \bigcup_i P_i$, with each P_i lying inside a G_n-coset. It follows from the independence of η_{P_i} for different i that $H(\eta_P) = \sum_i H(\eta_{P_i})$, and so

$$\begin{aligned} |H(\eta_P) - \#P \cdot H(\eta)| &\le \sum_i |H(\eta_{P_i}) - \#P_i \cdot H(\eta)| \\ &= \sum_i \left| \frac{1}{\#P_i} \cdot H(\eta_{P_i}) - H(\eta) \right| \#P_i \le \#P \cdot \frac{\varepsilon}{3}, \end{aligned}$$

and hence

$$\left| \frac{1}{\#P} \cdot H(\eta_P) - H(\eta) \right| < \frac{\varepsilon}{3}.$$

On the other hand, note that for any two finite partitions α, β one has $H(\alpha \vee \beta) = H(\alpha) + H(\beta \,|\, \alpha) = H(\beta) + H(\alpha \,|\, \beta)$, and hence $H(\alpha) - H(\beta) = H(\alpha \,|\, \beta) - H(\beta \,|\, \alpha)$. Apply this observation to our case as follows:

$$\begin{aligned} &\left| \frac{1}{\#P} \cdot H(\xi_P) - \frac{1}{\#P} \cdot H(\eta_P) \right| \\ &\quad = \frac{1}{\#P} |H(\xi_P \,|\, \eta_P) - H(\eta_P \,|\, \xi_P)| \le \frac{1}{\#P} (H(\xi_P \,|\, \eta_P) + H(\eta_P \,|\, \xi_P)) \\ &\quad \le \frac{1}{\#P} \sum_{g \in P} (H(g\xi \,|\, g\eta) + H(g\eta \,|\, g\xi)) = H(\xi \,|\, \eta) + H(\eta \,|\, \xi) < \frac{\varepsilon}{3}. \end{aligned}$$

Now it follows from the above observations that

$$\begin{aligned} \left| \frac{1}{\#P} \cdot H(\xi_P) - H(\xi) \right| &\le \left| \frac{1}{\#P} \cdot H(\xi_P) - \frac{1}{\#P} \cdot H(\eta_P) \right| \\ &\quad + \left| \frac{1}{\#P} \cdot H(\eta_P) - H(\eta) \right| + |H(\eta) - H(\xi)| < \varepsilon. \end{aligned}$$

That is, we are in the conditions of Theorem 1, so an application of that theorem yields $h(\xi, G) > 0$, which proves c.p.e. for the G-space Y. ■

Remark 2. The same argument as in the proof of Corollary 7 can be used to construct a c.p.e. non-Bernoullian action for a countable group G which has a normal Abelian subgroup A with an infinite order element and such that the quotient group G/A is locally finite. Also, this approach was used in [5] to produce c.p.e. non-Bernoullian actions for torsionfree nilpotent groups and a class of solvable groups.

Remark 3. It should be noted that in the above construction, one could choose a subgroup N generated by an automorphism Q from the uncountable family of non-conjugate non-Bernoullian transformations with completely positive entropy, produced in [5]. Thus, we also have an uncountable family of pairwise non-conjugate non-Bernoullian actions of a countable Abelian group G with at least one element of infinite order.

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