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CONSTRUCTION OF NON-CONSTANT AND ERGODIC COCYCLES

BY

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Dedicated to Prof. Anzelm Iwanik

Abstract. We construct continuous G-valued cocycles that are not cohomologous to any compact constant via a measurable transfer function, provided the underlying dynamical system is rigid and the range group G satisfies a certain general condition. For more general ergodic aperiodic systems, we also show that the set of continuous ergodic cocycles is residual in the class of **all** continuous cocycles provided the range group G is a compact connected Lie group. The first construction is based on the "closure of coboundaries technique", whereas the second result is proved by developing in addition a new approximation technique.

1. Introduction. Let (Y, T, μ) be a topological dynamical system, i.e. Y is a compact metric space, $T : Y \to Y$ is a homeomorphism and μ is a T-invariant, regular Borel probability measure on Y. Let G be a locally compact, second countable topological group. A G-valued (continuous) *cocycle* is a (continuous) map $\varphi : Y \to G$. Two continuous maps $\varphi_1, \varphi_2 : Y \to G$ are *cohomologous* (via a measurable transfer function ψ) if there exists a measurable map $\psi : Y \to G$ such that

$$\varphi_2(y) = \varphi_1 \cdot 1^{\psi}(y) \equiv \psi(Ty)\varphi_1(y)\psi(y)^{-1}, \quad \text{a.e. } y \in Y. \tag{1.1}$$

A continuous cocycle φ is *non-constant* if it is not cohomologous (via a measurable transfer function) to any constant cocycle (i.e. any constant map) from Y to G. In other words φ is non-constant if and only if the functional equation

$$\varphi(y) = \psi(Ty)c\psi(y)^{-1}, \quad \text{a.e. } y \in Y, \tag{1.2}$$

has no measurable solution ψ for any $c \in G$. In this note we shall produce such cocycles using a fairly simple "closure of coboundaries" construction procedure.

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Another related problem is that of constructing ergodic skew-products. Given a continuous cocycle $\varphi : Y \to G$, one defines a *skew-product transformation* $T_\varphi : G \times Y \to G \times Y$ by setting

$$T_\varphi(g, y) = (\varphi(y)g, Ty), \qquad (g, y) \in G \times Y. \tag{1.3}$$

The system $(G \times Y, T_\varphi, \nu \times \mu)$ is called a *skew-product extension* of (Y, T, μ), where ν is a left Haar measure on G. Whenever G is compact, we shall assume that ν is normalized so that $\nu(G) = 1$. Notice that $\nu \times \mu$ is T_φ-invariant.

Even though compact skew-product extensions (i.e. G is compact) have been extensively studied in ergodic theory and topological dynamics, several basic questions remain unanswered particularly when the fiber group G is *non-abelian*. One such question regards the density (or genericity, in the class of all continuous cocycles) of the set of those cocycles for which the skew-product transformation given in (1.3) is ergodic. We shall settle this question affirmatively if G is a compact connected Lie group. Since cohomologous cocycles generate isomorphic skew-product transformations, if G is compact and non-abelian, ergodicity of the skew-product implies that the corresponding cocycle is non-constant.

Our construction procedures are based on the well known technique of producing "wild coboundaries" in the class of closures of coboundaries. This technique (which originated in the works of [AK], Fathi–M. Herman and subsequently refined in [GW] and [N1], [N2]) will be referred to as the "closure of coboundaries technique". For the ergodicity lifting result, we need to develop a new "approximation technique" which together with a modified version of the closure of coboundaries technique yields generic results in the class of *all* continuous cocycles.

To state the results precisely, we introduce some notation.

1.1. Notation. (1) The set of all continuous cocycles will be identified with the complete, separable metric space $C(Y, G)$ of continuous functions from Y to G with the supremum metric.

(2) Let $\mathcal{C}$ denote the set of constant cocycles, i.e. constant functions from Y to G. Hence one may identify $\mathcal{C}$ with G. An element $c \in G$ is *compact* if the closure of the set $\{c^n \mid n \in \mathbb{Z}\}$ is compact. Let $\mathcal{C}_c$ denote the set of all elements of $\mathcal{C}$ whose values are compact elements of G.

(3) Let $\mathcal{N}$ denote the set of all non-constant cocycles in $C(Y, G)$ and $\mathcal{N}_c$ be the set of all cocycles in $C(Y, G)$ that are not cohomologous to any constant in $\mathcal{C}_c$ via a measurable transfer function.

(4) Let B denote the set of all coboundaries generated by continuous transfer functions, i.e.

$$B = \{1^\psi \mid 1^\psi(y) \equiv \psi(Ty)\psi(y)^{-1} \text{ for all } y \in Y, \text{ and } \psi \in C(Y, G)\}.$$

(5) If G is a matrix Lie group, say $G \subseteq \mathrm{GL}(n, \mathbb{C})$, given a $\varphi \in C(Y, G)$, define the operator V_φ on $L^2(Y, \mathbb{C}^n, \mu)$ by setting

$$V_\varphi f(y) = \varphi(y)^{-1} f(Ty). \tag{1.4}$$

If G is a subgroup of the unitary group $U(n)$, this representation is unitary and a good deal of information is available about its spectral properties in the ergodic theory and physics literature, particularly when $G = \mathbb{S}^1$, the circle group. As a corollary to the ergodicity lifting result, we shall show that the spectrum of V_φ is only continuous for a generic $\varphi \in C(Y, G)$ and only singular continuous for a generic $\varphi \in \overline{B}$ provided (Y, T, μ) is rigid.

This paper consists of two constructions. The first construction produces non-constant cocycles based on rigid dynamical systems and the second produces ergodic lifts when G is compact. Before stating the theorems we need to introduce the notion of rigidity and the "Property P".

1.2. Definition. A dynamical system (Y, T, μ) is *rigid* if there exists a sequence $q_n \in \mathbb{N}$ with $q_n \to \infty$ such that $T^{q_n} \to I$, where I is the identity automorphism of Y and the convergence is in the weak topology on the set of all μ-preserving Borel automorphisms of Y (see [H] for more on this topology). We shall refer to such a $\{q_n\}$ as a *rigidity sequence*.

A rotation transformation on the n-torus is a basic example of a rigid system. However there are examples of weakly mixing (both in the measure-theoretic as well as topological sense) rigid transformations (see [AK], [GM]). Other stronger notions of rigidity can be defined by either demanding that the convergence of T^{q_n} take place in other stronger topologies (e.g. the supremum topology) or by requiring a certain speed of convergence ([K]). Here, however, our (weak) notion of rigidity will enable us to obtain results in the category of continuous cocycles. Next, we introduce "Property P".

1.3. Definition. Let G be a locally compact, second countable topological group. Then G is said to have *Property P* if there exists a continuous finite-dimensional representation π of G and a one-parameter subgroup $H = \{h(t) \mid t \in \mathbb{R}\} \subset G$ (where $h : \mathbb{R} \to G$ is a continuous group homomorphism) such that $\Sigma(h(1)) \neq \{1\}$, where $\Sigma(g)$ denotes the spectrum (i.e. the set of eigenvalues) of $\pi(g)$, $g \in G$.

Clearly non-trivial compact connected Lie groups have Property P. More generally any connected Lie group with a semisimple element has Property P (e.g. $\mathrm{SL}(n, \mathbb{R})$). However, in general nilpotent Lie groups will not have Property P and consequently our results will not be applicable to this class of groups. Now we state the main results.

A. Theorem. *Assume that*

(1) *the topological dynamical system* (Y, T, μ) *is aperiodic and ergodic*,
(2) *the system* (Y, T, μ) *is rigid*, *and*
(3) *the group* G *has Property* P.

Then the set $\mathcal{N}_{\mathrm{c}} \cap \overline{B}$ *is residual in* $\overline{B}$.

Our second result concerns lifting ergodicity.

B. Theorem. *Suppose that*

(1) (Y, T, μ) *is aperiodic and ergodic*, *and*
(2) G *is a compact connected Lie group.*

Then the set

$$\mathcal{C}_{\mathrm{erg}} \equiv \{\varphi \in C(Y, G) \mid (G \times Y, T_\varphi, \nu \times \mu) \text{ is ergodic}\}$$

is a residual subset of $C(Y, G)$.

C. Corollary. *Assume that*

(1) (Y, T, μ) *is aperiodic and ergodic*, *and*
(2) G *is a compact*, *connected Lie group.*

Then $\mathcal{N}$ *is a residual subset of* $C(Y, G)$.

D. Corollary. *Assume that*

(1) (Y, T, μ) *is aperiodic and ergodic*,
(2) G *is a closed subgroup of the unitary group* $U(n)$, *and*
(3) G *does not fix any ray in the complex projective* n*-space* $P(\mathbb{C}^n)$.

(I) *Then the set*

$$\mathcal{C}_{\mathrm{cont}} = \{\varphi \in C(Y, G) \mid \text{the spectrum of } V_\varphi \text{ is only continuous}\},$$

is residual in $C(Y, G)$.

(II) *Furthermore*, *if in addition* (Y, T, μ) *is rigid*, *then the set*

$$\mathcal{C}_{\mathrm{sing}} = \{\varphi \in \overline{B} \mid \text{the spectrum of } V_\varphi \text{ is only singular continuous}\}$$

is residual in $\overline{B}$.

We remark that even though a generic ergodicity lifting result was known in the class $\overline{B}$ (see [N1], [N2]), the same question has remained unsettled in the class $C(Y, G)$ until now. Theorem B settles this question affirmatively and along with Corollary C it extends results of [JP] and [IS] to the (compact) non-abelian case. We also mention that a real-valued (smooth) non-constant cocycle was constructed under a stronger rigidity assumption in [K] and Katok and Stepin have also constructed cocycles φ into $\mathbb{Z}_2$ for which the spectrum of V_φ is singular continuous.

We prove Theorems A and B in Sections 2 and 3 respectively and discuss the "prevalence" of ergodic cocycles in the compact abelian case in Section 4.

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2. Proof of Theorem A. We begin by proving the following consequence of Property P.

2.1. LEMMA. *Suppose that the group G has Property P. Then there exists a continuous finite-dimensional representation π of G, a one-parameter subgroup $H=\{h(t)\mid t\in\mathbb{R}\}\subset G$, $\varepsilon>0$ and positive reals a,b with $a<b/2$ such that*

$$d_{\mathrm{H}}(\Sigma(h(s)),\Sigma(h(t)))>\varepsilon \tag{2.1}$$

for all $s\in[-a,-a/2]\cup[a/2,a]$ and for all $t\in[-b,-b/2]\cup[b/2,b]$, where d_{H} denotes the Hausdorff metric on the set of all non-empty, finite subsets of the complex plane.

Proof. Let π and H be as in Definition 1.3. Write $\Sigma(h(1))\setminus\{1\}$ as a disjoint union of three sets: $\Sigma(h(1))=\Sigma_{<}\cup\Sigma_1\cup\Sigma_{>}$, the sets of eigenvalues with absolute value less than 1, equal to 1 and greater than 1 respectively. Let

$$l=\max_{\lambda\in\Sigma_{>}}\ln|\lambda|\quad\text{and}\quad k=\max_{\lambda\in\Sigma_{<}}-\ln|\lambda|.$$

We set $l=0$ (resp. $k=0$) if $\Sigma_{>}=\emptyset$ (resp. $\Sigma_{<}=\emptyset$). Now we analyze the following cases.

CASE 1: $l>0$ (equivalently $\Sigma_{>}\neq\emptyset$). In this case select $a,b\in\mathbb{R}$ such that

$$0<a<b/2\quad\text{and}\quad\max\{1,ka\}<lb/2.$$

Now consider the following subcases.

CASE 1(a): $t\in[b/2,b]$ and $s\in[a/2,a]$. Note that in this case $\Sigma(h(s))$ is contained in the closed disk $\overline{\Delta}(0,e^{la})\equiv\{z\mid |z|\le e^{la}\}$ and at least one element of $\Sigma(h(t))$ is outside the open disk $\Delta(0,e^{lb/2})$. Hence

$$d_{\mathrm{H}}(\Sigma(h(s)),\Sigma(h(t)))>e^{lb/2}-e^{la}.$$

CASE 1(b): $t\in[-b,-b/2]$ and $s\in[-a,-a/2]$. In this case $\Sigma(h(s))$ lies outside $\Delta(0,e^{-la})$ and at least one element of $\Sigma(h(t))$ is inside $\overline{\Delta}(0,e^{-lb/2})$. Hence

$$d_{\mathrm{H}}(\Sigma(h(s)),\Sigma(h(t)))>e^{-al}-e^{-lb/2}.$$

CASE 1(c): $t\in[b/2,b]$ and $s\in[-a,-a/2]$. In this case $\Sigma(h(s))$ is contained in $\overline{\Delta}(0,r)$, where $r=1$ if $k=0$ and $r=e^{ak}$ otherwise; and at least one element of $\Sigma(h(t))$ is outside $\Delta(0, e^{lb/2})$. Hence

$$d_{\mathrm{H}}(\Sigma(h(s)), \Sigma(h(t))) > \begin{cases} e^{lb/2} - e^{ak} & \text{if } k \neq 0, \\ e^{lb/2} - 1 & \text{if } k = 0. \end{cases}$$

CASE 1(d): $t \in [-b, -b/2]$ and $s \in [a/2, a]$. In this case $\Sigma(h(s))$ lies outside $\Delta(0, r)$, where $r = 1$ if $k = 0$ and $r = e^{-ak}$ otherwise; and at least one element of $\Sigma(h(t))$ is inside $\overline{\Delta}(0, e^{-lb/2})$. Hence

$$d_{\mathrm{H}}(\Sigma(h(s)), \Sigma(h(t))) > \begin{cases} e^{-ka} - e^{-lb/2} & \text{if } k \neq 0, \\ 1 - e^{-lb/2} & \text{if } k = 0. \end{cases}$$

CASE 2: $l = 0,\ k > 0$. In this case select $a, b \in \mathbb{R}$ such that

$$0 < a < b/2 \quad \text{and} \quad \max\{1, ka\} < kb/2.$$

Again, we have the following subcases.

CASE 2(a): $t \in [b/2, b]$ and $s \in [a/2, a]$. In this case $\Sigma(h(s))$ lies outside $\Delta(0, e^{-ka})$, whereas $\Sigma(h(t))$ has a point inside $\overline{\Delta}(0, e^{-kb/2})$. Hence

$$d_{\mathrm{H}}(\Sigma(h(s)), \Sigma(h(t))) > e^{-ka} - e^{-kb/2}.$$

CASE 2(b): $l = 0,\ k > 0$ and $t \in [-b, -b/2]$, $s \in [-a, -a/2]$. In this case $\Sigma(h(s)) \subseteq \overline{\Delta}(0, e^{ka})$ and $\Sigma(h(t))$ has at least one point outside $\Delta(0, e^{bk/2})$, thus

$$d_{\mathrm{H}}(\Sigma(h(s)), \Sigma(h(t))) > e^{bk/2} - e^{ak}.$$

CASE 2(c): $l = 0,\ k > 0$ and $t \in [b/2, b]$, $s \in [-a, -a/2]$. In this case $\Sigma(h(t)) \subseteq \overline{\Delta}(0, 1)$ and $\Sigma(h(s))$ has at least one point outside $\Delta(0, e^{ak/2})$, thus

$$d_{\mathrm{H}}(\Sigma(h(s)), \Sigma(h(t))) > e^{ak/2} - 1.$$

CASE 2(d): $l = 0,\ k > 0$ and $t \in [-b, -b/2]$, $s \in [a/2, a]$. In this case $\Sigma(h(s)) \subseteq \overline{\Delta}(0, 1)$ and $\Sigma(h(t))$ has at least one point outside $\Delta(0, e^{bk/2})$, thus

$$d_{\mathrm{H}}(\Sigma(h(s)), \Sigma(h(t))) > e^{bk/2} - 1.$$

CASE 3: $l = k = 0$ (i.e. $\Sigma(h(1)) = \Sigma_1(h(1))$). Let $\{e^{2\pi i r_j} \mid j = 1, \ldots, k\}$ $(0 < r_1 < \ldots < r_k < 1)$ be an enumeration of the elements of $\Sigma(h(1)) \setminus \{1\}$ together with their (multiplicative) inverses. Select $p > 0$ such that the intervals $[r_j, 6pr_j]$ $(1 \leq j \leq k)$ are all pairwise disjoint and contained in $(0, 1)$. Let $a = 2p$ and $b = 6p$. Note that if $\Sigma(h(t)) \setminus \{1\}$ and $\Sigma(h(s)) \setminus \{1\}$ intersect then for some $j, l \in \{1, \ldots, k\}$, $e^{2\pi(tr_j - sr_l)} = 1$ and hence $tr_j - sr_l \in \mathbb{Z}$. Replacing r_j (and/or r_l) by $1 - r_j$ (and/or $1 - r_l$) it is enough to assume that $t \in [b/2, b] \equiv [3p, 6p]$ and $s \in [a/2, a] \equiv [p, 2p]$. If $j \neq l$, then $tr_j - sr_l \notin \mathbb{Z}$ (by our choice of intervals). If $j = l$, $tr_j - sr_l = (t - s)r_j$ and again by our choice of intervals this quantity cannot be an integer.

Thus in all the possible cases we have shown that the Hausdorff distance between $\Sigma(h(t))$ and $\Sigma(h(s))$ is strictly positive. Thus we can select an $\varepsilon > 0$ (which will depend on a and b) such that (2.1) is satisfied. ∎

Next, we fix a left invariant metric d on G. Let D denote the supremum metric that d generates on $C(Y,G)$. The identity element of G is denoted by e and the constant cocycle (map) $y \mapsto e$ will be denoted by 1. Given $\alpha \in C(Y,G)$ and $n \in \mathbb{N}$, we set

$$\alpha(y,n) \equiv \alpha(T^{n-1}y)\alpha(T^{n-2}y)\ldots\alpha(y), \qquad y \in Y.$$

Thus α is extended to a map on $Y \times \mathbb{N}$ so that its restriction to $Y \times \{1\}$ is the original α.

We begin by fixing a rigidity sequence q_n as in Definition 1.2. Since G has Property P, there exists an $\varepsilon > 0$, a representation π of G, a one-parameter subgroup H and positive numbers a, b with the properties described in Lemma 2.1.

Given $N \in \mathbb{N}$ define the set

$$\begin{aligned} R(N) = \{\alpha \in \overline{B} \mid \exists q_m \geq N &\text{ and compact subsets } U \equiv U(\alpha, N),\\ &V \equiv V(\alpha,N) \text{ of } Y \text{ such that } \mu(U) \geq 1/8,\ \mu(V) \geq 1/8 \text{ and}\\ &\text{if } x \in U,\ y \in V \text{ then } d_{\mathrm{H}}(\Sigma(\alpha(x,q_m)), \Sigma(\alpha(y,q_m))) > \varepsilon/3\}. \end{aligned}$$

2.2. Lemma. *With the above notation,* $\bigcap\{R(n) \mid n \in \mathbb{N}\} \subseteq \mathcal{N}_{\mathrm{c}}$.

Proof. Suppose the above assertion is false. Then there exists some measurable map $\psi : Y \to G$ and some compact constant $c \in G$ such that

$$\alpha(y) = \psi(Ty)c\psi(y)^{-1}, \quad \text{a.e. } y. \tag{2.2}$$

This implies that $\alpha(z,n) = \psi(T^n z)c^n\psi(z)^{-1}$ for all $n \in \mathbb{N}$ and $z \in Y \setminus Y_0$, where $\mu(Y_0) = 0$. Since G is σ-compact, there exists a compact set $K_1 \subseteq G$ such that if $\widetilde{F} = \{y \in Y \mid \psi(y) \in K_1\}$ then $\mu(\widetilde{F}) > 15/16$. Let K_2 be a compact set such that $\{c^n \mid n \in \mathbb{Z}\} \subset K_2$. Let $\delta_1 > 0$ be such that the set $K^* \equiv B_{\delta_1}(K_1K_2K_1^{-1})$—the closed δ_1-neighbourhood of the set $K_1K_2K_1^{-1}$ is compact.

Next, select δ such that $0 < \delta < \delta_1$ and

$$\text{if} \quad d(g_1,g_2) < \delta,\ g_1, g_2 \in K^* \quad \text{then} \quad d_{\mathrm{H}}(\Sigma(g_1), \Sigma(g_2)) < \varepsilon/6.$$

For $n \in \mathbb{N}$ set

$$F_n = \{y \in Y \mid d(\psi(T^{q_n}y)\psi(y)^{-1}, e) < \delta\}. \tag{2.3}$$

By the rigidity hypothesis, $\psi(T^{q_n}y)\psi(y)^{-1} \to e$ in measure and hence there exists $n_0 \in \mathbb{N}$ (see Proposition 12.1 of [K]) such that

$$\mu(F_n) > 15/16 \quad \text{if } q_n > n_0. \tag{2.4}$$

By the hypothesis $\alpha \in R(n_0)$, let q_m, U and V be as in the definition of $R(n_0)$. Observe that

$$\mu(U \cap F_m) \geq \mu(F_m) + \mu(U) - 1 > 1/16 > 0,$$

and

$$\mu(U \cap F_m \cap T^{-q_m}\widetilde{F}) \geq \mu(U \cap F_m) + \mu(\widetilde{F}) - 1 > 0.$$

In particular $(U \cap F_m \cap T^{-q_m}\widetilde{F}) \setminus Y_0 \neq \emptyset$ and similarly it follows that $(V \cap F_m \cap T^{-q_m}\widetilde{F}) \setminus Y_0 \neq \emptyset$. Now let $x \in (U \cap F_m \cap T^{-q_m}\widetilde{F}) \setminus Y_0$ and $y \in (V \cap F_m \cap T^{-q_m}\widetilde{F}) \setminus Y_0$. Using the left invariance of the metric and $x \in F_m$, we get

$$\begin{aligned} d(\alpha(x, q_m), \psi(T^{q_m}x)c^{q_m}\psi(T^{q_m}x)^{-1}) \\ &= d(\psi(T^{q_m}x)c^{q_m}\psi(x)^{-1}, \psi(T^{q_m}x)c^{q_m}\psi(T^{q_m}x)^{-1}) \\ &= d(\psi(x)^{-1}, \psi(T^{q_m}x)^{-1}) = d(\psi(T^{q_m}x)\psi(x)^{-1}, e) < \delta. \end{aligned}$$

Since $T^{q_m}x \in \widetilde{F}$ and $\delta < \delta_1$, it follows that $\psi(T^{q_m}x)c^{q_m}\psi(T^{q_m}x)^{-1}$ and $\alpha(x, q_m)$ are in K^* and hence by our choice of δ we have

$$d_{\mathrm{H}}(\Sigma(\alpha(x, q_m)), \Sigma(c^{q_m})) < \varepsilon/6.$$

The same argument also yields $d_{\mathrm{H}}(\Sigma(\alpha(y, q_m)), \Sigma(c^{q_m})) < \varepsilon/6$ (replace x by y). Thus,

$$d_{\mathrm{H}}(\Sigma(\alpha(x, q_m)), \Sigma(\alpha(y, q_m))) < \varepsilon/3.$$

This contradicts the fact that $x \in U$ and $y \in V$. ∎

Now the Baire category theorem reduces the proof of Theorem A to proving openness and density of $R(N)$ in $\overline{B}$, for each $N \in \mathbb{N}$. The openness can be easily verified, so we proceed to prove the density.

First, for each $N \in \mathbb{N}$, we define a set $\widetilde{R}(N)$ as follows. Fix a compact neighorhood K of the set $\{h(t) \mid t \in [-(b+1), b+1]\}$ throughout the rest of the proof. Then set

$$\begin{aligned} \widetilde{R}(N) = \{\alpha \in \overline{B} \mid\ & \exists q_m \geq N \text{ and compact subsets } U \equiv U(\alpha, N), \\ & V \equiv V(\alpha, N) \text{ of } Y \text{ such that } \mu(U) \geq 3/16,\ \mu(V) \geq 3/16 \text{ and} \\ & \text{if } x \in U,\ y \in V \text{ then } \alpha(x, q_m), \alpha(y, q_m) \in K \text{ and} \\ & d_{\mathrm{H}}(\Sigma(\alpha(x, q_m)), \Sigma(\alpha(y, q_m))) > \varepsilon\}. \end{aligned}$$

2.3. Lemma. *Suppose* $1 \in \overline{\widetilde{R}}(M)$ *for all* $M \in \mathbb{N}$. *Then* $R(N)$ *is dense in* $\overline{B}$ *for each* $N \in \mathbb{N}$.

Proof. Let $N \in \mathbb{N}$. To prove the density of $R(N)$ in $\overline{B}$, we need to show that given any $\psi \in C(Y, G)$ and $\gamma > 0$, we can find $\alpha \in R(N)$ such that $D(\alpha, 1^{\psi}) < \gamma$.

Let $\delta_1 > 0$ be such that the set $K^{**} \equiv B_{\delta_1}(\psi(Y)K\psi(Y)^{-1})$—the closed δ_1-neighbourhood of the set $\psi(Y)K\psi(Y)^{-1}$ is compact.

Next, choose a positive number δ such that if $d(g_1, g_2) < \delta,\ g_1, g_2 \in K^{**}$, then $d_{\mathrm{H}}(\Sigma(g_1), \Sigma(g_2)) < \varepsilon/3$. Then select $n_0 \in \mathbb{N},\ n_0 > N$ as in the previous lemma, so that (2.4) holds.

Select $\gamma_1 > 0$ such that $d(h, e) < \gamma_1$ implies that $d(h\psi(y)^{-1}, \psi(y)^{-1}) < \gamma$ for all $y \in Y$. Since $1 \in \overline{\widetilde{R}}(n_0)$, we can pick $\widehat{\alpha} \in \widetilde{R}(n_0)$ such that $D(\widehat{\alpha}, 1) < \gamma_1$. Set $\alpha = \widehat{\alpha} \cdot 1^{\psi}$. Then, by the left invariance of the metric and our choice of γ_1,

$$\begin{aligned} D(\alpha, 1^{\psi}) &= \sup_{y \in Y} d(\psi(Ty)\widehat{\alpha}(y)\psi(y)^{-1}, \psi(Ty)\psi(y)^{-1}) \\ &= \sup_{y \in Y} d(\widehat{\alpha}(y)\psi(y)^{-1}, \psi(y)^{-1}) < \gamma. \end{aligned}$$

Now let $q_m, \widetilde{U}$ and $\widetilde{V}$ be as in the definition of $\widetilde{R}(n_0)$. Set $U = \widetilde{U} \cap F_m, V = \widetilde{V} \cap F_m$ (where F_m is defined by (2.3)). Observe that $q_m > n_0 > N$ and

$$\mu(U) \geq \mu(\widetilde{U}) + \mu(F_m) - 1 > 3/16 + 15/16 - 1 = 1/8.$$

Similarly $\mu(V) > 1/8$. Finally if $x \in U$ then $x \in F_m$, hence

$$d(\alpha(x, q_m), \psi(T^{q_m}x)\widehat{\alpha}(x, q_m)\psi(T^{q_m}x)^{-1}) = d(\psi(T^{q_m}x)\psi(x)^{-1}, e) < \delta.$$

Thus, by our choice of δ, $d_{\mathrm{H}}(\Sigma(\alpha(x, q_m)), \Sigma(\widehat{\alpha}(x, q_m))) < \varepsilon/3$. Similarly for $y \in V$, $d_{\mathrm{H}}(\Sigma(\alpha(y, q_m)), \Sigma(\widehat{\alpha}(y, q_m))) < \varepsilon/3$. Since $x \in \widetilde{U}$ and $y \in \widetilde{V}$,

$$d_{\mathrm{H}}(\Sigma(\widehat{\alpha}(x, q_m)), \Sigma(\widehat{\alpha}(y, q_m))) > \varepsilon.$$

Hence $d_{\mathrm{H}}(\Sigma(\alpha(x, q_m)), \Sigma(\alpha(y, q_m))) > \varepsilon/3$. Thus $\alpha \in R(N)$. ∎

Thus the proof now reduces to the following lemma.

2.4. Lemma. *Given any $M \in \mathbb{N}$ and $\gamma > 0$, there exists a function $\psi \in C(Y, G)$ such that*

(a) $D(1^{\psi}, 1) < \gamma$ *and*

(b) $1^{\psi} \in \widetilde{R}(M)$.

Proof. We shall describe the detailed construction of ψ in a series of steps.

Step 1. Recall that we have fixed a continuous homomorphism $h : \mathbb{R} \to G$ and $\varepsilon > 0$ such that

$$d_{\mathrm{H}}(\Sigma(h(s)), \Sigma(h(t))) > \varepsilon \quad \text{if } s \in [-a, -a/2] \cup [a/2, a] \text{ and } t \in [-b, -b/2] \cup [b/2, b]. \tag{2.5}$$

Step 2. Let $\delta > 0$ be chosen so that if $|t-s| < \delta$ and $t, s \in [-(b+1), b+1]$ then $d(h(t), h(s)) < \gamma$.

STEP 3. Pick $N \in \mathbb{N}$ such that $L \equiv q_N$ satisfies

$$L > \max\{M, 2(b+1)/\delta\}. \tag{2.6}$$

The required map ψ will be composition of h and θ, where $\theta : Y \to \mathbb{R}$ will be defined shortly, using the "stacking and averaging" technique.

STEP 4. Next, using Rokhlin's lemma, pick a compact set $A \subseteq Y$ such that

$$A, TA, \ldots, T^{L^2-1}A \text{ are mutually disjoint and} \tag{2.7.a}$$

$$\mu\Big(\bigcup\{T^iA \mid 0 \le i \le L^2-1\}\Big) > 1-\varrho, \tag{2.7.b}$$

where $\varrho > 0$ is a small number such that

$$.9\frac{(1-\varrho)(L-4)(L-2)}{L^2} > \frac{3.1}{4}; \tag{2.7.c}$$

notice that such a choice of ϱ is possible if L is chosen sufficiently large in the first place. Without loss of generality, we shall assume that such a choice of L is made in Step 3.

STEP 5. Next, let B and C be disjoint compact subsets of A such that $\mu(B) = \mu(C) > .9(1-\varrho)/(2L^2)$, (this is possible since $\mu(A) > (1-\varrho)/L^2$ and μ is non-atomic).

STEP 6. We start by defining a map $\widetilde{\theta}$ on $\bigcup\{T^i(B\cup C) \mid 0 \le i \le L^2-1\}$ by setting it equal to 0 and a (resp. 0 and b) alternately on stacks of height L based on B (resp. C). More precisely, set

$$\widetilde{\theta} = \begin{cases} 0 & \text{on } T^{kL+s}(B\cup C),\ 0 \le s \le L-1,\ 0 \le k \le L-1, \text{ if } k \text{ is even,} \\ a & \text{on } T^{kL+s}(B),\ 0 \le s \le L-1,\ 0 \le k \le L-1, \text{ if } k \text{ is odd,} \\ b & \text{on } T^{kL+s}(C),\ 0 \le s \le L-1,\ 0 \le k \le L-1, \text{ if } k \text{ is odd.} \end{cases}$$

Set $\widetilde{\theta} = 0$ outside $\bigcup\{T^i(B\cup C) \mid 0 \le i \le L^2-1\}$.

STEP 7. Now by Lusin's approximation theorem select a continuous function $\overline{\theta} : Y \to [-(b+1), b+1]$ such that

$$\mu(Y \setminus \widetilde{F}) \ge 1 - \frac{.1}{32L},$$

where $Y \setminus \widetilde{F}$ is compact and $Y \setminus \widetilde{F} \subseteq \{y \in Y \mid \overline{\theta}(y) = \widetilde{\theta}(y)\}$. Set

$$F = \bigcup\{T^{-j}\widetilde{F} \mid 0 \le j \le 2L-1\}. \tag{2.8}$$

Then $Y \setminus F$ is compact and

$$\mu(Y \setminus F) \ge (1 - .1/16). \tag{2.9}$$

STEP 8. Now set

$$\theta(y) = \frac{1}{L}\sum_{i=0}^{L-1}\overline{\theta}(T^i y). \tag{2.10}$$

Then $\theta : Y \to [-(b+1), b+1]$ is continuous.

STEP 9. Finally set

$$\psi(y) = h(\theta(y)) \qquad (y \in Y).$$

We shall show that ψ is the required map. Clearly ψ is continuous and

$$\sup_{y\in Y} |\theta(Ty) - \theta(y)| = \frac{1}{L}\sup_{y\in Y} |\overline{\theta}(T^L y) - \overline{\theta}(y)| \le \frac{2(b+1)}{L} < \delta.$$

Now, (a) follows from our choice of δ in Step 2. To prove (b), we make the following observations.

STEP 10. First, consider the map θ_1 defined by

$$\theta_1(y) = \frac{1}{L}\sum_{i=0}^{L-1}\widetilde{\theta}(T^i y). \tag{2.11}$$

First we shall analyze the map θ_1 on the set $\bigcup\{T^i(B\cup C) \mid 0 \le i \le L^2 - L - 1\} \equiv \bigcup\{T^{kL+j}(B\cup C) \mid 0\le j\le L-1,\ 0\le k\le L-2\}$. We shall later show that θ_1 differs from θ only on a set of small measure.

Observe that

$$\theta_1(T^L y) - \theta_1(y) = a\left(\frac{L-j}{L}\right) - a\frac{j}{L} = a\left(1 - \frac{2j}{L}\right) \qquad \text{if } y \in T^j(B),\ 0 \le j \le L-1.$$

Since $\widetilde{\theta}$ is defined periodically (with period $2L$) on stacks, it follows that if $0\le j\le L-1,\ 0\le k\le L-2$, then

$$\theta_1(T^L y) - \theta_1(y) = a\left(1 - \frac{2j}{L}\right) \qquad \text{if } y \in T^{kL+j}B \text{ and } k \text{ is even.} \tag{2.12.i}$$

Similarly it is easy to verify that

$$\theta_1(T^L y) - \theta_1(y) = a\frac{j}{L} - a\left(\frac{L-j}{L}\right) = a\left(\frac{2j}{L} - 1\right) \qquad \text{if } y \in T^{kL+j}B \text{ and } k \text{ is odd,} \tag{2.12.ii}$$

$$\theta_1(T^L y) - \theta_1(y) = b\left(1 - \frac{2j}{L}\right) \qquad \text{if } y \in T^{kL+j}C \text{ and } k \text{ is even,} \tag{2.12.iii}$$

$$\theta_1(T^L y) - \theta_1(y) = b\left(\frac{2j}{L} - 1\right) \qquad \text{if } y \in T^{kL+j}C \text{ and } k \text{ is odd.} \tag{2.12.iv}$$

STEP 11. Now let $\widetilde{U}$ be the union of stacks of height $2L_0$ ($L_0 = [L/4]$ is the integral part of $L/4$), centered at the level $T^{kL}B$ (the exception is the very first stack with base B, which is of height L_0). More precisely, set

$$\widetilde{U} = \bigcup \{T^j B \mid 0 \le j \le L_0 - 1\} \cup \{T^{kL+j} B \mid -L_0 \le j \le L_0,\ 1 \le k \le L-2\}.$$

Similarly, define $\widetilde{V}$ by replacing B by C in the above definition.

Now we make the following observation. Let $y \in \widetilde{U}$. Clearly $\theta_1(y) \le a$. Now suppose $y \in T^{kL+j}B$ where $1 \le k \le L-2$, k is even and $-L_0 \le j \le L_0$. First supposing that $0 \le j \le L_0$, by (2.12.i) we have

$$\theta_1(T^L y) - \theta_1(y) = a\left(1 - \frac{2j}{L}\right) \ge a\left(1 - \frac{2L_0}{L}\right) \ge a\left(1 - \frac{2}{L}\left[\frac{L}{4}\right]\right) \ge \frac{a}{2}.$$

Now suppose $-L_0 \le j < 0$. Set $j' = -j$. Writing $kL + j = (k-1)L + (L - j')$ and then using (2.12.ii) we get

$$\theta_1(T^L y) - \theta_1(y) = a\left(\frac{2(L-j')}{L} - 1\right) \ge a\left(1 - \frac{2j'}{L}\right) \ge a\left(1 - \frac{2L_0}{L}\right) \ge \frac{a}{2}.$$

This argument shows that $\theta_1(T^L y) - \theta_1(y) \in [a/2, a]$ if $y \in T^{kL+j}B$ where $0 \le k \le L-2$, k is even and $-L_0 \le j \le L_0$. By a similar argument one can easily verify that $\theta_1(T^L y) - \theta_1(y) \in [-a, -a/2]$ if $y \in T^{kL+j}B$, where $1 \le k \le L-2$, k is odd and $-L_0 \le j \le L_0$, i.e.

$$\text{(2.13.i)} \qquad \theta_1(T^L y) - \theta_1(y) \in [-a, -a/2] \cup [a/2, a] \quad \text{if } y \in \widetilde{U}.$$

Similarly,

$$\text{(2.13.ii)} \qquad \theta_1(T^L y) - \theta_1(y) \in [-b, -b/2] \cup [b/2, b] \quad \text{if } y \in \widetilde{V}.$$

Next, using (2.7.b) and (2.7.c) and the fact $L - 4 \le 4L_0$, we have

$$\text{(2.14.i)} \qquad \begin{aligned} \mu(\widetilde{U}) &\ge \mu(B)(2L_0)(L-2) \ge \mu(B)\frac{2(L-4)}{4}(L-2) \\ &\ge .9\frac{1-\varrho}{2L^2} \cdot \frac{2(L-4)}{4}(L-2) \ge \frac{3.1}{16}. \end{aligned}$$

Similarly,

$$\text{(2.14.ii)} \qquad \mu(\widetilde{V}) \ge \frac{3.1}{16}.$$

Now we prove that $1^\psi \in \widetilde{R}(M)$. Let

$$\text{(2.15)} \qquad q_N \equiv L, \quad U = \widetilde{U} \cap (Y \setminus F), \quad V = \widetilde{V} \cap (Y \setminus F).$$

Then U, V are compact and using (2.14.i) we have

$$\mu(U) \ge \mu(\widetilde{U}) + \mu(Y \setminus F) - 1 > \frac{3.1}{16} + \left(1 - \frac{.1}{16}\right) - 1 = \frac{3}{16}.$$

Similarly (2.14.ii) yields $\mu(V) > 3/16$.

Now if $y \in U$ then by (2.8), $T^i y \notin \widetilde{F}$ for $0 \le i \le L-1$. Hence

$$\theta(y) = \frac{1}{L}\sum_{i=0}^{L-1} \overline{\theta}(T^i y) = \frac{1}{L}\sum_{i=0}^{L-1} \widetilde{\theta}(T^i y) = \theta_1(y).$$

In fact, again by (2.8), $y \notin F$ implies that $T^{L+i}y \notin \widetilde{F}$ for $0 \le i \le L-1$. Hence $\theta(T^L y) = \theta_1(T^L y)$ if $y \in U$. Thus by (2.13.i),

$$\theta(T^L y) - \theta(y) = \theta_1(T^L y) - \theta_1(y) \in [-a, -a/2] \cup [a/2, a] \quad \text{if } y \in U.$$

Similarly (2.13.ii) yields

$$\theta(T^L x) - \theta(x) = \theta_1(T^L x) - \theta_1(x) \in [-b, -b/2] \cup [b/2, b] \quad \text{if } x \in V.$$

Thus, if $y \in U$, and $x \in V$ then (2.5) implies that in the Hausdorff metric the sets $\Sigma(1^{\psi}(y, L))$ and $\Sigma(1^{\psi}(x, L))$ are at least ε apart. This shows that $1^{\psi} \in \widetilde{R}(M)$. ■

3. Proof of Theorem B. The non-commutativity of the fiber group G is the main obstacle in extending the closure of coboundaries technique to yield a generic lifting theorem in the class $C(Y, G)$ of all continuous cocycles. We develop a technique to overcome this problem. The first step involves introducing more "general skew-products", where one multiplies the elements of the fiber group G on both left and right side by the skewing function (i.e. by the cocycle). Our technique to analyze the generic ergodicity lifting in such extensions has two ingredients: (a) the "modified closure of coboundaries technique" (which proves Theorem 3.2) and (b) an approximation procedure, which will allow us to derive Theorem B from Theorem 3.2. This procedure consists of approximating the average of a given function under the "usual skew-product" flow by the average of its translate under the "generalized skew-product flow". We now give the precise details.

We begin by refining the notion of a skew-product transformation.

3.1. Definitions. As before, let (Y, T, μ) be a topological dynamical system and G be a topological group with identity e. The *generalized skew-product* corresponding to a given pair of maps $\varphi_1, \varphi_2 \in C(Y, G)$ is defined by

$$T_{(\varphi_1,\varphi_2)}(g, y) = (\varphi_1(y) g \varphi_2(y)^{-1}, Ty), \tag{3.1}$$

and its iterates are given by

$$T^n_{(\varphi_1,\varphi_2)}(g, y) = (\varphi_1(y, n) g \varphi_2(y, n)^{-1}, T^n y), \quad n \in \mathbb{N} \cup \{0\}, \tag{3.2}$$

where $\varphi(y, 0) = e$ and

$$\varphi(y, n) = \varphi(T^{n-1}y)\varphi(T^{n-2}y)\ldots\varphi(y), \quad n \in \mathbb{N}, \tag{3.3}$$

for any given $\varphi \in C(Y, G)$.

In this notation, Theorem B states that the set

$$\{\varphi \in C(Y,G) \mid (G\times Y, T_{(\varphi,1)}, \nu\times\mu) \text{ is ergodic}\}$$

is residual. As mentioned before, the proof of Theorem B has two main components. The first one—the modified closure of coboundaries technique—is described in the following theorem.

3.2. THEOREM. *Let (Y,T,μ) be a topological dynamical system and G be a metric topological group and let $\alpha \in C(Y,G)$ be a given cocycle. Suppose*

(1) *(Y,T,μ) is aperiodic and ergodic and*
(2) *G is compact and connected.*

Then the set

$$\mathcal{C}^{\alpha}_{\mathrm{erg}} \equiv \{\beta \in \overline{B} \mid (G\times Y, T_{(\alpha,\beta)}, \nu\times\mu) \textit{ is ergodic}\}$$

is a residual subset of $\overline{B}$.

We remark that if $\alpha = 1$, then the above theorem is proved in [N1].

Proof (of Theorem 3.2). The technique employed to prove this theorem is a modification of the closure of coboundaries technique (employed in Section 2). Those who are familiar with this technique will realize that the essential feature of this modification is the use of Proposition 3.4 below. We begin with the usual steps of this procedure (see [N2]).

Let $\mathcal{H} = L^2(G\times Y, \nu\times\mu)$ and $\mathcal{H}_0 = \{f \in \mathcal{H} \mid \int_{G\times Y} f\, d(\nu\times\mu) = 0\}$. Let $\|\,\|_2$ be the L^2 norm on $\mathcal{H}$. Given $\varphi_1, \varphi_2 \in C(Y,G)$, define a unitary operator $U_{(\varphi_1,\varphi_2)}$ on $\mathcal{H}$ by setting

$$U_{(\varphi_1,\varphi_2)} f = f \circ T_{(\varphi_1,\varphi_2)}. \tag{3.4}$$

Furthermore, let

$$W_n^{(\varphi_1,\varphi_2)} = \frac{1}{n}\sum_{i=0}^{n-1} U^i_{(\varphi_1,\varphi_2)}. \tag{3.5}$$

Let $\alpha \in C(Y,G)$ be the map given in the statement of Theorem 3.2. Given $f \in \mathcal{H}_0$, $\varepsilon > 0$ and $r \in \mathbb{N}$, define a set $E_\alpha(f,\varepsilon,r)$ as follows:

$$E_\alpha(f,\varepsilon,r) = \{\beta \in \overline{B} \mid \exists M \in \mathbb{N},\ M > r \text{ such that } \|W_M^{(\alpha,\beta)} f\|_2 < \varepsilon\}.$$

3.3. LEMMA. *Let $\{f_j \mid j \in \mathbb{N}\}$ be a dense subset of $\mathcal{H}_0$. If $\beta \in E_\alpha(f_j, 1/n, r)$ for all $j,n,r \in \mathbb{N}$, then $(G\times Y, T_{(\alpha,\beta)}, \nu\times\mu)$ is ergodic.*

Proof. Fix any $j \in \mathbb{N}$. By the L^2 ergodic theorem, as $n \to \infty$ the sequence $W_n^{(\alpha,\beta)} f_j$ converges in the L^2 norm to some $f_j^* \in \mathcal{H}$. Since $\beta \in E_\alpha(f_j, 1/n, r)$ for all $n, r \in \mathbb{N}$, f_j^* must be the zero function. Since j was arbitrary and $\{f_j \mid j \in \mathbb{N}\}$ is dense in $\mathcal{H}_0$, it follows that for each $f \in \mathcal{H}_0$, $W_n^{(\alpha,\beta)} f \to 0$ as $n \to \infty$. This proves ergodicity of $T_{(\alpha,\beta)}$. ∎

Thus, once openness and density of each $E_\alpha(f,\varepsilon,r)$ in $\overline{B}$ is established, the assertion of Theorem 3.2 follows from the Baire category theorem. Openness of each $E_\alpha(f,\varepsilon,r)$ can be easily checked, so we turn to the proof of its density.

Density of $E_\alpha(f,\varepsilon,r)$. We continue with the "usual steps" of the closure of coboundaries technique in our more general set-up. Given a $\phi \in C(Y,G)$, define a $\nu\times\mu$-preserving map $\widehat{R}_\phi : G\times Y \to G\times Y$ by setting

$$\widehat{R}_\phi(g,y) = (g\phi(y),y).$$

Let R_ϕ be the corresponding unitary operator induced on $\mathcal{H}$, i.e.

$$R_\phi f = f\circ \widehat{R}_\phi, \qquad f\in\mathcal{H}.$$

Now, the following identities can be easily verified:

$$\begin{aligned} \widehat{R}_\phi \circ T_{(\alpha,\beta\cdot 1^\phi)} &= T_{(\alpha,\beta)}\circ \widehat{R}_\phi, \\ U_{(\alpha,\beta\cdot 1^\phi)} &= R_\phi\circ U_{(\alpha,\beta)}\circ R_\phi^{-1}, \\ W_n^{(\alpha,\beta\cdot 1^\phi)} &= R_\phi \circ W_n^{(\alpha,\beta)}\circ R_\phi^{-1} \qquad \text{for all } n\in\mathbb{N}. \end{aligned}$$

Since R_ϕ is unitary, $\beta\cdot 1^\phi \in E_\alpha(f,\varepsilon,r)$ if and only if $\beta\in E_\alpha(R_\phi^{-1}f,\varepsilon,r)$.

Thus, if $1\in\overline{E_\alpha(g,\varepsilon,r)}$ for all $g\in\mathcal{H}_0$, $\varepsilon>0$ and $r\in\mathbb{N}$, given any $\delta>0$ and $\phi\in C(Y,G)$, setting $g=R_\phi^{-1}f$ and selecting $\beta\in E_\alpha(R_\phi^{-1}f,\varepsilon,r)$ with $D(\beta,1)<\delta$ we observe that

(1) $D(1^\phi,\beta\cdot 1^\phi) = D(\beta,1)<\delta$ and
(2) $\beta\cdot 1^\phi\in E_\alpha(f,\varepsilon,r)$.

In other words $E_\alpha(f,\varepsilon,r)$ is dense in $\overline{B}$. Thus to prove the density of $E_\alpha(f,\varepsilon,r)$ in $\overline{B}$, it is enough to prove that $1\in\overline{E_\alpha(g,\varepsilon,r)}$ for all $g\in\mathcal{H}_0$, $\varepsilon>0$ and $r\in\mathbb{N}$. Hence we need to show that given any $f\in\mathcal{H}_0$, $\varepsilon>0$, $r\in\mathbb{N}$ and $\delta>0$, there exists a $\psi\in C(Y,G)$ such that

(1) $D(1^\psi,1)\equiv \sup_{y\in Y} d(1^\psi(y),e)<\delta$ and
(2) $\|W_M^{(\alpha,1^\psi)}f\|_2<\varepsilon$ for some $M>r$.

Now observe that

$$\|W_M^{(\alpha,1^\psi)}f\|_2 = \|R_\psi\circ W_M^{(\alpha,1)}\circ R_\psi^{-1}f\|_2 = \|W_M^{(\alpha,1)}\circ R_\psi^{-1}f\|_2.$$

This computation shows that we need to analyze the ergodic averages of functions under the transformation $T_{(\alpha,1)}$. This is done via the following result, which describes the ergodic components of the invariant measure $\nu\times\mu$ in terms of the "Mackey range" G_α of the cocycle α (see [Sch], [Z] for a proof).

3.4. Proposition. *Let* (Y, T, μ) *be ergodic and let* G *be a compact group. Then corresponding to a given cocycle* α, *there exist a closed subgroup* $G_\alpha \subseteq G$ *and a Borel measurable map* $\xi : Y \to G$ *such that*

(1) $\alpha \cdot 1^\xi \equiv \xi(Ty)\alpha(y)\xi(y)^{-1} \in G_\alpha$, *a.e.* $y \in Y$, *and*
(2) $(G_\alpha \times Y, T_{(\alpha\cdot 1^\xi, 1)}, \nu_\alpha \times \mu)$ *is ergodic*,

where ν_α *is the normalized Haar measure on* G_α.

Applying this proposition to our given map α, we get a closed subgroup G_α and a Borel measurable map $\xi : Y \to G$ satisfying the conclusion of Proposition 3.4.

Next, define a map $\widehat{L}_\xi$ on $G \times Y$ by setting

$$\widehat{L}_\xi(g, y) = (\xi(y)^{-1}g, y)$$

and let L_ξ be the corresponding unitary operator induced on $\mathcal{H}$. Then the following identities can be easily verified:

$$\begin{aligned} T_{(\alpha,1)} &= \widehat{L}_\xi \circ T_{(\alpha\cdot 1^\xi,1)} \circ \widehat{L}_\xi^{-1}, \\ U_{(\alpha,1)} &= L_\xi^{-1} \circ U_{(\alpha\cdot 1^\xi,1)} \circ L_\xi, \\ W_M^{(\alpha,1)} &= L_\xi^{-1} \circ W_M^{(\alpha\cdot 1^\xi,1)} \circ L_\xi. \end{aligned}$$

Returning to the proof of Theorem 3.2, we have

$$\begin{aligned} \|W_M^{(\alpha,1^\psi)} f\|_2 = \|W_M^{(\alpha,1)} \circ R_\psi^{-1} f\|_2 &= \|L_\xi^{-1} W_M^{(\alpha\cdot 1^\xi,1)} L_\xi \circ R_\psi^{-1} f\|_2 \\ &= \|W_M^{(\alpha\cdot 1^\xi,1)} (L_\xi \circ R_\psi^{-1} f)\|_2. \end{aligned}$$

Now by the ergodicity of the system $(G_\alpha \times Y, T_{(\alpha\cdot 1^\xi,1)}, \nu_\alpha \times \mu)$, we know that the sequence of operators $n \mapsto W_n^{(\alpha\cdot 1^\xi,1)}$ converges strongly to the projection operator P_α given by

$$(P_\alpha \varrho)(g, y) = \int_Y \int_{G_\alpha} \varrho(kg, y)\, d\nu_\alpha(k)\, d\mu(y), \qquad \varrho \in \mathcal{H}.$$

Thus as $M \to \infty$,

$$W_M^{(\alpha\cdot 1^\xi,1)}(L_\xi(R_\psi^{-1} f)) \to \int_Y \int_{G_\alpha} f(\xi(y)^{-1} k g \psi(y)^{-1}, y)\, d\nu_\alpha(k)\, d\mu(y).$$

Set

$$f^*(g, y) = \int_{G_\alpha} f(\xi(y)^{-1} k g, y)\, d\nu_\alpha(k).$$

Then $f^* \in \mathcal{H}_0$ and as $M \to \infty$,

$$W_M^{(\alpha\cdot 1^\xi,1)}(L_\xi(R_\psi^{-1}f)) \to \int_Y f^*(R_\psi^{-1}(g,y))\,d\mu(y) = \int_Y f^*(g\psi(y)^{-1},y)\,d\mu(y).$$

Hence, if we can choose a $\psi \in C(Y,G)$ such that

(1) $D(1^\psi,1) < \delta$ and
(2) $\|\int_Y f^*(g\psi(y)^{-1},y)\,d\mu(y)\|_2 < \varepsilon/2$,

then by choosing M large enough, we can make $\|W_M^{(\alpha,1^\psi)}f\|_2 < \varepsilon$. Thus we have reduced the proof of Theorem 3.2 to proving the following lemma.

3.5. Lemma. *Under the hypothesis of Theorem* 3.2 *and with the above notation*, *given any* $f^* \in \mathcal{H}_0$, $\varepsilon > 0$ *and* $\delta > 0$, *there exists a map* $\psi \in C(Y,G)$ *such that*

(1) $D(1^\psi,1) < \delta$ *and*
(2) $\|\int_Y f^*(g\psi(y),y)\,d\mu(y)\|_2 < \varepsilon$.

This lemma is proved in [N1] (Lemma 3.9). In fact, in the continuous case the proof is given for far more general ergodic dynamical systems (Y,T,μ) where the "acting group T" is allowed to be any "reasonable amenable group". This completes the proof of Theorem 3.2. ■

Approximation procedure and proof of Theorem B. Now we develop the approximation technique to derive Theorem B from Theorem 3.2. As before, given $f \in \mathcal{H}_0$, $\varepsilon > 0$ and $r \in \mathbb{N}$, define the set $F(f,\varepsilon,r)$ by setting

$$F(f,\varepsilon,r) = \{\varphi \in C(Y,G) \mid \exists M \in \mathbb{N},\ M > r \text{ such that } \|W_M^{(\varphi,1)}f\|_2 < \varepsilon\}.$$

Once again the arguments of Lemma 3.3 remain valid and the proof reduces to showing:

3.6. Lemma. *Each* $F(f,\varepsilon,r)$ *is dense in* $C(Y,G)$.

Proof. Since the set of continuous maps is dense in $\mathcal{H}_0$, there is no loss of generality in assuming that the map $f : G \times Y \to \mathbb{R}$ is continuous. Let $\alpha \in C(Y,G)$ and $\delta > 0$ be given. We want to construct a map $\psi \in C(Y,G)$ such that if we set $\varphi = \alpha\psi$ then

(1) $D(\alpha,\varphi) = D(1,\psi) < \delta$ and
(2) $\varphi \in F(f,\varepsilon,r)$.

Now, we carry out this construction in a series of steps.

Step 1. First, select $\delta_1 > 0$ such that

$$d(h,e) < \delta_1,\ h \in G \quad \text{implies} \quad d(ghg^{-1},e) < \delta \text{ for all } g \in G.$$

Step 2. Then applying Theorem 3.2, we get a map $\beta \in C(Y,G)$ such that

(1) $D(1,\beta) < \delta_1$ and
(2) $(G \times Y, T_{(\alpha,\beta)}, \eta \times \mu)$ is ergodic.

Step 3. The following lemma is a consequence of ergodicity of $T_{(\alpha,\beta)}$ and the fact that this transformation is an "isometric extension" of T. We shall use it in the next constructive step and prove it later.

3.7. Lemma. *Let $\alpha,\beta \in C(Y,G)$ be such that $T_{(\alpha,\beta)}$ is ergodic. Then, given $r \in \mathbb{N}$, $f \in \mathcal{H}_0 \cap C(G\times Y)$ and $\varepsilon > 0$, there exists $N \in \mathbb{N}$ and a Borel set $Y^* \subset Y$ such that*

(1) *$r < N$ and $1/N < \varepsilon^2/(4K^2)$,*
(2) *$m(Y^*) > 1 - \varepsilon^2/(4K^2)$, and*
(3) *$|W_N^{(\alpha,\beta)} f_h(g,y)| < \varepsilon/2$ for all $g,h \in G$ and $y \in Y^*$,*

where $f_h(g,y) = f(gh,y)$ and $K = \|f\|_\infty$ is the sup-norm of f.

The uniformity (in $(g,h,y) \in G\times G\times Y^*$) in inequality (3) will be crucial to our proof.

Thus, applying this lemma with f, r, ε, α and β as in Step 2, we get $N \in \mathbb{N}$ and a Borel set Y^* satisfying the conclusion of Lemma 3.7.

Now, if G is commutative, then $T_{(\alpha,\beta)} = T_{(\alpha\beta^{-1},1)}$ and β^{-1} will be our desired function ψ. When G is non-abelian, the key idea in the construction of the desired ψ is to make sure that for "most of the points $(h,y) \in G\times Y$" the average of f over the first N iterates under $T_{(\alpha\psi,1)}$ can be approximated by the average of its "h-translate" under $T_{(\alpha,\beta)}$ at (e,y). This is a sort of "approximation procedure" which we shall carry out by a "stacking construction" using Rokhlin's lemma.

Step 4. Using Rokhlin's lemma, select a Borel set $E \subset Y$ such that the sets $E, TE, \ldots, T^{N^2-1}E$ are pairwise disjoint and

$$\mu\Big(\bigcup\{T^iE \mid 0 \le i \le N^2-1\}\Big) > 1 - \frac{\varepsilon^2}{4K^2}. \tag{3.6}$$

Since μ is regular, we shall also assume that E is compact.

Step 5. Now we define ψ on $E, TE, \ldots, T^{N^2-1}E$ successively by the requirement that

$$(T_{(\alpha\psi,1)})^n(e,y) = (T_{(\alpha,\beta)})^n(e,y) \quad \text{for all } y \in E \text{ and } n = 0,1,\ldots,N^2.$$

Using the notation introduced in (3.2), observe that for each $y \in Y$ and $n = 0,1,\ldots,N^2$,

$$T^n_{(\alpha\psi,1)}(e,y) = ((\alpha\psi)(y,n), T^n y),$$
$$T^n_{(\alpha,\beta)}(e,y) = (\alpha(y,n)\beta(y,n)^{-1}, T^n y).$$

(Warning: in general $(\alpha\psi)(y,n) \neq \alpha(y,n)\psi(y,n)$.) Thus, we define ψ on E by requiring that

$$(\alpha\psi)(y,n+1) = \alpha(y,n+1)\beta(y,n+1)^{-1} \tag{3.7}$$

for all $y \in E$ and $n = 0,1,\dots,N^2-1$. By the cocycle identity, this is equivalent to requiring that

$$\alpha(T^n y)\psi(T^n y)(\alpha\psi)(y,n) = \alpha(T^n y)\alpha(y,n)\beta(y,n)^{-1}\beta(T^n y)^{-1}$$

for all $y \in E$ and $n = 0,1,\dots,N^2-1$. This allows us to define ψ inductively on $\bigcup\{T^n E \mid 0 \le n \le N^2-1\}$ by first setting

$$\psi(y) = \beta(y)^{-1} \quad \text{if } y \in E,$$

and then inductively defining

$$\begin{aligned} \psi(T^n y) &= \alpha(y,n)\beta(y,n)^{-1}\beta(T^n y)^{-1}(\alpha\psi)(y,n)^{-1} \\ &= \alpha(y,n)\beta(y,n)^{-1}\beta(T^n y)^{-1}\beta(y,n)\alpha(y,n)^{-1} \qquad \text{(by (3.7))} \\ &= ad_{\alpha(y,n)\beta(y,n)^{-1}}(\beta(T^n y)^{-1}) \end{aligned} \tag{3.8}$$

for all $y \in E$ and $n = 1,\dots,N^2-1$. (Note that if $y \in E$ then $\alpha(y,n)$ and $\beta(y,n)$ are determined by values of α and β on $\bigcup\{T^i E \mid 1 \le i \le n-1\}$.) Now the choice of β in Step 2 and δ_1 in Step 1 implies that

$$d(\psi(y),e) < \delta \quad \text{for all } y \in T^k E,\ 0 \le k \le N^2-1. \tag{3.9}$$

Now extend ψ continuously to all of Y so that $D(\psi,1) < \delta$. This is possible by viewing the δ-neighbourhood of e as a Euclidean ball and then applying Tietze's extension theorem. Thus we have constructed $\varphi = \alpha\psi \in C(Y,G)$ such that $D(\alpha,\varphi) < \delta$.

Now, we verify that $\varphi \in F(f,\varepsilon,r)$. We begin by observing that if $y \in E$ and $n \in [0,N-1]$, then

$$\begin{aligned} T^n_{(\varphi,1)}(e,y) &= ((\alpha\psi)(y,n), T^n y) \\ &= (\alpha(y,n)\beta(y,n)^{-1}, T^n y) \qquad \text{(by (3.7))} \\ &= T^n_{(\alpha,\beta)}(e,y). \end{aligned} \tag{3.10}$$

Recalling that $f_h(g,y) = f(gh,y)$, we have

$$\begin{aligned} (W_N^{(\varphi,1)} f)(h,y) &= \frac{1}{N}\sum_{n=0}^{N-1} f_h(T^n_{(\alpha\psi,1)}(e,y)) \\ &= \frac{1}{N}\sum_{n=0}^{N-1} f_h(T^n_{(\alpha,\beta)}(e,y)) \qquad \text{(by (3.10))} \\ &= (W_N^{(\alpha,\beta)} f_h)(e,y). \end{aligned} \tag{3.11}$$

Thus, Lemma 3.7(3) (with $g = e$) implies that

$$|(W_N^{(\varphi,1)} f)(h, y)| < \varepsilon/2 \quad \text{if } y \in E \cap Y^*. \tag{3.12}$$

Now we show that the same estimate holds for any $y \in T^k E \cap Y^*$, for any $k \in [0, N^2 - N]$. Fix a $z = T^k y$, where $y \in E$ and $k \in [0, N^2 - N]$, and $n \in [0, N-1]$ and consider

$$\begin{aligned} T^n_{(\alpha\psi,1)}(e, z) &= T^n_{(\alpha\psi,1)}(e, T^k y) \\ &= ((\alpha\psi)(T^k y, n), T^{n+k} y) \\ &= ((\alpha\psi)(y, k+n)(\alpha\psi)(y, k)^{-1}, T^{n+k} y) \quad \text{(by the cocycle identity)} \\ &= (\alpha(y, k+n)\beta(y, k+n)^{-1}(\alpha(y, k)\beta(y, k)^{-1})^{-1}, T^{n+k} y) \text{ (by (3.7))}. \end{aligned}$$

Writing $g_z = \alpha(y, k)\beta(y, k)^{-1}$, we get

$$\begin{aligned} T^n_{(\alpha\psi,1)}(e, z) &= (\alpha(y, k+n)\beta(y, k+n)^{-1} g_z^{-1}, T^{n+k} y) \\ &= (\alpha(T^k y, n)\alpha(y, k)\beta(y, k)^{-1}\beta(T^k y, n)^{-1} g_z^{-1}, T^{n+k} y) \\ &= (\alpha(z, n) g_z \beta(z, n)^{-1} g_z^{-1}, T^n(z)). \end{aligned} \tag{3.13}$$

Thus,

$$\begin{aligned} (W_N^{(\varphi,1)} f)(h, z) &= \frac{1}{N}\sum_{n=0}^{N-1} f_h(T^n_{(\alpha\psi,1)}(e, z)) \\ &= \frac{1}{N}\sum_{n=0}^{N-1} f_{g_z^{-1}h}(T^n_{(\alpha,\beta)}(g_z, y)) \quad \text{(by (3.13))} \\ &= (W_N^{(\alpha,\beta)} f_{g_z^{-1}h})(g_z, y). \end{aligned}$$

Again by applying Lemma 3.7(3) (with $g = g_z$ and h replaced by $g_z^{-1}h$) we get

$$|(W_N^{(\varphi,1)} f)(h, z)| < \varepsilon/2 \quad \text{if } z \in T^k E \cap Y^*,\ k \in [0, N^2 - N]. \tag{3.14}$$

Now set $\widetilde{Y} = Y^* \cap \bigcup\{T^k E \mid 0 \le k \le N^2 - N\}$. Then

$$\begin{aligned} \mu(Y \setminus \widetilde{Y}) &\le \mu(Y \setminus Y^*) + \mu\Big(Y \setminus \bigcup\{T^k E \mid k \in [0, N^2 - 1]\}\Big) \\ &\quad + (N-1)\mu(E) \\ &< 2\frac{\varepsilon^2}{4K^2} + (N-1)\left(\frac{1}{N^2}\right) \quad \text{(by (3.6) and Lemma 3.7(2))} \\ &< 2\frac{\varepsilon^2}{4K^2} + \frac{\varepsilon^2}{4K^2} = \frac{3\varepsilon^2}{4K^2} \quad \text{(by Lemma 3.7(1))}. \end{aligned} \tag{3.15}$$

With (3.14) and (3.15) we have

$$\begin{aligned}\|W_N^{(\varphi,1)} f\|_2^2 &= \int\limits_{G\times Y} |W_N^{(\alpha\psi,1)} f(h,y)|^2\, d\nu(h)\times d\mu(y)\\ &\le \int\limits_{G\times \widetilde{Y}} |W_N^{(\alpha\psi,1)} f(h,y)|^2\, d\nu(h)\times d\mu(y) + K^2\mu(Y\setminus \widetilde{Y})\\ &< \frac{\varepsilon^2}{4} + \frac{3\varepsilon^2}{4} < \varepsilon^2.\end{aligned}$$

Since $N > r$ (by Lemma 3.7(1)), the above computation shows that $\varphi \in F(f,\varepsilon,r)$. Thus, the proof of Lemma 3.6 and consequently that of Theorem B is complete. ∎

Proof of Lemma 3.7. First select $\delta > 0$ such that if $d(g_1,g_2) < \delta$ then

(3.16) $\qquad |f_h(g_1,y) - f_h(g_2,y)| < \varepsilon/4 \quad$ for all $y \in Y$ and $h \in G$.

Notice that

$$W_m^{(\alpha,\beta)} f_h(g,y) = \frac{1}{m}\sum_{k=0}^{m-1} f(\alpha(y,k)g\beta(y,k)^{-1}h, T^k y).$$

Now, since G is compact, it admits a bi-invariant metric and hence the transformation $T_{(\alpha,\beta)}$ is an "isometry on fibers". This along with the compactness of Y and continuity of f implies the following:

- If $d(g_1,g_2) < \delta$, then for all $y\in Y, h\in G$, and $m \in \mathbb{N}$,

(3.17$_\mathrm{a}$) $\qquad |W_m^{(\alpha,\beta)} f_h(g_1,y) - W_m^{(\alpha,\beta)} f_h(g_2,y)| < \varepsilon/4.$

- If $d(h_1,h_2) < \delta$, then for all $y \in Y$, $h\in G$, $m\in\mathbb{N}$,

(3.17$_\mathrm{b}$) $\qquad |W_m^{(\alpha,\beta)} f_{h_1}(g,y) - W_m^{(\alpha,\beta)} f_{h_2}(g,y)| < \varepsilon/4.$

Next select a finite subset $H = \{h_1,\dots,h_L\} \subset G$ such that H is δ-dense in G (i.e. every δ-ball in G intersects H).

Next, pick η $(0<\eta<1)$ such that for $K = \|f\|_\infty$,

(3.18$_\mathrm{a}$) $\qquad 2\eta < \varepsilon^2/(4K^2),$

(3.18$_\mathrm{b}$) if $F \subset G$ is any Borel set with $\nu(F) > 1-\eta/L$, then F is δ-dense in G.

Now by the ergodicity of $T_{(\alpha,\beta)}$, $W_m^{(\alpha,\beta)} f_h \to 0$ pointwise a.e. for each $h\in H$. Hence applying Egoroff's theorem, for each i $(1\le i\le L)$ we get a Borel set $P_i \subset G\times Y$ such that

(3.19) $\qquad \nu\times\mu(P_i) > 1-(\eta/L)^2,$

(3.20) $\qquad W_m^{(\alpha,\beta)} f_{h_i} \to 0 \quad$ uniformly on P_i, as $m\to\infty$.

Thus, (given $\varepsilon > 0$) there exists $N \in \mathbb{N}$ such that

$$N > \max(r, 4K^2/\varepsilon^2), \tag{3.21}$$

$$|W_N^{(\alpha,\beta)} f_{h_i}(g,y)| < \varepsilon/4 \quad \text{for all } (g,y) \in P_i,\ 1 \le i \le L. \tag{3.22}$$

Next, for each i ($1 \le i \le L$) set

$$Y_i = \{y \in Y \mid \nu\{g \in G \mid |W_N^{(\alpha,\beta)} f_{h_i}(g,y)| \ge \varepsilon/4\} > \eta/L\},$$

We claim that $\mu(Y_i) < \eta/L$, for if not then

$$\begin{aligned}\nu \times \mu\{(g,y) \in G \times Y \mid |W_N^{(\alpha,\beta)} f_{h_i}(g,y)| \ge \varepsilon/4\} \\ \ge \nu \times \mu\{(g,y) \in G \times Y_i \mid |W_N^{(\alpha,\beta)} f_{h_i}(g,y)| \ge \varepsilon/4\} \ge (\eta/L)^2,\end{aligned}$$

which contradicts (3.19) and (3.22), proving the claim. Now set

$$Y^* = \bigcap_{i=1}^{L} \pi(P_i) \setminus Y_i,$$

where π is the projection $\pi(g,y) = y$. Then

$$\begin{aligned}\mu(Y \setminus Y^*) &\le \sum_{i=1}^{L} [\mu(Y \setminus \pi(P_i)) + \mu(Y_i)] \le L[(\eta/L)^2 + \eta/L] \\ &\le 2\eta < \frac{\varepsilon^2}{4K^2} \quad \text{(by } (3.18_{\rm a})).\end{aligned}$$

Now fix $y \in Y^*$ and $i \in \{1, \ldots, L\}$. Then, since $y \notin Y_i$,

$$\nu\{g \in G \mid |W_N^{(\alpha,\beta)} f_{h_i}(g,y)| > \varepsilon/4\} \le \eta/L.$$

Hence,

$$\nu\{g \in G \mid |W_N^{(\alpha,\beta)} f_{h_i}(g,y)| \le \varepsilon/4\} > 1 - \eta/L;$$

and therefore by our choice of η as in $(3.18_{\rm b})$, the set

$$\{g \in G \mid |W_N^{(\alpha,\beta)} f_{h_i}(g,y)| \le \varepsilon/4\}$$

is δ-dense in G, for each $i \in \{1, \ldots, L\}$. Now our choice of δ along with $(3.17_{\rm a})$ implies that

$$|W_N^{(\alpha,\beta)} f_{h_i}(g,y)| < \varepsilon/2 \quad \text{for all } g \in G,\ y \in Y^*\ (1 \le i \le L).$$

Using $(3.17_{\rm b})$ we conclude that

$$|W_N^{(\alpha,\beta)} f_h(g,y)| < \varepsilon \quad \text{for all } g \in G,\ h \in H \text{ and } y \in Y^*. \ \blacksquare$$

Proof of Corollary C. If G is abelian, this result appears in [IS], and if G is non-abelian and φ lifts ergodicity then $\varphi \in \mathcal{N}$. Thus the corollary follows from Theorem B.

Proof of Corollary D. First we observe that if φ is ergodic (i.e. the skew-product transformation T_φ is ergodic) and G satisfies condition (2) and (3) of Corollary D then V_φ does not have any discrete spectrum. To see this suppose $f \in L^2(Y, \mathbb{C}^n, \mu)$ is an eigenvector of V_φ. Since the representation is unitary, there exists a $\lambda \in \mathbb{R}$ such that

$$(V_\varphi f)(y) = e^{i\lambda} f(y), \quad \text{a.e. } y \in Y.$$

Consider the map $\varrho : U(n) \times Y \to P(\mathbb{C}^n)$ defined by

$$\varrho(g, y) = \pi(g^{-1} f(y)),$$

where $\pi : \mathbb{C}^n \to P(\mathbb{C}^n)$ is the canonical projection onto the projective space. Then

$$\varrho(T_\varphi(g, y)) = \varrho(\varphi(y)g, Ty) = \pi(g^{-1}\varphi(y)^{-1} f(Ty)) = \pi(g^{-1} f(y)) = \varrho(g, y).$$

Thus ϱ is T_φ-invariant and hence by the ergodicity assumption, it is constant a.e. (g, y). Thus there is a non-zero vector $v \in \mathbb{C}^n$ such that

$$\pi(g^{-1} f(y)) = \pi(v), \qquad \text{a.e. } (g, y).$$

In particular, this means that almost all $g \in G$ map the ray $\pi(v)$ into a fixed ray. This implies that the stabilizer of this fixed ray is a closed subgroup with full Haar measure and hence must be all of G. This contradicts assumption (3) of Corollary D. This observation along with Theorem B completes the proof of part (I).

To prove part (II) consider the set

$$S(f, M) = \{\varphi \in \overline{B} \mid \text{there exists } q_n > M \text{ such that } |\langle V_\varphi^{q_n} f, f\rangle| > 1/2\},$$

where $f \in L^2(Y, \mathbb{C}^n, \mu)$ with $\int_Y f\, d\mu = \overline{0}$ and $\|f\| = 1$ and $M \in \mathbb{N}$. Notice that if $\varphi \in S(f, M)$ for all such f's and all $M \in \mathbb{N}$, then f cannot be in the eigenspace belonging to the absolutely continuous component of the spectrum of V_φ (since $|\langle V_\varphi^n f, f\rangle|$ does not tend to zero as $n \to \infty$). Note that each $S(f, M)$ is open and dense in $\overline{B}$. The density follows from the observation that for any $1^\psi \in B$,

$$\langle V_{1^\psi}^{q_n} f, f\rangle = \langle V_1^{q_n}(L_\psi f), L_\psi f\rangle \to \|L_\psi f\|^2 = \|f\|^2 = 1$$

as $q_n \to \infty$ (where $L_\psi f(y) = \psi(y)^{-1} f(y)$). This shows that $B \subset S(f, M)$. Combining this observation with part (I) yields the proof of part (II).

4. Prevalence of ergodic cocycles in the abelian case. The “measure-theoretic counterpart” of residuality is the notion of prevalence introduced in [HSY]. It is natural to ask whether the set of non-constant cocycles (or ergodic cocycles) is prevalent in $C(Y, G)$. In the following, we shall affirmatively answer this question when the fiber group is compact abelian (a weaker result appears in [M]). We begin by briefly reviewing some definitions and facts (the reader is referred to [HSY] for details).

4.1. Definition. A Borel set $R \subseteq C(Y,G)$ is *prevalent* if there is a compactly supported Borel probability measure η on $C(Y,G)$ such that

$$\eta\{\beta \mid \alpha * \beta \in R\} = 1 \quad \text{for any } \alpha \in C(Y,G),$$

where $\alpha*\beta(y)=\alpha(y)\beta(y)$. A general set is prevalent if it contains a prevalent Borel set. The complement of a prevalent set is said to be *shy*.

We remark that prevalent sets are dense and a countable intersection of prevalent sets is prevalent.

Let $\widehat{G}$ be the dual group. For $\gamma \in \widehat{G} \setminus \{1\}$, set

$$C_\gamma = \{\varphi \in C(Y,G) \mid \gamma \circ \varphi \text{ is measurably cohomologous to a constant cocycle}\}.$$

A cocycle φ is called a *weakly mixing* cocycle if $\varphi \notin C_\gamma$ for every $\gamma \in \widehat{G}\setminus\{1\}$. It is well known (see [IS], [JP]) that if φ is weakly mixing then T_φ is a weakly mixing extension of the base dynamical system. In particular, if the base is ergodic and φ is weakly mixing then T_φ is ergodic.

4.2. Proposition. *Let (Y,T,μ) be a dynamical system and G be a metric, topological group. Suppose that*

(1) (Y,T,μ) *is aperiodic and ergodic, and*
(2) G *is a compact connected abelian group.*

Then the set of weakly mixing (and hence ergodic) cocycles is prevalent in $C(Y,G)$.

This result follows at once from the following lemma.

4.3. Lemma. *The set $C(Y,G) \setminus C_\gamma$ is prevalent for each $\gamma \in \widehat{G} \setminus \{1\}$.*

Proof. Recall that $\mathcal{C} \subset C(Y,G)$ is the set of constant maps and hence can be identified with G itself. This allows us to think of the normalized Haar measure ν on G as a measure on $\mathcal{C}$. We show that $(\mathcal{C},\nu)$ is a "probe" for the set $D(\gamma) \equiv C(Y,G)\setminus C_\gamma$ (see [HSY] for detailed definition of a "probe"). In short we need to show that

$$\nu(\varphi * C_\gamma) = 0 \quad \text{for all } \varphi \in C(Y,G),$$

where (as in Definition 4.1) the operation $*$ denotes the group multiplication on $C(Y,G)$.

Notice that $c_0 \in \varphi * C_\gamma \cap \mathcal{C}$ if and only if there exists a Borel measurable map $\xi : Y \to \mathbb{S}^1$ such that $\gamma(c_0) = \gamma(\varphi(y))\xi(Ty)\xi^{-1}(y)$, a.e. y. We want to know how many c_0's can arise this way by varying ξ over Borel maps from Y to G. Suppose d_0 is another such constant map, i.e. $d_0 \in \varphi * C_\gamma \cap \mathcal{C}$. Then $\gamma(d_0) = \gamma(\varphi(y))\eta(Ty)^{-1}\eta(y)$, a.e. y, for some Borel measurable map $\eta : Y \to \mathbb{S}^1$. Thus,

$$(\xi\cdot\eta^{-1})(Ty) = \lambda(\xi\cdot\eta^{-1})(y), \quad \text{a.e. } y,$$

where $\lambda = \gamma(c_0)\gamma(d_0)^{-1} = \gamma(c_0 d_0^{-1})$. Hence λ must be an eigenvalue of the transformation T.

Let Λ be the set of all eigenvalues of T. Then such possible d_0's are contained in the set $c_0\gamma^{-1}(\Lambda^{-1})$. Since $\gamma \neq 1$ and G is connected, $\mathrm{Ker}(\gamma)$ is a proper closed subgroup of G with Haar measure zero. Furthermore since Λ is countable, it follows that $\nu(\varphi * C_\gamma) = 0$. ■

The question of whether the set of ergodic cocycles is prevalent or not remains open for non-compact as well as compact non-abelian groups.

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