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A NOTE ON A GENERALIZED COHOMOLOGY EQUATION

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Dedicated to the memory of Anzelm Iwanik

Abstract. We give a necessary and sufficient condition for the solvability of a generalized cohomology equation, for an ergodic endomorphism of a probability measure space, in the space of measurable complex functions. This generalizes a result obtained in [7].

Throughout the paper $(X, \mathfrak{M}, \mu)$ will mean a probability measure space and φ will be an endomorphism of it.

All functions and linear spaces will be complex.

$M(\mu)$ will be the space of all $\mathfrak{M}$-measurable functions on X.

$\langle\cdot|\cdot\rangle$ will be the inner product in $L^2(\mu)$ and $\|\cdot\|_2$ be the $L^2(\mu)$-norm.

I_A will be the indicator function of $A \in \mathfrak{M}$.

If $f \in M(\mu), \lambda \in S^1$ and $n \in \mathbb{N}$, then

$$S_n^\lambda(f) = \sum_{k=0}^{n-1} \lambda^{-k} f \circ \varphi^k.$$

The following definition was given in [3].

It is said that $f \in M(\mu)$ is (φ, λ)-*cohomologous* to 0 in $M(\mu)$ if there exists $g \in M(\mu)$, called a (φ, λ)-*coboundary* of f in $M(\mu)$, such that

$$f = \lambda g - g \circ \varphi \quad \mu\text{-a.e.} \tag{1}$$

For applications of these notions, see [3–6]. We will prove a theorem which generalizes Proposition 3 of [7] obtained in the case of an ergodic automorphism and $\lambda = 1$.

THEOREM. *Let φ be ergodic, $f \in M(\mu)$ and $\lambda \in S^1$. Then f is (φ, λ)-cohomologous to* 0 *in $M(\mu)$ if and only if there exists $A \in \mathfrak{M}$ with $\mu(A) > 0$ and $M \in \mathbb{R}_+$ such that*

$$|S_n^\lambda(f)(x)| \le M \tag{$*$}$$

for any $n \in \mathbb{N}$ and μ-a.e. $x \in A$ such that $\varphi^n(x) \in A$.

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To prove the Theorem we need the following

LEMMA. *If* (f_n) *is a* μ*-a.e. pointwise bounded sequence in* $M(\mu)$*, then there exist strictly increasing sequences* $(n_m),(m_k)$ *in* $\mathbb{N}$ *such that the sequence*

$$m_k^{-1}\sum_{i=1}^{m_k} f_{n_i}$$

converges μ*-a.e.*

Proof. Since a closed ball in the Hilbert space of square integrable functions is sequentially weakly compact (cf. Corollary 8, p. 425 and Theorem 1, p. 430 of [2]), by passing to a subsequence if necessary, we may assume that

(2) there exist an increasing sequence (A_l) in $\mathfrak{M}$ with $\mu(\bigcup_{l=1}^\infty A_l)=1$ and $f\in M(\mu)$ such that

$$f_n|A_l\to f|A_l \quad \text{weakly in } L^2(\mu|A_l) \text{ for any } l\in\mathbb{N}.$$

Without loss of generality we may assume that $f=0$. Then we prove that

(3) there exists a subsequence (f_{n_m}) such that

$$g_m=m^{-1}\sum_{i=1}^{m} f_{n_i}\to 0 \quad \text{in } L^2(\mu|A_l) \text{ for any } l\in\mathbb{N}.$$

In the proof we use some ideas from the proof of the Banach–Saks theorem for the Hilbert space of square integrable functions (cf. Théorème I of [1] and Théorème, p. 80 of [8]). By (2), there exists a subsequence (f_{n_m}) such that for any $j\in\mathbb{N}$,

$$|\langle f_{n_i}\mid f_{n_{j+1}}I_{A_l}\rangle|\le j^{-1},$$

where $i,l=1,\ldots,j$. Now let $l\in\mathbb{N}$ and $m\ge l$. Then we have

$$\Big\|m^{-1}\sum_{i=l}^{m} f_{n_i}I_{A_l}\Big\|_2^2\le m^{-2}[(m-l+1)\sup_{n\ge1}\|f_nI_{A_l}\|_2^2+2(m-l)].$$

This proves (3). In view of (3) there exists a subsequence (g_{m_k}) such that

$$\|g_{m_k}I_{A_k}\|_2\le 2^{-k}$$

for $k\in\mathbb{N}$. It is easy to see that $g_{m_k}\to 0$ μ-a.e. This completes the proof of the Lemma.

We now proceed to the proof of the Theorem. The "only if" part is easy and is similar to that of the proof of Proposition 3 of [7]. For the sake of completeness, we give a proof. Let g be a (φ,λ)-coboundary of f in $M(\mu)$. Then there exists $M'\in\mathbb{R}_+$ such that $\mu(B)>0$, where

(4) $$B=\{x\in X:|g(x)|\le M'\}.$$

From (1) it follows that

$$S_n^\lambda(f)(x) = \lambda g(x) - \lambda^{n-1} g(\varphi^n(x))$$

for $x \in X \setminus B_0$ and $n \in \mathbb{N}$, where $B_0 \in \mathfrak{M}$, $\mu(B_0) = 0$. This yields $(*)$ for $A = B \setminus B_0$ and $M = 2M'$. For the proof of the "if" part notice that, by the ergodicity of φ, there exists $B \in \mathfrak{M}$ with $\mu(B) = 1$ such that if $x \in B$, then $\varphi^r(x) \in A$ for infinitely many $r \in \mathbb{N}$. Let $(r_n(x))$ be the strictly increasing sequence of all those r. We now show that

(5) for any $x \in B$ the sequence $(S_{r_n(x)}^\lambda(f)(x))$ is bounded and

(6) for any $n \in \mathbb{N}$ the function $S_{r_n(\cdot)}^\lambda(f)(\cdot)$ is $\mathfrak{M}$-measurable on B.

For the proof of (5) let $x \in B$ and $n \in \mathbb{N}$, $n > 1$. Since

$$S_{r_n(x)}^\lambda(f)(x) = S_{r_1(x)}^\lambda(f)(x) + \lambda^{-r_1(x)} S_{r_n(x)-r_1(x)}^\lambda(f)(\varphi^{r_1}(x))$$

and $\varphi^{r_1(x)}(x), \varphi^{r_n(x)-r_1(x)}(\varphi^{r_1(x)}(x)) \in A$, we have

$$|S_{r_n(x)}^\lambda(f)(x)| \le |S_{r_1(x)}^\lambda(f)(x)| + M.$$

This completes the proof of (5).

To prove (6) we first show that for any strictly increasing finite sequence $(l_i)_{i=1,\ldots,m}$ in $\mathbb{N}$,

(7) $B_{l_1,\ldots,l_m} = \{x \in B : r_i(x) = l_i,\ i = 1, \ldots, m\}$ is $\mathfrak{M}$-measurable.

This follows by induction from the following equalities:

$$B_{l_1} = \varphi^{-l_1}(A) \cap B \setminus \bigcup_{l=1}^{l_1-1} B_l,$$

$$B_{l_1,\ldots,l_m,l_{m+1}} = \varphi^{-l_{m+1}}(A) \cap B_{l_1,\ldots,l_m} \setminus \bigcup_{l=l_m+1}^{l_{m+1}-1} B_{l_1,\ldots,l_m,l}.$$

We claim that

$$\{x \in B : r_n(x) = l\} \in \mathfrak{M} \tag{8}$$

for $n, l \in \mathbb{N}$. This follows from (7), since the left-hand side of (8) is equal to

$$\bigcup_{l_1 < \ldots < l_{n-1} < l} B_{l_1,\ldots,l_{n-1},l}$$

if $n > 1$. The condition (8) yields (6).

Now define (f_n) as follows:

$$f_n(x) = \begin{cases} n^{-1} \sum_{i=1}^n \widehat{S}_{r_i(x)}^\lambda(f)(x) & \text{for } x \in B, \\ 0 & \text{for } x \notin B, \end{cases}$$

where $\widehat{S}_n^\lambda(f) = \lambda^{-1} S_n^\lambda(f)$. By (5) and (6), the sequence (f_n) satisfies the assumptions of the Lemma. Let $(n_m), (m_k)$ be the sequences from the Lemma applied to (f_n). Put

$$g_k = h_{m_k} \quad \text{for } k \in \mathbb{N},$$

where

$$h_m = m^{-1} \sum_{i=1}^{m} f_{n_i} \quad \text{for } m \in \mathbb{N}.$$

Then (g_k) converges μ-a.e. to, say, $g \in M(\mu)$. We now prove that g is a (φ, λ)-coboundary of f in $M(\mu)$. In the proof we use the equality

$$\lambda \widehat{S}_m^\lambda(f)(x) - \widehat{S}_{m-1}^\lambda(f)(\varphi(x)) = f(x) \tag{9}$$

for $x \in X$ and $m > 1$. Notice that there exists $B_0 \subset B$, $B_0 \in \mathfrak{M}$ with $\mu(B_0) = 1$ such that $\varphi(B_0) \subset B_0$ and

$$g_k(x) \to g(x) \quad \text{for } x \in B_0. \tag{10}$$

We will show that for $x \in B_0$,

$$f(x) = \lambda g(x) - g(\varphi(x)). \tag{11}$$

First consider the case

$$r_1(x) = 1. \tag{12}$$

We then have

$$r_i(\varphi(x)) = r_{i+1}(x) - 1 \quad \text{for } i \in \mathbb{N}. \tag{13}$$

From (9) and (13) it follows that for $n \in \mathbb{N}$,

$$\lambda \sum_{i=2}^{n} \widehat{S}_{r_i(x)}^\lambda(f)(x) - \sum_{i=1}^{n-1} \widehat{S}_{r_i(\varphi(x))}^\lambda(f)(\varphi(x)) = (n-1)f(x).$$

This yields

$$\lambda f_n(x) - f_n(\varphi(x)) = n^{-1}(n-1)f(x) + \lambda n^{-1} f(x) - n^{-1} \widehat{S}_{r_n(\varphi(x))}^\lambda(f)(\varphi(x))$$

for $n \in \mathbb{N}$, which, in turn, gives

$$\lambda h_m(x) - h_m(\varphi(x)) = m^{-1} \sum_{i=1}^{m} a_i(x) \tag{14}$$

for $m \in \mathbb{N}$, where

$$a_i(x) = n_i^{-1}(n_i - 1)f(x) + \lambda n_i^{-1} f(x) - n_i^{-1} \widehat{S}_{r_{n_i}(\varphi(x))}^\lambda(f)(\varphi(x))$$

for $x \in B_0$ and $i \in \mathbb{N}$. From (14) it follows that

$$\lambda g_k(x) - g_k(\varphi(x)) = m_k^{-1} \sum_{i=1}^{m_k} a_i(x)$$

for $k \in \mathbb{N}$. From this, (5) and (10) we obtain (11) under the assumption (12).

Assume now that (12) is not satisfied. Then

$$r_i(\varphi(x)) = r_i(x) - 1 \quad \text{for } i \in \mathbb{N}.$$

This and (9) give, for $n \in \mathbb{N}$,

$$\lambda \sum_{i=1}^{n} \widehat{S}^{\lambda}_{r_i(x)}(f)(x) - \lambda \sum_{i=1}^{n} \widehat{S}^{\lambda}_{r_i(\varphi(x))}(f)(\varphi(x)) = nf(x).$$

Hence

$$\lambda f_n(x) - f_n(\varphi(x)) = f(x),$$

which, in turn, yields

$$\lambda g_k(x) - g_k(\varphi(x)) = f(x)$$

for $k \in \mathbb{N}$. Together with (10) this implies (11). Thus the proof of the Theorem is complete.

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