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STRETCHING THE OXTOBY–ULAM THEOREM

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Abstract. On a manifold X of dimension at least two, let μ be a nonatomic measure of full support with $\mu(\partial X) = 0$. The Oxtoby–Ulam Theorem says that ergodicity of μ is a residual property in the group of homeomorphisms which preserve μ. Daalderop and Fokkink have recently shown that density of periodic points is residual as well. We provide a proof of their result which replaces the dependence upon the Annulus Theorem by a direct construction which assures topologically robust periodic points.

Introduction. The classical Oxtoby–Ulam Theorem [10] says that for a general class of measures (the OU measures) on a compact, connected manifold of dimension at least two the ergodic homeomorphisms form a dense G_δ subset of the completely metrizable group of all homeomorphisms preserving the measure. A beautiful exposition of the theorem together with a number of generalizations can be found in Alpern and Prasad's forthcoming book [2]. Using their language we will call a nonzero, finite, Borel measure μ on a compact manifold X an *Oxtoby–Ulam*, or OU, *measure* if it is nonatomic, of full support and is zero on the boundary.

A key step in the proof is the Homeomorphic Measures Theorem which says that if μ and ν are OU measures on a topological ball B such that $\mu(B) = \nu(B)$ then there exists a homeomorphism h on B which restricts to the identity on the boundary sphere ∂B and which maps μ to ν, i.e. $h_*\mu = \nu$ ([10], Theorem 2; see also [2], Appendix 2).

Recently Daalderop and Fokkink [4] have shown that the condition of dense periodic points is residual as well. Since an ergodic homeomorphism for an OU measure is topologically transitive, there exist, for every positive ε, orbits which are ε-dense in X. Such an orbit is easily perturbed to obtain a closed orbit. The difficulty is to obtain closed orbits which persist under further perturbation. To obtain them Daalderop and Fokkink use strong theorems from algebraic topology including the Annulus Theorem of Kirby and Freedman. It is our purpose here to provide a simple direct construction. By following the original argument of Oxtoby [7] (see also [9]), it requires only a little additional work to provide a relatively self-contained

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proof (modulo the Homeomorphic Measures Theorem) of the genericity of topological transitivity as well. The stronger ergodicity results require much more work.

1. Transitive systems on manifolds. A dynamical system (X, f), where f is a homeomorphism on a compact metric space X, is called *topologically transitive*, or just *transitive*, if for some x in X the forward orbit $\{f(x), f^2(x), \ldots\}$ is dense in X. Equivalently, the nonwandering relation

$$\mathcal{N}f = \overline{\bigcup_{n=1}^{\infty} f^n} \subset X \times X \tag{1.1}$$

is all of $X \times X$ (see [1], Theorem 4.12). It is sufficient that for every $\varepsilon > 0$ there exists a point with an ε-dense forward orbit. The system is called *central* when every point is nonwandering, i.e. when the diagonal 1_X is contained in $\mathcal{N}f$. It is *totally transitive* when (X, f^k) is transitive for every positive integer k. Finally, the system is *weak mixing* when the product system $(X \times X, f \times f)$ is transitive. Any weak mixing system is totally transitive and the converse is true when the periodic points are dense (see [3], Theorem 1.1). A point x is *periodic* for (X, f) if $f^p(x) = x$ for some positive integer p.

Following the notation of [1] we denote by $\mathcal{C}f$ the *chain relation* of f: $(x, y) \in \mathcal{C}f$ iff for every $\varepsilon > 0$, there is an ε-chain from x to y, that is, a sequence $\{x_0, \ldots, x_k\}$ with $x_0 = x$, $k > 0$ and $x_k = y$ such that $d(f(x_{i-1}), x_i) \le \varepsilon$ for $i = 1, \ldots, k$. The system (X, f) is called *chain transitive* when $\mathcal{C}f = X \times X$.

By a *measure* μ on a compact metric space X we mean a nonzero, finite, Borel measure. The measure has *full support* if $\mu(U) > 0$ for every nonempty open subset U of X. If $\mu(\{x\}) = 0$ for every point $x \in X$ then μ is called *nonatomic*. If $f_*\mu = \mu$ for a homeomorphism f on X then we say that f *preserves* μ or μ is *invariant* for f. Such an invariant measure μ is called *ergodic* if for every Borel subset A of X, $f(A) = A$ implies $\mu(A)$ equals 0 or $\mu(X)$.

1.1. PROPOSITION. *Let f be a homeomorphism on a compact, connected, metric space X. If f preserves a measure μ of full support then the system (X, f) is central and chain transitive. If, in addition, μ is ergodic then (X, f) is transitive.*

Proof. We briefly review the proofs of these largely well known results.

That the system is central is Poincaré's Recurrence Theorem. This observes that if A is a wandering Borel set, i.e. the sequence $\{f^k(A) : k = 0, 1, \ldots\}$ is pairwise disjoint, then $\mu(A) = 0$ for any invariant measure μ. So if μ has full support, every nonempty open subset is nonwandering.

For any connected space X, the trivial system $(X, 1_X)$ is chain transitive (see [1], Exercise 1.9b). Now we apply [1], Proposition 1.11(c), which says that the operator $\mathcal{C}$ on closed relations is idempotent. Hence

$$X \times X = \mathcal{C}(1_X) \subset \mathcal{C}(\mathcal{N}f) \subset \mathcal{C}(\mathcal{C}f) = \mathcal{C}f. \tag{1.2}$$

Thus, (X, f) is chain transitive.

If, in addition, μ is ergodic then any invariant nonempty open subset U of X satisfies $\mu(U) = \mu(X)$ and so U is dense. This implies (X, f) is transitive, because it is central. ■

For a *manifold* X, i.e. a finite-dimensional topological manifold equipped with a metric d, we denote the boundary by ∂X and its complement, $X \backslash \partial X$, by $\operatorname{Int} X$. For a topological ball B, ∂B is the boundary sphere and $\operatorname{Int} B$ is the interior open ball. By a *ball* in a manifold we will mean a closed, codimension zero topological ball. If B is a ball in a manifold X then by the Invariance of Domain Theorem, $\operatorname{Int} B$ is open in $\operatorname{Int} X$ and so is contained in the topological interior of B with equality when $B \subset \operatorname{Int} X$. An *OU measure* μ on X is a nonatomic measure of full support such that $\mu(\partial X) = 0$. If μ is an OU measure on X and B is a ball in X then $\mu(\operatorname{Int} B) > 0$ and we will call B a *μ-ball* if $\mu(\partial B) = 0$. For any ball B in X we can choose a homeomorphism h from B onto a *rectangular region* R in $\mathbb{R}^n$, i.e. a product of n closed bounded intervals of positive length. Only countably many hyperplanes which are parallel to the coordinate hyperplanes can intersect R in sets of positive measure with respect to $h_*\mu$. So we can shrink R slightly to obtain a rectangular region R' in R with $h_*\mu(\partial R') = 0$. Thus, we can obtain $B' = h^{-1}(R')$ a μ-ball contained in and arbitrarily close to B. By composing h with a dilation we can assume that $\mu(B') = \lambda(B')$ where λ is Lebesgue measure. So by the Homeomorphic Measures Theorem we can take h to be a homeomorphism from the μ-ball B' to a rectangular region R' such that h maps μ on B' to λ on R'.

For a compact manifold X and an OU measure μ we let $H(X)$ denote the completely metrizable group of homeomorphisms equipped with the sup metric and $H_\mu(X)$ denote the closed subgroup of μ-preserving homeomorphisms. Our goal is the following theorem of Daalderop and Fokkink.

1.2. Theorem. *Let μ be an Oxtoby–Ulam measure on a compact, connected manifold X of dimension at least 2. The set*

$$\{f \in H_\mu(X) : (X, f) \text{ is a weak mixing dynamical system with periodic points dense in } X\} \tag{1.3}$$

contains a dense G_δ subset H^* of $H_\mu(X)$. In fact, if $H'_\mu(X) = \{f \in H_\mu(X) : f|\partial X = 1_{\partial X}\}$ then in each coset of the closed subgroup $H'_\mu(X)$ in $H_\mu(X)$ the points of H^* are dense.

Our main tool will be an open condition which assures the occurrence of fixed points. For example, suppose a ball B in a manifold X is *inward* for a homeomorphism f on X, that is, $f(B) \subset \operatorname{Int} B$. Then B is still inward for maps close enough to f and so such maps have fixed points by the Brouwer Fixed Point Theorem. However, a measure preserving map has no proper inward subsets. We consider instead a topological version of hyperbolicity.

Let D be a ball of dimension at least 1. Let I be the interval $[-2, 2]$ with $I_- = [-2, -1]$, $I_0 = [-1, 1]$, $I_+ = [1, 2]$. A continuous function $g : I_0 \times D \to I \times D$ is called a *stretch map* if it satisfies the following conditions:

$$g(I_0 \times D) \subset \operatorname{Int}(I \times D), \qquad g(\{\pm 1\} \times D) \subset \operatorname{Int}(I_\pm \times D). \tag{1.4}$$

If for $(t, z) \in I_0 \times D$ we write

$$g(t, z) = (\tau(t, z), G(t, z)), \tag{1.5}$$

then the conditions (1.4) are equivalent to

$$G(t, z) \in \operatorname{Int} D \quad \text{ for } (t, z) \in I_0 \times D \tag{1.6}$$

and

$$1 < \pm\tau(\pm 1, z) < 2 \quad \text{ for } z \in D. \tag{1.7}$$

The proof of the required fixed point result was suggested to me by my colleague Hironori Onishi.

1.3. Lemma. *Every stretch map has a fixed point.*

Proof. We can assume that D is the unit ball in the Euclidean space $\mathbb{R}^n$.

If g fails to have a fixed point then, for some $\varepsilon > 0$, $d(g(x), x) > \varepsilon$ for all $x \in I_0 \times D$. Then any function close enough to g also fails to have a fixed point. Also, the conditions (1.4) are open conditions on the continuous map g. Thus, by perturbing slightly we can assume that g is smooth. By Sard's Theorem we can further perturb by a translation to reduce to the case where $G : I_0 \times D \to \operatorname{Int} D$ is smooth and 0 is a regular value for $V : I_0 \times D \to \mathbb{R}^n$ defined by

$$V(t, z) = z - G(t, z), \tag{1.8}$$

as well as for the restrictions of V to $\{\pm 1\} \times D$. It then follows that

$$\operatorname{Fix} = \{(t, z) : G(t, z) = z\}, \tag{1.9}$$

the zero-set of V, is a smooth manifold of dimension 1 whose boundary lies in $\partial(I_0 \times D)$. By (1.6) the boundary is in fact in $\{-1, 1\} \times D$. Condition (1.6) also says that for each $t \in I_0$, $z \mapsto V(t, z)$ is an outward pointing vector field on D. Thus, Fix consists of a finite number of smooth loops, contained in $\operatorname{Int}(I_0 \times D)$, and a finite number of arcs which intersect $\{-1, 1\} \times D$ in the endpoints. If for every such arc either both ends were in $\{1\} \times D$ or both in $\{-1\} \times D$ then $\operatorname{Fix} \cap (\{+1\} \times D)$ and $\operatorname{Fix} \cap (\{-1\} \times D)$ would each consist of an even number of points. But the mod 2 version of the Poincaré–Hopf Index Theorem says that an outward pointing, smooth vector field on D with nondegenerate zeros has an odd number of zeros (see [5], Chapter 6). Hence some arc in Fix has endpoints $\{-1\} \times D$ and $\{+1\} \times D$. By (1.7), $\tau(t,z) - t$ is negative on $\{-1\} \times D$ and is positive on $\{+1\} \times D$. Applying the Intermediate Value Theorem to the arc in Fix which spans these sets, we obtain $(t^*, z^*) \in$ Fix such that $\tau(t^*, z^*) = t^*$. That is, (t^*, z^*) is a fixed point for g. ∎

Assume f is a homeomorphism on a manifold X of dimension at least 2. A triple of subsets (B_-, B_0, B_+) with union a ball B in X is called a *stretch* for (X, f) if $f(B_0) \subset B$ and there exists a homeomorphism $h : B \to I \times D$, where $I = [-2, 2]$ and D is a ball so that

$$\begin{gathered} h(B_\pm) = I_\pm \times D, \qquad h(B_0) = I_0 \times D, \\ h \circ f \circ h^{-1} : I_0 \times D \to I \times D \text{ is a stretch map.} \end{gathered} \tag{1.10}$$

Notice that if (B_-, B_0, B_+) is a stretch for f, then, by using the same homeomorphism h, the triple is a stretch for any homeomorphism close enough to f. So Lemma 1.3 implies that each such homeomorphism has a fixed point in B_0.

Now let X and μ be fixed as in the hypotheses of Theorem 1.2. For each positive ε and positive integer k we define the subset $G_{k,\varepsilon}$ of $H_\mu(X)$ as follows: $f \in G_{k,\varepsilon}$ if there exists a positive integer p and a triple (B_-, B_0, B_+) of subsets of X such that

(1.11) (B_-, B_0, B_+) is a stretch for (X, f^{pk}),

(1.12) $\bigcup_{i=0}^{p-1} f^{ik}(B_0)$ is ε-dense in X, i.e. every point in X has d-distance less than ε from some point in the union,

(1.13) for $i = 0, \ldots, p-1$, the d-diameter of $f^{ik}(B_0)$ is less than ε.

In the following section we will prove:

1.4. Claim. *$G_{k,\varepsilon}$ has a dense intersection with each coset of $H'_\mu(X)$ in $H_\mu(X)$.*

Proof of Theorem 1.2. Each $G_{k,\varepsilon}$ is an open subset of $H_\mu(X)$. Intersecting over positive integers k and rationals ε we obtain a G_δ set H^* which by Claim 1.4 and the Baire Category Theorem intersects each coset in a dense subset. If $f \in G_{k,\varepsilon}$ and p, (B_-, B_0, B_+) satisfy (1.11), (1.12) and (1.13) then f^{pk} has a fixed point $x \in B_0$ and by (1.12) and (1.13) the f^k-orbit of x is a 2ε-dense subset of X. Thus, if $f \in H^*$ then for each positive integer k, f^k is transitive with dense periodic points. That f is weak mixing then follows from [3], Theorem 1.1. ∎

2. Perturbation constructions. We begin with a property closely related to that of generalized homogeneity in [1], Exercise 7.40 (see also [10], Lemma 13, [2], Theorem 2.4, and [6]).

2.1. Proposition. *Let X be a compact manifold equipped with a metric d. Assume that the dimension of X is at least 2. For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\{x_1, \ldots, x_m, y_1, \ldots, y_m\}$ is any list of $2m$ distinct points in $\operatorname{Int} X$ satisfying $d(x_i, y_i) < \delta$ for $i = 1, \ldots, m$ then there exists a list $\{B_1, \ldots, B_m\}$ of pairwise disjoint balls contained in $\operatorname{Int} X$, each of d-diameter less than ε and satisfying $\{x_i, y_i\} \subset \operatorname{Int} B_i$ for $i = 1, \ldots, m$.*

Proof. First assume the dimension of X is at least 3. In the piecewise linear (= p.l.) case we connect the pairs $\{x_i, y_i\}$ by arcs and then use general position to make the arcs disjoint. For the general result we will use local linear structures given by charts.

Consider the covering of X by all topological balls of diameter at most ε. Let $\delta > 0$ be the Lebesgue number of the cover by the topological interiors of these balls. So for each pair $\{x_i, y_i\}$ in the list we can choose a topological ball $\widetilde{B}_i$ with $\{x_i, y_i\} \subset \operatorname{Int} \widetilde{B}_i$. By choosing some homeomorphism of $\widetilde{B}_i$ into Euclidean space we can put a linear structure on $\widetilde{B}_i$. Of course, the linear structures induced on the overlap between different balls need not be even p.l. equivalent.

By induction on k for $1 \le i \le k \le m$, find arcs $A_i^k \subset \operatorname{Int} \widetilde{B}_i$ with endpoints $\{x_i, y_i\}$ such that $\{A_1^k, \ldots, A_k^k\}$ are pairwise disjoint. Each A_i^k meets $\operatorname{Int} \widetilde{B}_{k+1}$ in a countable union of disjoint intervals with endpoints in $\partial \widetilde{B}_{k+1} \cup \{x_i, y_i\}$. Let A_{k+1}^{k+1} be the segment in $\operatorname{Int} \widetilde{B}_{k+1}$ connecting x_{k+1} and y_{k+1}. Only finitely many of the $A_i^k \cap \operatorname{Int} \widetilde{B}_{k+1}$ intervals meet A_{k+1}^{k+1}. Change A_i^k on each of these open intervals to obtain an arc with the same endpoints which is p.l. relative to $\widetilde{B}_{k+1}$. Using general position and dimension $\widetilde{B}_{k+1} \ge 3$ we can ensure that the adjusted A_i^k's now labelled A_i^{k+1} are disjoint from A_{k+1}^{k+1} for $i = 1, \ldots, k$. Since the adjustments can be made arbitrarily small we can preserve the containments $A_i^{k+1} \subset \operatorname{Int} \widetilde{B}_i$ for $i = 1, \ldots, k$ and disjointness among $\{A_1^{k+1}, \ldots, A_k^{k+1}\}$. Finally, having obtained pairwise disjoint arcs $\{A_1^m, \ldots, A_m^m\}$ we can make one final adjustment to obtain $\{A_1, \ldots, A_m\}$ pairwise disjoint such that $A_i \subset \operatorname{Int} \widetilde{B}_i$ and $\widetilde{B}_i$ is a p.l. arc with endpoints $\{x_i, y_i\}$. Thicken these up to obtain balls B_i with $A_i \subset \operatorname{Int} B_i$, $B_i \subset \operatorname{Int} \widetilde{B}_i$ with B_i p.l. relative to $\widetilde{B}_i$. With the balls chosen close enough to the arcs we obtain $\{B_1, \ldots, B_m\}$ pairwise disjoint.

For dimension 2 we cannot use the general position argument which let us separate arcs by arbitrarily small changes. Instead we use the fact that a 2-dimensional compact manifold can be triangulated as a p.l. manifold and then adapt the proof of Oxtoby's Gerrymandering Theorem from [8].

Assume K is a 2-dimensional simplicial complex triangulating a manifold. For each simplex σ of K, let $D(\sigma)$ be the *regular neighborhood* of σ, i.e. the union of those simplices of the barycentric subdivision K' of K which meet σ. Then $D(\sigma_1) \cap D(\sigma_2)$ is nonempty iff σ_1 and σ_2 have a common face $\tau = \sigma_1 \cap \sigma_2$, in which case the intersection is $D(\tau)$. Each $D(\sigma)$ is a p.l. disc. Now suppose $\sigma_1 \cap \sigma = \tau_1$ and $\sigma_2 \cap \sigma = \tau_2$ with dimension $\sigma = 2$. There exists an edge τ_3 of σ which is neither τ_1 nor τ_2. Let σ_3 be a simplex of maximum dimension such that $\sigma_3 \cap \sigma = \tau_3$. So $\dim \sigma_3 = 2$ unless τ_3 is a boundary edge, in which case $\sigma_3 = \tau_3$. In either case, the barycenter of σ_3 lies in the boundary of $D(\sigma)$ but not in $\operatorname{Int} D(\sigma_1)$ or $\operatorname{Int} D(\sigma_2)$. Now we can explain the following:

2.2. LEMMA. *Let K be a simplicial complex triangulating a 2-dimensional p.l. manifold X. Let U be an open subset of X, let $F_1, \ldots, F_m$ be disjoint finite sets and let $\sigma_1, \ldots, \sigma_m$ be 2-simplices of K. There exist disjoint p.l. arcs $A_1, \ldots, A_m$ such that $F_i \subset A_i \subset U \cap \operatorname{Int} D(\sigma_i)$ for $i = 1, \ldots, m$ iff for each i, F_i is contained in a single component of $U \cap \operatorname{Int} D(\sigma_i)$.*

Proof. This result is a manifold version of Theorem 1 of [8]. Necessity is obvious and the proof of sufficiency simply mimics Oxtoby's inductive proof with the following adjustments. Because of the special nature of these discs we need not worry about their diameters nor pick out the largest diameter disc upon which to build the induction. The subarc C_i in Oxtoby's proof is defined to be the arc of the boundary circle which is contained in $\operatorname{Int} D(\sigma_j)$ for some j. By the above remarks for any pair of points on $\partial D(\sigma_i)$ at most one of the two arcs connecting the pair is contained in some $\operatorname{Int} D(\sigma_j)$. It is easy to check that $\operatorname{Int} D(\sigma_j) \cap \partial D(\sigma_i)$ is either an open arc or is empty. The details are left to the industrious reader.

To complete the proof of the dimension 2 case we triangulate X and then subdivide to obtain K such that for each $\sigma \in K$, $\operatorname{diam}(D(\sigma)) < \varepsilon$. Let $\delta > 0$ be the Lebesgue number of the open cover consisting of the topological interiors of $D(\sigma)$ as σ varies over the 2-simplices of K. For each pair $\{x_i, y_i\}$ from our list we can choose a 2-simplex σ_i of K such that $\{x_i, y_i\} \subset \operatorname{Int} D(\sigma_i)$, $i = 1, \ldots, m$. Apply Lemma 2.2 with $F_i = \{x_i, y_i\}$ and $U = \operatorname{Int} X$ to obtain the disjoint arcs $\{A_1, \ldots, A_m\}$. Finally, thicken them up to obtain disjoint p.l. balls. ∎

The volume preserving homeomorphisms we will use are obtained by integrating divergence free vector fields on $\mathbb{R}^n$ (with $n \geq 2$). It will be convenient to distinguish the first coordinate, writing $(t, z) = (t, z_1, \ldots, z_{n-1})$ for a typical point so that $t \in \mathbb{R}$ and $z \in \mathbb{R}^{n-1}$. For $r > 0$ define smooth functions $a, b : \mathbb{R} \to \mathbb{R}$ such that

$$a(t) = \begin{cases} 1 & \text{if } |t| \le r, \\ 0 & \text{if } |t| \ge 2r, \end{cases} \tag{2.1}$$

$$b(t) = a(t)t \quad \text{for all } t. \tag{2.2}$$

On $\mathbb{R}^n$ we define the vector field $(\widetilde{T}, \widetilde{Z})$ by

$$\begin{aligned} \widetilde{T}(t,z) &= a(t) \prod_{j=1}^{n-1} b'(z_j), \\ \widetilde{Z}_i(t,z) &= -a'(t)b(z_i) \prod_{j=1,\, j\neq i}^{n-1} b'(z_j)\,/(n-1). \end{aligned} \tag{2.3}$$

The vector field vanishes outside $(I_{2r})^n$ while in $(I_r)^n$, $(\widetilde{T}, \widetilde{Z}) = (1, 0, 0, \ldots, 0)$ where $I_s = [-s, s]$ for $s > 0$. Hence near a point in $\mathrm{Int}(I_r)^n$ the flow is just translation at unit speed until the solution path leaves $(I_r)^n$. By rotating and translating this picture we see that we can move any point to any other within a ball in $\mathbb{R}^n$ by using diffeomorphisms which are the identity on the boundary. Because the vector fields have divergence zero, these diffeomorphisms preserve Lebesgue measure.

Our stretch will be built using the vector field (T, Z):

$$\begin{aligned} T(t,z) &= (n-1)b(t) \prod_{j=1}^{n-1} b'(z_j), \\ Z_i(t,z) &= -b'(t)b(z_i) \prod_{j=1,\, j\neq i}^{n-1} b'(z_j). \end{aligned} \tag{2.4}$$

This vector field, also vanishes outside $(I_{2r})^n$ while for $(t, z) \in (I_r)^n$ we have

$$\begin{aligned} T(t,z) &= (n-1)t, \\ Z_i(t,z) &= -z_i \quad \text{for } i = 1, \ldots, n-1. \end{aligned} \tag{2.5}$$

Let $Q^s : \mathbb{R}^n \to \mathbb{R}^n$ be the time s map of the associated flow. Again, Q^s preserves Lebesgue measure because the vector field (T, Z) has divergence zero. If $|t| \le e^{-(n-1)s}r$ and $|z_i| \le r$ for $i = 1, \ldots, n-1$, then

$$Q^s(t,z) = (e^{(n-1)s}t, e^{-s}z). \tag{2.6}$$

Now we are ready to prove Claim 1.4: Given a positive integer k, positive reals $\varepsilon, \delta > 0$ and $f \in H_\mu(X)$ we will construct $\widetilde{f} \in G_{k,\varepsilon}$ such that $\widetilde{f}|\partial X = f|\partial X$ and $d(f, \widetilde{f}) < \delta$. We are using the sup metric on $H_\mu(X)$ induced from d on X.

First choose $\delta_1 > 0$ so that, as in Proposition 2.1, disjoint pairs of diameter less than δ_1 can be enclosed in disjoint balls of diameter less than $\delta/4$.

By Proposition 1.1, f^k is a chain transitive homeomorphism on X. Choose a finite subset of $\operatorname{Int} X$ which is ε-dense in X. Then choose δ_1-chains for f^k from one point to the next and then back to the first element. Next fill in by f-orbits of length k. Thus, there exists for some positive integer p a sequence $\{x_0, x_1, \ldots, x_{pk}\}$ in $\operatorname{Int} X$ such that

$$d(f(x_{i-1}), x_i) < \delta_1 \quad \text{for } i = 1, \ldots, pk, \tag{2.7}$$

$$x_{pk} = x_0, \tag{2.8}$$

$$\{x_0, x_k, \ldots, x_{(p-1)k}\} \quad \text{is } \varepsilon\text{-dense in } X. \tag{2.9}$$

Since X has no isolated points we can move the points slightly so that $\{x_1, \ldots, x_{pk}\}$ consists of pk distinct points. Since f is injective and $x_0 = x_{pk}$, $\{f(x_0), f(x_1), \ldots, f(x_{pk-1})\}$ consists of pk distinct points as well. Finally, choose $\{y_1, \ldots, y_{pk}\}$ in $\operatorname{Int} X$ so that

(2.10) $d(f(x_{i-1}), y_i), d(y_i, x_i) < \delta_1$ for $i = 1, \ldots, pk$,

(2.11) $\{f(x_0), \ldots, f(x_{pk-1}), y_1, \ldots, y_{pk}\}$, $\{y_1, \ldots, y_{pk}, x_1, \ldots, x_{pk}\}$ each consist of $2pk$ distinct points.

By the choice of δ_1 there exist two lists $\{A_1, \ldots, A_{pk}\}$ and $\{B_1, \ldots, B_{pk}\}$ of pairwise disjoint balls contained in $\operatorname{Int} X$, of diameter less than $\delta/4$ and such that

$$\{f(x_{i-1}), y_i\} \subset \operatorname{Int} A_i, \qquad \{y_i, x_i\} \subset \operatorname{Int} B_i. \tag{2.12}$$

By shrinking slightly we can assume A_i and B_i are μ-balls and so by the Homeomorphic Measures Theorem and the use of vector fields like (2.3) we can construct $q_{1/2}$, $q_1 \in H_\mu(X)$ with $q_{1/2}$ the identity outside $\bigcup_i \operatorname{Int} A_i$ and q_1 the identity outside $\bigcup_i \operatorname{Int} B_i$ and so that in A_i, $q_{1/2}(f(x_{i-1})) = y_i$, and in B_i, $q_1(y_i) = x_i$ for $i = 1, \ldots, pk$. Since the balls have diameter less than $\delta/4$, $d(q_{1/2}, 1_X)$ and $d(q_1, 1_X)$ are both less than $\delta/4$. So $f_1 \equiv q_1 \circ q_{1/2} \circ f \in H_\mu(X)$ satisfies

$$d(f_1, f) < \delta/2, \tag{2.13}$$

$$f_1(x_{i-1}) = x_i, \qquad i = 1, \ldots, pk. \tag{2.14}$$

Thus, x_0 is a periodic point for f_1^k whose f_1^k-orbit is ε-dense in X. Because $\{x_0, \ldots, x_{pk-1}\}$ consists of pk distinct points we can choose B a μ-ball contained in $\operatorname{Int} X$ such that $x_0 \in \operatorname{Int} B$ and

$$B, f_1(B), \ldots, f_1^{pk-1}(B) \quad \text{are disjoint}, \tag{2.15}$$

$$\text{diameter}(B) < \delta/2, \tag{2.16}$$

$$\text{diameter}(f_1^{ik}(B)) < \varepsilon \qquad \text{for } i = 0, \ldots, p-1. \tag{2.17}$$

By the Homeomorphic Measures Theorem, for some $r > 0$ and there exists a homeomorphism of B on $(I_{2r})^n$ which maps μ on B to the Lebesgue measure on the cube. We will regard the homeomorphism as a coordinatization writing $B = (I_{2r})^n = I_{2r} \times D_{2r}$ where for $s > 0$, $D_s = (I_s)^{n-1}$ is a cube in $\mathbb{R}^{n-1}$ where the z-coordinate of the point (t, z) lives. Furthermore, by moving the coordinates x_0 if necessary, we can assume it is at the origin, i.e.

$$x_0 = (0,0) \quad \text{in } I_{2r} \times D_{2r}. \tag{2.18}$$

So we can choose r_1 such that $0 < r_1 < r$ and

$$f_1^{pk}(I_{r_1} \times 0) \subset \mathrm{Int}(I_r \times D_r) \subset B. \tag{2.19}$$

Now define $q_2 \in H_\mu(X)$ to be the identity outside $I_r \times D_r$ with $q_2(x_0) = x_0$ and $q_2(f_1^{pk}(\pm r_1, 0)) = (\pm r_1, 0)$. Define $f_2 = q_2 \circ f_1 \in H_\mu(X)$. Since q_2 is supported by B, (2.15) implies

$$\begin{aligned} f_2^i|B &= f_1^i|B \quad \text{for } i = 1, \ldots, pk-1, \\ f_2^{pk}|B &= q_2 \circ (f_1^{pk}|B). \end{aligned} \tag{2.20}$$

In particular, it follows that

$$f_2^{pk}(0,0) = (0,0) \quad \text{and} \quad f_2^{pk}(\pm r_1, 0) = (\pm r_1, 0) \tag{2.21}$$

and

$$f_2^{pk}(I_{r_1} \times 0) \subset \mathrm{Int}(I_r \times D_r). \tag{2.22}$$

Now we can choose r_2 such that $0 < r_2 < r$ and

$$f_2^{pk}(I_{r_1} \times D_{r_2}) \subset \mathrm{Int}(I_r \times D_r). \tag{2.23}$$

Furthermore, by (2.21) we can choose r_2 small enough for the first coordinate to be positive on $f_2^{pk}(\{r_1\} \times D_{r_2})$ and negative on $f_2^{pk}(\{-r_1\} \times D_{r_2})$. We can then choose r_3 so that $0 < r_3 < r_1$ and

$$\begin{aligned} f_2^{pk}(\{r_1\} \times D_{r_2}) &\subset \mathrm{Int}([r_3, r] \times D_r), \\ f_2^{pk}(\{-r_1\} \times D_{r_2}) &\subset \mathrm{Int}([-r, -r_3] \times D_r). \end{aligned} \tag{2.24}$$

Now we use the flow of the vector field (2.4). Choose s so that

$$e^{(n-1)s} r_3 > r_1, \tag{2.25}$$

$$e^{-s} r < r_2. \tag{2.26}$$

That is, choose s larger than $\ln(r/r_2)$ and $(n-1)^{-1} \ln(r_1/r_3)$. Define r_4 and r_5 by

$$e^{(n-1)s} r_4 = r_1 \quad \text{and} \quad e^{-s} r = r_5, \tag{2.27}$$

so that $0 < r_4 < r_3$ and $0 < r_5 < r_2$.

Because Q^s is the identity outside $I_{2r} \times D_{2r} = B$ in $\mathbb{R}^n$ we can define $\widetilde{q} \in H_\mu(X)$ to be Q^s on B and the identity on $X \backslash B$. Define $\widetilde{f} = \widetilde{q} \circ f_2 \in$

$H_\mu(X)$. As in (2.20) we have

$$\begin{aligned} \widetilde{f}^i|B &= f_2^i|B \quad \text{for } i = 1, \ldots, pk-1, \\ \widetilde{f}^{pk}|B &= Q^s \circ (f_2^{pk}|B). \end{aligned} \tag{2.28}$$

In particular,

$$\widetilde{f}^i(x_0) = x_i \quad \text{for } i = 0, \ldots, pk. \tag{2.29}$$

The stretch for $\widetilde{f}^{pk}$ is defined by

$$\begin{aligned} B_- &= Q^s([-r, -r_4] \times D_r), \\ B_0 &= Q^s(I_{r_4} \times D_r), \\ B_+ &= Q^s([r_4, r] \times D_r). \end{aligned} \tag{2.30}$$

By (2.6) and (2.27), $Q^s(I_{r_4} \times D_r) = I_{r_1} \times D_{r_5}$. Hence, by (2.23), (2.28) and (2.30),

$$\begin{aligned} \widetilde{f}^{pk}(B_0) &= \widetilde{f}^{pk}(Q^s(I_{r_4} \times D_r)) \\ &\subset Q^s f_2^{pk}(I_{r_1} \times D_{r_2}) \subset Q^s(\mathrm{Int}(I_r \times D_r)) \\ &= \mathrm{Int}(B_- \cup B_0 \cup B_+). \end{aligned} \tag{2.31}$$

Because $B_0 \cap B_\pm = Q^s(\{\pm r_4\} \times D_r)$ we similarly show from (2.24) that

$$\widetilde{f}^{pk}(B_0 \cap B_\pm) \subset \mathrm{Int}\, B_\pm. \tag{2.32}$$

Thus, (B_-, B_0, B_+) is a stretch for $\widetilde{f}^{pk}$. By (2.17), (2.20) and (2.28),

$$\mathrm{diameter}(\widetilde{f}^{ik}(B_0)) < \varepsilon \quad \text{for } i = 0, 1, \ldots, p-1.$$

Since $x_{ik} = \widetilde{f}^{ik}(x_0) \in \widetilde{f}^{ik}(B_0)$ it follows that $\bigcup_{i=0}^{p-1} \widetilde{f}^{ik}(B_0)$ is ε-dense. Thus, $\widetilde{f} \in G_{k,\varepsilon}$. Because $\widetilde{q} \circ q_2$ is supported by B, it follows that $d(\widetilde{f}, f_1) < \delta/2$ by (2.16). So by (2.13), $d(\widetilde{f}, f) < \delta$. Finally, $q_{1/2}$, q_1, q_2 and $\widetilde{q}$ are all supported by unions of balls contained in $\mathrm{Int}\, X$. Thus, on ∂X, $\widetilde{f}$ equals f. This completes the construction required by Claim 1.4.

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