DISJOINTNESS OF THE CONVOLUTIONS FOR CHACON’S AUTOMORPHISM

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Abstract. The purpose of this paper is to show that if $\sigma$ is the maximal spectral type of Chacon’s transformation, then for any $d \neq d'$ we have $\sigma^{*d} \perp \sigma^{*d'}$. First, we establish the disjointness of convolutions of the maximal spectral type for the class of dynamical systems that satisfy a certain algebraic condition. Then we show that Chacon’s automorphism belongs to this class.

Let us consider a measure preserving invertible transformation $T$ of the Lebesgue space $(X, \mu)$. We associate with $T$ the unitary operator $\hat{T} : f(x) \mapsto f(Tx)$ on $L^2(X, \mu)$. Let $\sigma$ be the maximal spectral type of $\hat{T}$ restricted to the subspace $H$ of functions with zero mean.

It is an important problem of spectral theory of dynamical systems to investigate properties of convolutions of the maximal spectral type $\sigma$ (see [2], [3] and [6]–[8]). This question originates from Kolmogorov’s well-known problem concerning the group property of the spectrum. It was discovered that for some automorphisms the spectral type $\sigma$ and the convolution $\sigma \ast \sigma$ are mutually singular (see [5]–[8]). An example is the so-called $\kappa$-mixing automorphism, i.e. a transformation $T$ with the following property: there exists a subsequence $k_j$ such that $\hat{T}^{k_j}$ converges weakly to the operator $\kappa\Theta + (1 - \kappa)I$, where $\Theta$ is the orthoprojection onto the subspace of constants and $I$ is the identity operator. This property is known to be generic for measure preserving transformations (see [8]).

Another generic property of automorphisms is the existence of a subsequence $k_j$ such that $\hat{T}^{k_j} \to \frac{1}{2}I \ast \frac{1}{2}T$. This property implies $\sigma \perp \sigma \ast \sigma$ as well. (This fact was established first by Lemańczyk. Parreau extended this observation by showing that $\sigma \perp \sigma^{*d}$ for all $d$. Ryzhikov also obtained the same result and used it for solving Rokhlin’s problem on homogeneous spectrum (see [2]). Ageev deduced this statement as a consequence of his results concerning spectral multiplicity of $T \times T$.)

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It is known that Chacon’s well-known automorphism has the property mentioned above. The following question (raised by del Junco and Lemańczyk [3]) has remained open: are all the $d$-fold convolutions $\sigma^d \ast d$ of the maximal spectral type $\sigma$ pairwise singular for Chacon’s map? In this paper we show that the answer is affirmative. Namely, we establish (Section 2) that the closure of the powers of Chacon’s automorphism contains a sequence of symmetric square polynomials which tends to the operator \((\frac{1}{2}I + \frac{1}{2}\hat{T})^2\), and we show that this condition implies the disjointness of the convolutions.

1. Disjointness of convolutions. Let $\text{Cl}(T)$ be the set of all operators $cK$, where $c$ is a positive number and $K$ belongs to the weak closure of the powers of the operator $\hat{T}$.

**Theorem 1.1.** Let $\sigma$ be the maximal spectral type of a weakly mixing automorphism $T$. Suppose that for some sequence $a_n$ of distinct positive numbers the set $\text{Cl}(T)$ contains the polynomials $Q_n(\hat{T}) = I + a_n\hat{T} + \hat{T}^2$, where $I$ is the identity operator.

Then all the convolutions $\sigma^d \ast d$ are mutually singular.

**Proof.** Let us fix integers $d' > d > 1$ and show that $\sigma^d \ast d \perp \sigma^{d'} \ast d'$. Suppose that an operator $J : H^{\otimes d} \to H^{\otimes d'}$ satisfies

$$J \hat{T} \otimes \ldots \otimes \hat{T}^{d} = \hat{T} \otimes \ldots \otimes \hat{T}^{d'} J,$$

where $H$ is the subspace in $L^2(X, \mu)$ of functions with zero mean. It is enough to prove that $J = 0$. Indeed, it is evident that $\sigma^{d'}$ is the spectral type of the operator $\hat{T}^{\otimes d}$ restricted to the subspace $H^{\otimes d}$. Suppose that $\sigma^{d'} \not\perp \sigma^{d'}$. Then there are two cyclic subspaces $C_1 \subset H^{\otimes d}$ and $C_2 \subset H^{\otimes d'}$ with the same spectral measure. Let $J$ be an operator establishing a unitary equivalence between the restriction of $\hat{T}^{\otimes d}$ to $C_1$ and the restriction of $\hat{T}^{\otimes d'}$ to $C_2$ which is zero on $C_1^\perp$. Then, evidently, $J\hat{T}^{\otimes d} = \hat{T}^{\otimes d'} J$ and $J \neq 0$.

For any $K \in \text{Cl}(T)$ we have $JK^{\otimes d} = \gamma(K)K^{\otimes d'} J$, where $\gamma(K)$ is a positive constant that depends on $K$. In particular, for $K = Q_n(\hat{T})$,

$$J(I + a_n\hat{T} + \hat{T}^2)^{\otimes d} = \gamma_n (I + a_n\hat{T} + \hat{T}^2)^{\otimes d'} J, \quad \gamma_n = \frac{1}{2 + a_n}d^d - d'.
$$

The left part of this equation can be represented in the form $J \sum a_k W^{(d)}_k$, where

$$W^{(d)}_k = \sum_{r_k \in \{-1, 0, 1\}, \sum |r_k| = d - i} \hat{T}^{1 + r_1} \otimes \ldots \otimes \hat{T}^{1 + r_d}.
$$

Since the dimension of the space spanned by $W^{(d)}_k$ is not greater than $d + 1$,
there exists a non-trivial sequence of reals \( c_i \) such that
\[
J \sum_{n=1}^{d+2} c_n Q_n(\hat{T})^\otimes d = 0.
\]

This implies that
\[
\sum_{n=1}^{d+2} \gamma_n c_n Q_n(\hat{T})^\otimes d J = 0.
\]

We will show that the operators \( W_i(d') J \) are linearly independent. It will follow that the operators \( Q_n(\hat{T})^\otimes d J, 1 \leq n \leq k \), are linearly independent if and only if \( k \leq d' + 1 \). (This follows directly from the representation \( Q_n(\hat{T})^\otimes d' = \sum_{i=0}^{d'} a_n^i W_i(d') \) and the fact that the \( a_n \) are distinct.) Thus, the linear combination above cannot be zero because \( d + 2 = (d + 1) + 1 \leq d' + 1 \) (recall that \( d < d' \)). This contradiction completes the proof.

The only thing we must show is that the \( W_i(d') J \) are linearly independent. Indeed, any non-trivial linear combination \( \sum_i c_i W_i(d') J \) has the form \( V(\hat{T}, \ldots, \hat{T}) J = 0 \), where \( V \) is some non-trivial polynomial of \( d' \) variables. If \( J \neq 0 \), then there exists a function \( f \) such that \( Jf \neq 0 \). Let us pass to the spectral representation of \( \hat{T} \). Namely, set
\[
U : L^2(T, \sigma) \to L^2(T, \sigma) : \phi(z) \mapsto z \phi(z)
\]
and let \( \Phi : L^2(X, \mu) \to L^2(T, \sigma) \) be the unitary operator that conjugates \( \hat{T} \) and \( U \) : \( \Phi \hat{T} U = U \Phi \).

Then for the function \( F = \Phi \otimes d' Jf \) on \( T^d' \) we have
\[
0 = \Phi \otimes d' V(\hat{T}, \ldots, \hat{T}) Jf = V(z_1, \ldots, z_{d'}) F.
\]
Thus, \( F \) is supported on the manifold \( \mathcal{N} = \{ V(z_1, \ldots, z_{d'}) = 0 \} \). It is not hard to prove that, since \( V \) is a polynomial, we have \( \sigma \otimes d'(\mathcal{N}) = 0 \). Indeed, suppose, for simplicity, that \( d' = 2 \). Then there are finitely many points \( z_1^{(j)} \) such that \( \mathcal{N} \cap (\{ z_1^{(j)} \} \times T) \) is not finite. It is known that a transformation is weakly mixing iff it has continuous spectrum. Hence, \( (\sigma \times \sigma)(\mathcal{N}) = 0 \), because \( \hat{T} \) is weakly mixing. Thus, \( Jf = 0 \) and \( J \) must be zero; but \( J \neq 0 \), and we have proved that the \( W_i(d') \) are linearly independent.

2. Chacon’s automorphism. Let \( h_1 = 1 \) and \( h_{j+1} = 3h_j + 1 \) be the sequence of heights. Note that \( h_j = (3^j - 1)/2 \). Chacon’s automorphism \( T \) is the rank-1 transformation that is built via a cutting-and-stacking construction described below (see [4] and [1]). At the \( j \)th stage we cut a tower of height \( h_j \) into 3 equal subtowers, add one spacer to the top of the middle subtower and stack these towers together.
Our purpose is to prove the following

**Theorem 2.1.** Let $\sigma$ be the maximal spectral type of Chacon's automorphism. Then for any $d \neq d'$ we have $\sigma^d \perp \sigma^{d'}$.

This theorem is a direct corollary of Theorem 1.1 and Lemma 2.3.

We begin with a definition of Chacon's map which will be more convenient in what follows. Namely, for each $j \geq 1$, we may consider $T$ as an integral automorphism over the 3-adic rotation, by identifying the base $B_j$ of the $j$th tower with the group $\mathbb{Z}_3$ of 3-adic integers in the following way. $\mathbb{Z}_3$ may be considered as the set of all sequences $a_1a_2\ldots$, where $a_k \in \{0, 1, 2\}$. Consider a point $x \in B_j$. When cutting the $j$th tower into 3 subtowers we get a partition $B_j = B_{j, 0} \sqcup B_{j, 1} \sqcup B_{j, 2}$ such that

$$B_{j, 0} \overset{T_{h_j}}{\longrightarrow} B_{j, 1} \overset{T_{h_j+1}}{\longrightarrow} B_{j, 2} \ldots \overset{T}{\longrightarrow} B_{j, 0}.$$ 

Suppose that $x \in B_{j, 0} \simeq [0, 1]$. We associate with $x$ its ternary decomposition $a_1a_2a_3\ldots$ (A more geometric way is to put $a_1 = a$ if $x \in B_{j, a}$, and to define $a_2a_3\ldots$ similarly considering $x - a/3 \in B_{j, 0} = B_{j+1}$ instead of $x$.) Then $T$ can be viewed as the integral automorphism over the map

$$R : \mathbb{Z}_3 \to \mathbb{Z}_3 : a_1a_2a_3\ldots \mapsto a_1a_2a_3 + 100\ldots$$

with the ceiling function $h_j + \phi$, where

$$\phi(a) = \begin{cases} 
0 & \text{if } a = 22\ldots20*, \\
1 & \text{if } a = 22\ldots21*, 
\end{cases}$$

where $*$ designates an arbitrary element of $\{0, 1, 2\}$. (Note that the conditional measure $\mu(\cdot | B_j)$ coincides after identification with the Haar measure $\lambda$ on $\mathbb{Z}_3$.)
It is convenient to redefine the function $\phi$ so that $\phi(a) = 0$ if $a = 00\ldots01\ldots$. The new system is conjugate to Chacon’s automorphism. Let us describe precisely the sets where $\phi$ is constant:

$$\phi(a) = \begin{cases} 0 & \text{if } a \in (0)1\ldots, \\ 1 & \text{if } a \in (0)2\ldots, \end{cases}$$

where $(0)1\ldots$ and $(0)2\ldots$ abbreviate the following two sets:

$(0)1\ldots: 1\star$  $(0)2\ldots: 2\star$

$01\star$  $02\star$

$001\star$  $002\star$

$0001\star$  $0002\star$

$\ldots$  $\ldots$

Each of these two tables should be meant as a code of a partition of some set in $\mathbb{Z}_3$. A row of a table designates an element of a partition, for example, $01\star$ is the set of sequences $a_1a_2\ldots$ such that $a_1 = 0$ and $a_2 = 1$. Here $\star$ means an arbitrary element of $\{0, 1, 2\}$ (more exactly, we assume that any symbol can appear at this position), and a $\star$ at the end of a line abbreviates $\star\star\ldots\star$.

It is a simple corollary from the definition of Chacon’s transformation that

$$\hat{T}^{-h_j} \xrightarrow{w} \lambda((0)1\star)\mathbb{I} + \lambda((0)2\star)\hat{T} = \frac{1}{2}\mathbb{I} + \frac{1}{2}\hat{T},$$

where $\lambda$ is the Haar measure on $\mathbb{Z}_3$, and $\mathbb{I}$ is the identity operator. Indeed, fix measurable sets $A$ and $C$. Since Chacon’s map is a rank-1 transformation, for any $\varepsilon > 0$ there exists $j_0$ such that for all $j \geq j_0$ we have $\mu(A \Delta A_j) < \varepsilon$ and $\mu(C \Delta C_j) < \varepsilon$, where $A_j$ and $C_j$ are the unions of levels of the $j$th tower. Then the base $B_j$ can be uniquely divided into sets $B^{(0)}_j$ and $B^{(1)}_j$ so that for any level $L = T^kB_j$ except one, the set $T^hL$ has the form $L^{(0)} \sqcup T^{-1}L^{(1)}$, where $L = L^{(0)} \sqcup L^{(1)}$ and $L^{(\alpha)} = T^kB^{(\alpha)}_j$. Moreover, $\mu(B^{(0)}_j|B_j) = \lambda((0)1\star) = \mu(B^{(1)}_j|B_j) = \lambda((0)2\star) = 1/2$. It follows directly from this picture that

$$\mu(T^hA_j \cap C_j) \approx \frac{1}{2}\mu(A_j \cap C_j) + \frac{1}{2}\mu(T^{-1}A_j \cap C_j)$$

with precision $1/h_j$. Taking into account the fact that $A_j$ and $C_j$ approximate $A$ and $C$ respectively we get the desired convergence

$$\hat{T}^{-h_j} = (\hat{T}^{h_j})^* \xrightarrow{w} \frac{1}{2}\mathbb{I} + \frac{1}{2}(\hat{T}^{-1})^* = \frac{1}{2}\mathbb{I} + \frac{1}{2}\hat{T}.$$  

It is also not hard to check using the same technique that

$$\hat{T}^{-kh_j} \xrightarrow{w} P_k(\hat{T}) = \int_{\mathbb{Z}_3} \hat{T}^{\phi^{(h_j)}(a)} d\lambda(a) = \sum_{t=0}^{k} c_{k,t}\hat{T}^t,$$

where $(0)1\ldots$ and $(0)2\ldots$ abbreviate the following two sets:

$(0)1\ldots: 1\star$  $(0)2\ldots: 2\star$

$01\star$  $02\star$

$001\star$  $002\star$

$0001\star$  $0002\star$

$\ldots$  $\ldots$
where

$$\phi^{(k)}(a) = \sum_{t=0}^{k-1} \phi(R^{-t}a).$$

(Here we have used the fact that \( \lambda \) is invariant under \( R \).) Note that \( P_k(T) \) is a polynomial in \( T \). Let \( \tilde{P}_k(T) = T^{-r_k} P_k(T) \), where \( r_k \) is the smallest power of \( T \) in \( P_k(T) \). Evidently, \( \tilde{P}_k(T) \in \text{Cl}(T) \) as well. Below several polynomials \( \tilde{P}_k(T) \) are given \(^{(1)}\):

\[
\begin{align*}
\tilde{P}_1(T) &= \frac{1}{2} T + \frac{1}{2}, \\
\tilde{P}_2(T) &= \frac{1}{6} T^2 + \frac{1}{6} T + \frac{1}{6}, \\
\tilde{P}_3(T) &= \frac{1}{18} T^3 + \frac{8}{18} T^2 + \frac{8}{18} T + \frac{1}{18}.
\end{align*}
\]

One can notice that all the polynomials \( \tilde{P}_k \) coincide with \( P_k \) (Lemma 2.2). The deeper Lemma 2.3 proves the following observation: the polynomials \( \tilde{P}_{3n+1} \) are symmetric square polynomials that tend to \( P_k^2 \) as \( n \to \infty \).

**Lemma 2.2.** Let \( l_n = (3^n - 1)/2 \). Then

\[
\phi^{(n)}(a) = \begin{cases} 
  l_n & \text{if } a \in \ast^n(0)1^*, \\
  l_n + 1 & \text{if } a \in \ast^n(0)2^*,
\end{cases}
\]

where \( \ast^n = \ast \ldots \ast \)

and \( P_{3n}(T) = \frac{1}{7} T^{l_n} + \frac{1}{7} T^{l_n+1} \).

**Proof.** This lemma is proved by induction on \( n \). The case \( n = 0 \) is trivial. We will establish the lemma for \( n = 1 \). The proof for arbitrary \( n \) is completely analogous. Consider three translations of the function \( \phi \):

<table>
<thead>
<tr>
<th>( t = 0 )</th>
<th>( t = 1 )</th>
<th>( t = 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0^* )</td>
<td>( 1^* )</td>
<td>( 2^* )</td>
</tr>
<tr>
<td>( 001^* )</td>
<td>( 11^* )</td>
<td>( 21^* )</td>
</tr>
<tr>
<td>( 02^* )</td>
<td>( 12^* )</td>
<td>( 22^* )</td>
</tr>
<tr>
<td>( 002^* )</td>
<td>( 102^* )</td>
<td>( 202^* )</td>
</tr>
</tbody>
</table>

Let \( A_k^\nu \) be the set on which \( \phi(R^{-t}a) = v \). Fixing \( v_0, v_1, v_2 \) we calculate \( A_k^0 \cap A_k^1 \cap A_k^2 \). It can be easily checked that it is non-empty only when \( v_1 + v_2 + v_3 \) is either 1 or 2. Suppose that \( v_0 = v_1 = 0 \) and \( v_2 = 1 \). Then the only non-trivial intersection is \( 1^* \cap 1(0)1^* \cap 1^* = 1(0)1^* \). Moreover, in all similar chains sets are ordered. In the intersection considered we have \( 1^* \subset 11^*, 101^*, \ldots \). So, any intersection is uniquely described by the longer code, e.g., \( 1(0)1^* \). All intersections in our case are represented in the

\(^{(1)}\) See www.geocities.com/apri7 for the first 122 polynomials \( P_k(z) \).
following table:

<table>
<thead>
<tr>
<th></th>
<th>0, 0, 1</th>
<th>1, 1, 0</th>
<th>0, 1, 0</th>
<th>1, 0, 0</th>
<th>0, 1, 1</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1(0)1*</td>
<td>0(0)2*</td>
<td>0(0)1*</td>
<td>2(0)1*</td>
<td>1(0)2*</td>
</tr>
<tr>
<td></td>
<td>0, 1, 0</td>
<td>1, 0, 1</td>
<td>2, 1, 0</td>
<td>0, 1, 1</td>
<td>1, 0, 0</td>
</tr>
</tbody>
</table>

∪: *(0)1*, ∪: *(0)2*.

It is evident that \( \phi^{(3)}(a) = 1 \) iff \( a \in *(0)1* \).

**Lemma 2.3.** \( \hat{T}^{-l_n} P_{3^n+1}(\hat{T}) \) are square polynomials,

\[
\hat{T}^{-l_n} P_{3^n+1}(\hat{T}) = \frac{(3^{n+1} - 1) + 2(3^{n+1} + 1)\hat{T} + (3^{n+1} - 1)\hat{T}^2}{4 \cdot 3^{n+1}} \rightarrow \left(\frac{1}{2} + \frac{1}{2} \hat{T}\right)^2, \quad n \rightarrow \infty.
\]

**Proof.** First, note that

\[
\phi^{(3^n+1)}(a) = \phi^{(3^n)}(a) + \phi(R^{-3^n} a).
\]

Since both \( \phi^{(3^n)} \) and \( \phi \circ R^{-3^n} \) take two values, these functions are uniquely described by the two corresponding partitions (see the discussion above). Let us see how these partitions look (Figs. 2 and 3).

![Fig. 2. Partitions for \( \phi^{(3^n)} \)](attachment:fig2.png)

![Fig. 3. Partitions for \( \phi \circ R^{-3^n} \)](attachment:fig3.png)

Suppose that \( \phi^{(3^n)} - l_n \) and \( \phi \circ R^{-3^n} \) equal \( v \) on the sets \( C_v \) and \( A_v \) respectively. It can be easily seen from Figures 2 and 3 that
\[ C_0 \cap A_0 = \bigcup_{p=0}^{n-1} 0^p1^{n-1-p}(0)1^* \cup 0^n1(0)1^*, \]
\[ C_1 \cap A_1 = \bigcup_{p=0}^{n-1} 0^p2^{n-1-p}(0)2^* \cup 0^n0(0)2^*, \]
and that
\[ \lambda(C_0 \cap A_0) = \lambda(C_1 \cap A_1) = \frac{1}{2} \sum_{p=0}^{n-1} \frac{1}{3^{p+1}} + \frac{1}{2} \cdot \frac{3^{n+1} - 1}{3^{n+1}} \to \frac{1}{4} \]
as \( n \to \infty \). To complete the proof we only have to recall that if \( P_k(z) = \sum_{t=0}^k c_{k,t} z^t \), then \( \sum_{t=0}^k c_{k,t} = 1 \).

Proof of Theorem 2.1. It is shown in Lemma 2.3 that \( 1 + (2 + \epsilon_n)\hat{T} + \hat{T}^2 \in \text{Cl}(T) \) with distinct \( \epsilon_n \). Thus, Theorem 2.1 follows immediately from Theorem 1.1.

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