C O L L O Q U I U M   M A T H E M A T I C U M

VOL. 84/85 2000 PART 1

TIME WEIGHTED ENTROPIES

BY

JÖRG SCHMELING (BERLIN)

Dedicated to the memory of Anzelm Iwanik

Abstract. For invertible transformations we introduce various notions of topological entropy. For compact invariant sets these notions are all the same and equal the usual topological entropy. We show that for non-invariant sets these notions are different. They can be used to detect the direction in time in which the system evolves to highest complexity.

1. Introduction. The notion of entropy plays a crucial role in the theory of dynamical systems. However, it is usually used in the context of invariant sets or measures and of stationary processes. In this paper we are interested in generalizing the notion of topological entropy to non-invariant sets. This will have applications to non-equilibrium systems.

In the early seventies R. Bowen [2] introduced a notion of topological entropy for non-compact sets. This notion was further studied by Pesin and Pitskel [4] and also applied to non-invariant sets. While it coincides with the usual definition of topological entropy for invariant compact sets it has the disadvantage that it is not symmetric in time—i.e. the entropy into the "future" may differ from those into the "past". Moreover, non-equilibrium systems may have an "inner time" which differs from the time steps used to measure the entropy. Sometimes we measure too slow or too fast. Therefore we propose to measure entropy in different time scales. This leads to the notion of *p-weighted entropies* where the weight is the ratio of the "inner time scale" to "our time scale". Taking into account the physical agreement that a system evolves to maximal complexity we can detect the "inner time" of a system as the maximum of the p-weighted entropies. For example we see that a stable manifold is "directed into the past"—i.e. its time scale is infinitely slower than the measurement—and the unstable manifold evolves according to our measurement.

2000 *Mathematics Subject Classification*: 37A35, 37A60, 37B10.

Key words and phrases: topological entropy, symbolic dynamics.

The author was supported by the Leopoldina-Forderpreis.

In this paper we study some basic properties of the weighted entropies. We also introduce a notion of time independent entropy (*unweighted entropy*). The latter notion has full symmetry in time and should replace Bowen's entropy for non-compact non-invariant sets.

In [6] we show that the notions introduced in this paper are preserved by coding via a finite Markov partition. Hence they can be calculated in the symbolic space associated with an Axiom A basic set.

2. Various definitions of topological entropy. Let X be a finite-dimensional compact metric space, and $f\colon X \to X$ a homeomorphism. The problem with the definition of topological entropy introduced in [2] (see also [4]) is that $h(f|Z)$ and $h(f^{-1}|Z)$ may not coincide for an arbitrary set $Z \subset X$; however, if Z is compact and f-invariant, then $h(f|Z) = h(f^{-1}|Z)$.

We introduce new notions of topological entropy which take into account the "complexity" both in the "future" and in the "past".

For each finite partition $\mathcal{U}$ of X, we denote by $\mathcal{W}_n(\mathcal{U})$ the collection of strings $\mathbf{U} = U_0 \ldots U_n$ of sets $U_0, \ldots, U_n \in \mathcal{U}$. For each $\mathbf{U} \in \mathcal{W}_n(\mathcal{U})$, we call the integer $m(\mathbf{U}) = n$ the *length* of $\mathbf{U}$, and define the cylinder set

$$X(\mathbf{U}) = \{x \in X : f^k x \in U_k \text{ for } k = 0, \ldots, n\}$$

and

$$X^-(\mathbf{U}) = f^{m(\mathbf{U})} X(\mathbf{U}).$$

For every set $Z \subset X$, $0 \le p \le 1$ and every real number α, we set

$$N^p(Z, \alpha, \mathcal{U}) = \lim_{n\to\infty} \inf_{\Gamma_n^p} \sum_{(\mathbf{U},\mathbf{V})\in\Gamma_n^p} \exp[-\alpha m(\mathbf{U}) - \alpha m(\mathbf{V})], \tag{1}$$

where the infimum is taken over all finite or countable collections $\Gamma_n^p \subset \bigcup_{k>n} \mathcal{W}_{[pk]}(\mathcal{U}) \times \mathcal{W}_{[(1-p)k]}(\mathcal{U}) =: \Omega_n^p(\mathcal{U})$ such that

$$\bigcup_{(\mathbf{U},\mathbf{V})\in\Gamma_n^p} X(\mathbf{U}) \cap X^-(\mathbf{V}) \supset Z.$$

We say that the family Γ_n^p *induces a covering* of Z. We note that the sets involved in the definition of the outer measures above have "$100p$ percent" increments from the past and "$100(1-p)$ percent" increments from the future. By a simple modification of the construction of Carathéodory dimension characteristics (see [3]), when α goes from $-\infty$ to $+\infty$, the quantity in (1) jumps from $+\infty$ to 0 at a unique critical value. Hence, we can define the number

$$h^p(f|Z, \mathcal{U}) = \inf\{\alpha : N^p(Z, \alpha, \mathcal{U}) = 0\} = \sup\{\alpha : N^p(Z, \alpha, \mathcal{U}) = \infty\}.$$

One can show that the following limit exists (compare with the proof of Theorem 11.1 in [3]):

$$h^p(f|Z) = \lim_{\operatorname{diam}\mathcal{U}\to 0} h^p(f|Z,\mathcal{U}).$$

We call $h^p(f|Z)$ the *p-weighted topological entropy* of f on the set Z. These notions were first introduced in [6].

Now we will give a version of the notion of topological entropy that chooses the minimal "local complexity" independent of time. This quantity was first introduced in [1]. For $Z \subset X$ and every real number α, we set

$$N^*(Z,\alpha,\mathcal{U}) = \lim_{n\to\infty} \inf_{\Gamma_n^*} \sum_{(\mathbf{U},\mathbf{V})\in\Gamma_n^*} \exp[-\alpha m(\mathbf{U}) - \alpha m(\mathbf{V})], \tag{2}$$

where the infimum is taken over all finite or countable collections $\Gamma_n^* \subset \bigcup_{k+l>n} \mathcal{W}_k(\mathcal{U}) \times \mathcal{W}_l(\mathcal{U}) =: \Omega_n^*(\mathcal{U})$ such that

$$\bigcup_{(\mathbf{U},\mathbf{V})\in\Gamma_n^*} X(\mathbf{U}) \cap X^-(\mathbf{V}) \supset Z.$$

Again when α goes from $-\infty$ to $+\infty$, the quantity in (2) jumps from $+\infty$ to 0 at a unique critical value. Hence, we can define the number

$$h^*(f|Z,\mathcal{U}) = \inf\{\alpha : N(Z,\alpha,\mathcal{U}) = 0\} = \sup\{\alpha : N(Z,\alpha,\mathcal{U}) = +\infty\}$$

and the following limit exists (compare with the proof of Theorem 11.1 in [3]):

$$h^*(f|Z) = \lim_{\operatorname{diam}\mathcal{U}\to 0} h^*(f|Z,\mathcal{U}).$$

We call $h^*(f|Z)$ the *unweighted topological entropy* of f on the set Z.

Remark 1. As for the usual topological entropy we have, for a generating partition $\mathcal{U}$,

$$h^p(f|Z) = h^p(f|Z,\mathcal{U}) \quad \text{and} \quad h^*(f|Z) = h^*(f|Z,\mathcal{U}).$$

Moreover,

$$h^1(f|Z) = h(f|Z) \quad \text{and} \quad h^0(f|Z) = h(f^{-1}|Z)$$

where h denotes the usual topological entropy for non-compact sets introduced by Bowen [2].

We also introduce the box-counting versions of the above definitions. For $Z \subset X$ and $n \in \mathbb{N}$ let $\Lambda_n^p = \mathcal{W}_{[pn]}(\mathcal{U}) \times \mathcal{W}_{[(1-p)n]}(\mathcal{U})$ and $A_n^p(f,Z,\mathcal{U})$ be the minimal number of pairs $(\mathbf{U}_i,\mathbf{V}_i) \in \Lambda_n^p$ such that $\bigcup_i X(\mathbf{U}_i) \cap X^-(\mathbf{V}_i) \supset Z$. Similarly, we define $A_n^*(f,Z,\mathcal{U})$ to be the minimal number of pairs $(\mathbf{U}_i,\mathbf{V}_i) \in \Lambda_n^* = \bigcup_{k+l=n} \mathcal{W}_k(\mathcal{U}) \times \mathcal{W}_l(\mathcal{U})$ such that $Z \subset \bigcup_i X(\mathbf{U}_i) \cap X^-(\mathbf{V}_i)$. The *upper*, respectively *lower*, *p-weighted box-counting entropy* and the *unweighted box-counting entropy* are defined as

$$\overline{h}_B^{\#}(f|Z,\mathcal{U}) := \limsup_{n\to\infty} \frac{\log A_n^{\#}(f,Z,\mathcal{U})}{n}$$

and

$$\underline{h}_B^{\#}(f|Z,\mathcal{U}) := \liminf_{n\to\infty} \frac{\log A_n^{\#}(f,Z,\mathcal{U})}{n}$$

where $\# = p$, $0 \le p \le 1$, or $\# = *$. As in Theorem 11.1 in [3] it can be shown that the following limits exist:

$$\overline{h}_B^{\#}(f|Z) = \lim_{\operatorname{diam}\mathcal{U}\to 0} \overline{h}_B^{\#}(f|Z,\mathcal{U})$$

and

$$\underline{h}_B^{\#}(f|Z) = \lim_{\operatorname{diam}\mathcal{U}\to 0} \underline{h}_B^{\#}(f|Z,\mathcal{U}).$$

Remark 2. It is well-known that for a compact invariant set Z we have $h^{\#}(f|Z) = \underline{h}_B^{\#}(f|Z) = \overline{h}_B^{\#}(f|Z) = h(f|Z)$, $\# \in [0,1]$ or $\# = *$.

Let μ be a (not necessarily f-invariant) probability measure on X. We define its *upper*, respectively *lower*, *p-weighted local entropy* and its *unweighted local entropy* at the point $x \in X$ as

$$\overline{d}_{\mu,\mathcal{U}}^{\#}(x) = \limsup_{n\to\infty} -\frac{\log \mu(X(\mathbf{U}) \cap X^-(\mathbf{V}))}{n}$$

and

$$\underline{d}_{\mu,\mathcal{U}}^{\#}(x) = \liminf_{n\to\infty} -\frac{\log \mu(X(\mathbf{U}) \cap X^-(\mathbf{V}))}{n}$$

where $x \in X(\mathbf{U}) \cap X^-(\mathbf{V})$ and $(\mathbf{U},\mathbf{V}) \in \Lambda_n^{\#}$.

Remark 3. If μ is an ergodic f-invariant measure, then the Shannon–McMillan–Breiman Theorem states that $\overline{d}_{\mu,\mathcal{U}}^{\#}(x) = \underline{d}_{\mu,\mathcal{U}}^{\#}(x) = h_\mu(\mathcal{U})$ for μ-a.e. $x \in X$, where $h_\mu(\mathcal{U})$ is the usual metric entropy of the partition $\mathcal{U}$.

3. Properties of topological entropies. The following basic properties are immediate consequences of the definitions.

Theorem 3.1. *The topological entropies have the following properties*:

(1) *for* $Z_1 \subset Z_2 \subset X$ *we have* $h^{\#}(f|Z_1) \le h^{\#}(f|Z_2)$,
(2) *for* $Z = \bigcup_i Z_i$ *we have* $h^{\#}(f|Z) = \sup_i h^{\#}(f|Z_i)$,
(3) $h^*(f|Z) = h^*(f^{-1}|Z)$,
(4) $h^p(f|Z) = h^{1-p}(f^{-1}|Z)$,
(5) $h^{\#}(f|Z) \le \underline{h}_B^{\#}(f|Z) \le \overline{h}_B^{\#}(f|Z)$,

where $\# = *$ *or* $\# = p$, $0 \le p \le 1$. *Moreover*,

$$h^*(f|Z) \le h^p(f|Z).$$

From now on we assume that $f : X \to X$ has finite topological entropy.

Theorem 3.2. $h^p(f|Z) : [0,1] \to \mathbb{R}$ *is a Lipschitz continuous function in* p.

THEOREM 3.3. *We have*

$$h^{\#}(f|Z) = \sup_{\mathcal{U}} \sup_{\mu} \{\operatorname{ess\,sup} \underline{d}^{\#}_{\mu,\mathcal{U}}(x) : \operatorname{supp} \mu \subset Z\}$$

where the supremum is over all finite partitions.

THEOREM 3.4. *We have*

$$h^*(f|Z) = \inf \Big\{ \max \min_{0 \le p \le 1} h^p(f|Z_i) : Z = \bigcup_{i=1}^{N} Z_i \Big\}.$$

THEOREM 3.5. *We have*

$$h^p(f|Z) \le \min\{p\overline{h}^1_B(f|Z) + (1-p)h^0(f|Z), ph^1(f|Z) + (1-p)\overline{h}^0_B(f|Z)\}.$$

THEOREM 3.6. $h^p(f|Z) \ge \max\{ph^1(f|Z), (1-p)h^0(f|Z)\}$.

COROLLARY 3.7. *If* $\overline{h}^0_B(f|Z) = h^0(f|Z)$ *and* $\overline{h}^1_B(f|Z) = h^1(f|Z)$ *then*

$$h^p(f|Z) = ph^1(f|Z) + (1-p)h^0(f|Z).$$

The general situation can be summarized as in Figure 1. Here $y_1 = \min_{0\le p\le 1} \max\{ph^1, (1-p)h^0\}$, $y_2 = \max_{0\le p\le 1} \min\{p\overline{h}^1_B + (1-p)\overline{h}^0_B, ph^1 + (1-p)\overline{h}^0_B\}$. The graph of h^p is located inside the area bounded by the bold line segments.

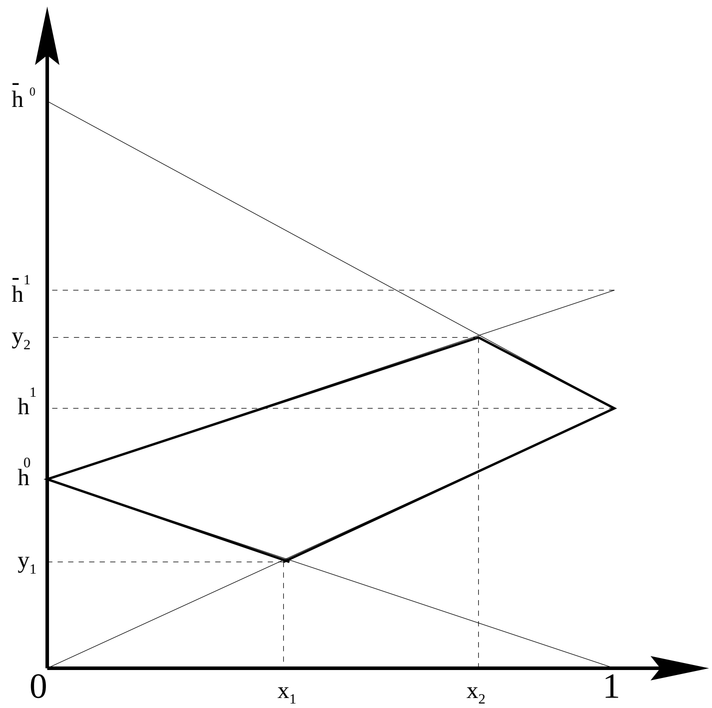

Fig. 1

4. Proofs

4.1. *Proof of Theorem 3.2.* We will need the following lemma.

LEMMA 4.1. *Let* $h_{\mathrm{top}}(f) < \infty$ *and* $\mathcal{U}$ *a finite partition. Then for each* $\varepsilon > 0$ *there is an* $n_0 = n_0(\mathcal{U}, \varepsilon)$ *such that* $\operatorname{card}(\{\mathbf{U} \in \mathcal{W}_n(\mathcal{U}) : X(\mathbf{U}) \neq \emptyset\}) < \exp[(h_{\mathrm{top}}(f) + \varepsilon)n]$ *for all* $n \geq n_0$.

Proof. By Remark 2 we have $\overline{h}_B^0(f|X) = h_{\mathrm{top}}(f) < \infty$. But

$$\begin{aligned}\overline{h}_B^0(f|X) &\geq \overline{h}_B^0(f|X, \mathcal{U})\\ &= \limsup_{n\to\infty} \frac{\log \operatorname{card}\{\mathbf{U} \in \mathcal{W}_n(\mathcal{U}) : X(\mathbf{U}) \neq \emptyset\}}{n}. \quad \blacksquare\end{aligned}$$

Let $Z \subset X$ and $\mathcal{U}$ be fixed. For $0 \leq p < q \leq 1$ we consider $h^p(f|Z)$ and $h^q(f|Z)$. We fix $\varepsilon > 0$. For $[qn] - [pn] - 1 > n_0(\mathcal{U}, \varepsilon)$ let $\Gamma_n^p \subset \Omega_n^p(\mathcal{U})$ with $\bigcup_{(\mathbf{U},\mathbf{V})\in\Gamma_n^p} X(\mathbf{U}) \cap X^-(\mathbf{V}) \supset Z$. For $s \geq t$ we consider the map $\varrho_{st} : \mathcal{W}_s(\mathcal{U}) \to \mathcal{W}_t(\mathcal{U})$ mapping the word $\mathbf{W} = W_0 \ldots W_s \in \mathcal{W}_s(\mathcal{U})$ to the word $\varrho_{st}(\mathbf{W}) = W_0 \ldots W_t \in \mathcal{W}_t(\mathcal{U})$. We note that $[pk] \leq [qk]$ and $[(1-p)k] \geq [(1-q)k]$ for $k \geq 0$. With a pair $(\mathbf{U}, \mathbf{V}) \in \Gamma_n^p$ we associate the set of words $\varrho((\mathbf{U}, \mathbf{V}))$ given by

$$\{(\mathbf{U}', \mathbf{V}') : \mathbf{U}' \in \varrho_{[qm][pm]}^{-1}(\mathbf{U});\ X(\mathbf{U}') \neq \emptyset,\ \mathbf{V}' = \varrho_{[(1-p)m][(1-q)m]}(\mathbf{V})\}$$

where $m = m(\mathbf{U}, \mathbf{V}) = m(\mathbf{U}) + m(\mathbf{V})$.

Clearly, $\varrho(\Gamma_n^p) := \{\varrho((\mathbf{U}, \mathbf{V})) : (\mathbf{U}, \mathbf{V}) \in \Gamma_n^p\}$ induces a cover of Z with $\varrho(\Gamma_n^p) \subset \Omega_n^q(\mathcal{U})$ and in view of Lemma 4.1 the cardinality

$$\begin{aligned}\operatorname{card}(\varrho((\mathbf{U}, \mathbf{V}))) &\leq \exp[(h_{\mathrm{top}}(f) + \varepsilon)([qm] - [pm])]\\ &\leq \exp[(h_{\mathrm{top}}(f) + \varepsilon)((q-p)m + 1)].\end{aligned}$$

Moreover, $|(m(\mathbf{U})+m(\mathbf{V}))-(m(\mathbf{U}')+m(\mathbf{V}'))| \leq 2$. Let $C_\varepsilon := \exp[h_{\mathrm{top}}(f)+\varepsilon]$. Then

$$\begin{aligned}&\sum_{(\mathbf{U},\mathbf{V})\in\Gamma_n^p} \exp[-\alpha m(\mathbf{U}, \mathbf{V})]\\ &\quad\geq \sum_{(\mathbf{U},\mathbf{V})\in\Gamma_n^p} C_\varepsilon^{-(q-p)m(\mathbf{U},\mathbf{V})-1} \sum_{(\mathbf{U}',\mathbf{V}')\in\varrho((\mathbf{U},\mathbf{V}))} \exp[-\alpha m(\mathbf{U}, \mathbf{V})]\\ &\quad\geq e^{-4} \sum_{(\mathbf{U}',\mathbf{V}')\in\varrho(\Gamma_n^p)} C_\varepsilon^{-(q-p)m(\mathbf{U},\mathbf{V})-1} \exp[-\alpha m(\mathbf{U}', \mathbf{V}')]\\ &\quad\geq K \sum_{(\mathbf{U}',\mathbf{V}')\in\varrho(\Gamma_n^p)} \exp[-(\alpha + (q-p)\log C_\varepsilon)m(\mathbf{U}', \mathbf{V}')]\\ &\quad\geq K \inf_{\Gamma_n^q} \sum_{(\mathbf{U}',\mathbf{V}')\in\Gamma_n^q} \exp[-(\alpha + (q-p)\log C_\varepsilon)m(\mathbf{U}', \mathbf{V}')],\end{aligned}$$

where the infimum is taken over all $\Gamma_n^q \subset \Omega_n^q(\mathcal{U})$ that induce a cover of Z and $K := e^{-4-(2(q-p)+1)(h_{\rm top}(f)+\varepsilon)}$. This implies that $h^q(f|Z,\mathcal{U}) \le h^p(f|Z,\mathcal{U}) + (q-p)h_{\rm top}(f)$. Similarly, $h^p(f|Z,\mathcal{U}) \le h^q(f|Z,\mathcal{U}) + (q-p)h_{\rm top}(f)$. Letting $\operatorname{diam}(\mathcal{U}) \to 0$ this implies that h^p is Lipschitz continuous with constant $h_{\rm top}(f)$.

4.2. *Proof of Theorem 3.3.* Let $Z \subset X$. We are going to prove the statement in the case of $h^*(f|Z)$. The other cases are a straightforward modification of the proof.

Let μ be a probability measure supported on Z, $\mathcal{U}$ a finite partition of X with $\operatorname{diam}(\mathcal{U})$ sufficiently small and $0 \le s < \operatorname{ess\,sup} \underline{d}^*_{\mu,\mathcal{U}}(x)$ (the case $\operatorname{ess\,sup} \underline{d}^*_{\mu}(x) = 0$ is trivial). Then there is a number n_1 and a set $Z' \subset Z$ of positive measure with

$$\mu(X(\mathbf{U}) \cap X^-(\mathbf{V})) < \exp[-ns]$$

provided $x \in X(\mathbf{U}) \cap X^-(\mathbf{V})$ and $m(\mathbf{U},\mathbf{V}) = n > n_1$. For $n > n_1$ we consider a subset $\Gamma_n^* \subset \Lambda_n^*$ that induces a cover of Z'. We get

$$0 < \sum_{(\mathbf{U},\mathbf{V})\in\Gamma_n^*} \mu(X(\mathbf{U}) \cap X^-(\mathbf{V})) < \sum_{(\mathbf{U},\mathbf{V})\in\Gamma_n^*} \exp[-sm(\mathbf{U}) - sm(\mathbf{V})].$$

Hence, $h^*(f|Z) \ge h^*(f|Z,\mathcal{U}) \ge h^*(f|Z',\mathcal{U}) \ge s$.

For the other direction let $s < h^*(f|Z)$ and $\mathcal{U}$ be a partition of X such that $s < h^*(f|Z,\mathcal{U})$. Then $N^*(Z,s,\mathcal{U}) = \infty$ and by standard arguments (see Theorem 54 in [5]) there is a compact subset $Z' \subset Z$ with $N^*(Z',s,\mathcal{U}) = 1$. We set $\mu = N^*(Z',s,\mathcal{U})|Z'$. Thus μ is a probability measure supported on Z. Let $d = \operatorname{ess\,sup} \underline{d}^*_{\mu,\mathcal{U}}(x)$ and $\varepsilon > 0$. For each $x \in Z'$ and each $n \in \mathbb{N}$ there is a pair $(\mathbf{U},\mathbf{V}) \in \Omega_n^*$ with $x \in X(\mathbf{U}) \cap X^-(\mathbf{V})$ and

$$\mu(X(\mathbf{U}) \cap X^-(\mathbf{V})) > \exp[-(d+\varepsilon)m(\mathbf{U},\mathbf{V})]. \tag{3}$$

By the net structure of the sets $X(\mathbf{U}) \cap X^-(\mathbf{V})$ we can find for any $n \in \mathbb{N}$ a family $\{(\mathbf{U}_i,\mathbf{V}_i) : m(\mathbf{U}_i,\mathbf{V}_i) \ge n\}$ such that (3) holds for each i and induces a disjoint covering of Z'. This yields

$$1 = \sum_i \mu(X(\mathbf{U}_i) \cap X^-(\mathbf{V}_i)) > \sum_i \exp[-(d+\varepsilon)(m(\mathbf{U}_i) + m(\mathbf{V}_i))].$$

This implies $N^*(Z',d+\varepsilon,\mathcal{U}) < N^*(Z',s,\mathcal{U}) = 1$, i.e. $d+\varepsilon > s$. Hence, $h^*(f|Z,\mathcal{U}) \le \sup_\mu\{\operatorname{ess\,sup} \underline{d}^*_{\mu,\mathcal{U}}(x) : \operatorname{supp}\mu \subset Z\}$.

4.3. *Proof of Theorem 3.4.* Let $Z \subset X$ and $Z = \bigcup_{i=1}^N Z_i$. In view of Theorem 3.1(1) and (5) we see that $h^*(f|Z_i) \le \min_{0\le p\le 1} h^p(f|Z_i)$ for all i (we note that actually there is a minimum since h^p is continuous). Then by (2) of the same theorem we have $h^*(f|Z) \le \max_i\{\min_{0\le p\le 1} h^p(f|Z_i)\}$ for all representations of Z as finite unions of subsets.

For the other direction let $s > h^*(f|Z)$ and $\mathcal{U}$ be a finite partition. Then for any $n \in \mathbb{N}$ there is a set $\Gamma_n^* \subset \Omega_n^*$ which induces a covering of Z and

$$\sum_{(\mathbf{U},\mathbf{V})\in\Gamma_n^*} \exp[-sm(\mathbf{U},\mathbf{V})] \le \frac{1}{2^n}.$$

Hence

$$\sum_{(\mathbf{U},\mathbf{V})\in\Gamma^*} \exp[-sm(\mathbf{U},\mathbf{V})] \le 1$$

where $\Gamma^* := \bigcup_{n\ge 1} \Gamma_n^*$. For $N \in \mathbb{N}$, $0 \le i < N$ we set

$$\Gamma(i,N)_n := \left\{(\mathbf{U},\mathbf{V}) \in \Gamma_n^* : \frac{i}{N} \le \frac{m(\mathbf{U})}{m(\mathbf{U},\mathbf{V})} \le \frac{i+1}{N}\right\}.$$

Then $\Gamma(i,N)_n \subset \bigcup_{i/N\le p\le (i+1)/N} \Omega_n^p$. We also define

$$Z_{(i,N)} := \bigcap_{K\ge 1} \bigcup_{n\ge K} \bigcup_{(\mathbf{U},\mathbf{V})\in\Gamma(i,N)_n} X(\mathbf{U}) \cap X^-(\mathbf{V}).$$

By construction for all $n \in \mathbb{N}$ we find a subset $\Gamma_n^{(i,N)} \in \bigcup_{l\ge n} \Gamma(i,n)_l$ which induces a covering of $Z_{(i,N)}$ with

$$\sum_{(\mathbf{U},\mathbf{V})\in\Gamma_n^{(i,N)}} \exp[-sm(\mathbf{U},\mathbf{V})] \le 1.$$

With $(\mathbf{U},\mathbf{V}) \in \Gamma_n^{(i,N)}$, $\mathbf{V} = V_0 \ldots V_{m(\mathbf{V})}$, we associate the pair $(\mathbf{U},\mathbf{V}') \in \Omega_n^{i/N}$ with $\mathbf{V}' = V_0 \ldots V_{[(1-p)k]}$ where $k = k(\mathbf{V})$ is the smallest integer such that $[pk]=m(\mathbf{U})$. This map is well defined since for $(\mathbf{U},\mathbf{V})\in\Gamma_n^{(i,N)}$ we have $[(1-p)k] \le m(\mathbf{V})$. Moreover, $\Gamma_n^p(i,N) := \{(\mathbf{U},\mathbf{V}') : (\mathbf{U},\mathbf{V}) \in \Gamma(i,N)_n\}$ induces a covering of $Z_{(i,N)}$. Therefore

$$\begin{aligned}
\sum_{(\mathbf{U},\mathbf{V}')\in\Gamma_n^p(i,N)} & \exp[-tm(\mathbf{U},\mathbf{V}')] \\
&\le \sum_{(\mathbf{U},\mathbf{V})\in\Gamma_n^{(i,N)}} \exp[-tm(\mathbf{U},\mathbf{V})] \exp[t(m(\mathbf{V}) - k(\mathbf{V}))] \\
&\le \sum_{(\mathbf{U},\mathbf{V})\in\Gamma_n^{(i,N)}} \exp[-tm(\mathbf{U},\mathbf{V})] \exp\left[t\frac{m(\mathbf{U},\mathbf{V})+1}{N}\right] \\
&\le \exp\left[\frac{t}{N}\right] \sum_{(\mathbf{U},\mathbf{V})\in\Gamma_n^{(i,N)}} \exp\left[-\left(t-\frac{t}{N}\right)m(\mathbf{U},\mathbf{V})\right].
\end{aligned}$$

The right-hand side is bounded uniformly in n if $t(1-1/N) \ge s$. This implies that $h^{i/N}(f|Z_{(i,N)},\mathcal{U}) \le h^*(f|Z) + 1/N$. Letting $\operatorname{diam}(\mathcal{U}) \to 0$ and $N \to \infty$ gives the desired result.

4.4. *Proof of Theorem 3.5.* Let $Z \subset X$, $0 < p < 1$ and $0 \le s < h^p(f|Z)$ (again the cases $h^p(f|Z) = 0$ and $p \in \{0,1\}$ are trivial), and let $\mathcal{U}$ be a finite partition with $h^p(f|Z,\mathcal{U}) > s$.

Given $n \in \mathbb{N}$ and $\varepsilon > 0$ there is a family $\Gamma_n^0 \subset \Omega_n^0$ with

$$\sum_{\mathbf{V}\in\Gamma_n^0} \exp[-(h^0(f|Z)+\varepsilon)m(\mathbf{V})] \le 1. \tag{4}$$

We note that for $(\mathbf{U},\mathbf{V}) \in \Omega_n^0$ the word $\mathbf{U}$ is empty. For each such $\mathbf{V} \in \Omega_n^0$ we define $l = l(\mathbf{V})$ to be the smallest integer such that $[(1-p)l] = m(\mathbf{V})$. We consider all words

$$W^+(\mathbf{V}) := \{\mathbf{U} \in \mathcal{W}_{[pl]}(\mathcal{U}) : X(\mathbf{U}) \cap Z \ne \emptyset\}.$$

By definition of $\overline{h}{}^1_B(f|Z)$ the cardinality

$$\operatorname{card}(W^+(\mathbf{V})) < \exp[(\overline{h}{}^1_B(f|Z)+\delta)pl]$$

provided n is large enough ($l > n/(1-p)$). By construction $\Gamma_l^p := \{(\mathbf{U},\mathbf{V}) : \mathbf{V} \in \Gamma_n^0,\ \mathbf{U} \in W^+(\mathbf{V})\} \subset \Omega_l^p$ induces a covering of Z. Moreover, $m(\mathbf{U}) + m(\mathbf{V}) > l(\mathbf{V}) - 2$. For sufficiently large n (or l) and $\delta > 0$ we can estimate

$$\begin{aligned}
&\sum_{(\mathbf{U},\mathbf{V})\in\Gamma_l^p} \exp[-(p\overline{h}{}^1_B(f|Z)+(1-p)h^0(f|Z)+2\varepsilon)l]\\
&\quad\le \sum_{\mathbf{V}\in\Gamma_n^0} \exp[(\overline{h}{}^1_B(f|Z)+\delta)pl-(p\overline{h}{}^1_B(f|Z)+(1-p)h^0(f|Z)+2\varepsilon)l]\\
&\quad\le \sum_{\mathbf{V}\in\Gamma_n^0} \exp[\delta pl-(h^0(f|Z)+2\varepsilon)(m(\mathbf{V})+1)]\\
&\quad\le \exp[h^0(f|Z)+2\varepsilon]\sum_{\mathbf{V}\in\Gamma_n^0} \exp\left[\left(\frac{2p}{1-p}\delta\right)-h^0(f|Z)-2\varepsilon)m(\mathbf{V})\right].
\end{aligned}$$

In view of (4) the right-hand side term in the inequality is bounded by $\exp[h^0(f|Z)+2\varepsilon]$ provided $\varepsilon > (2p/(1-p))\delta$. Hence,

$$h^p(f|Z) \le p\overline{h}{}^1_B(f|Z,\mathcal{U}) + (1-p)h^0(f|Z,\mathcal{U}).$$

Similarly, one gets

$$h^p(f|Z) \le ph^1(f|Z,\mathcal{U}) + (1-p)\overline{h}{}^0_B(f|Z,\mathcal{U}).$$

Letting $\operatorname{diam}(\mathcal{U}) \to 0$ yields the assertion.

4.5. *Proof of Theorem 3.6.* Let $0 \le p \le 1$ and $\varepsilon > 0$ and $\mathcal{U}$ a finite partition. In view of Theorem 3.3 there are probability measures μ^1 and μ^0 on $Z \subset X$ with

$$\operatorname*{ess\,sup}_{\mu^1} \liminf_{m(\mathbf{U})\to\infty} -\frac{\log \mu\{X(\mathbf{U}) : x \in X(\mathbf{U})\}}{m(\mathbf{U})} \geq h^1(f|Z, \mathcal{U}) - \varepsilon$$

and

$$\operatorname*{ess\,sup}_{\mu^0} \liminf_{m(\mathbf{V})\to\infty} -\frac{\log \mu\{X^-(\mathbf{V}) : x \in X^-(\mathbf{V})\}}{m(\mathbf{V})} \geq h^0(f|Z, sU) - \varepsilon.$$

If $(\mathbf{U}, \mathbf{V}) \in \Omega_n^p$ then there is a $k = k(\mathbf{U}, \mathbf{V})$ such that $[pk] = m(\mathbf{U})$ and $[(1-p)k] = m(\mathbf{V})$. Then

$$\begin{aligned} -\frac{1}{k} &\log \mu(X(\mathbf{U}) \cap X^-(\mathbf{V})) \\ &\geq -\frac{1}{k} \log \min(\mu(X(\mathbf{U})), \mu(X^-(\mathbf{V}))) \\ &\geq \max\left(\frac{[pk]}{k}\left(-\frac{\log \mu(X(\mathbf{U}))}{m(\mathbf{U})}\right), \frac{[(1-p)k]}{k}\left(-\frac{\log \mu(X^-(\mathbf{V}))}{m(\mathbf{V})}\right)\right). \end{aligned}$$

Hence,

$$\max_{i=0,1} \operatorname*{ess\,sup}_{\mu^i} \underline{d}^p_{\mu,\mathcal{U}}(x) \geq \max(ph^1(f|Z, \mathcal{U}), (1-p)h^0(f|Z, \mathcal{U})) - \varepsilon.$$

Since ε and $\mathcal{U}$ were arbitrary the assertion follows from Theorem 3.3.

5. Examples. In this section we consider the case where $X = \Sigma := \{0, 1, 2, 3, 4\}^{\mathbb{Z}}$ is the space of all bi-infinite sequences of five symbols endowed with the product topology of the discrete topology on the symbols. Then the shift map $\sigma : \Sigma \to \Sigma$ defined by $(\sigma\mathbf{i})_n = i_{n+1}$, $\mathbf{i} = \ldots i_n i_{n+1} \ldots \in \Sigma$, $(\mathbf{i})_n = i_n$, is a homeomorphism. The partition $\mathcal{U} = \{[0], [1], [2], [3], [4]\}$, $[k] := \{\mathbf{i} \in \Sigma : i_0 = k\}$, is generating and hence $h^\#(\sigma|\cdot) = h^\#(\sigma|\cdot, \mathcal{U})$, with $\# = *$ or $\# = p$, $0 \leq p \leq 1$. For $(\mathbf{U}, \mathbf{V}) \in \mathcal{W}_n(\mathcal{U}) \times \mathcal{W}_m(\mathcal{U})$ we write $C_m^n(\mathbf{U}, \mathbf{V}) = X(\mathbf{U}) \cap X^-(\mathbf{V})$.

Example 1. We have $h^*(\sigma|Z) \leq \min_{0\leq p\leq 1} h^p(\sigma|Z)$ and this inequality may be strict. For example, if $Z_1 = \{\mathbf{i} \in \Sigma : i_n = 0,\ n < 0\}$ and $Z_0 = \{\mathbf{i} \in \Sigma : i_n = 0,\ n \geq 0\}$ then $h^1(\sigma|Z_1) = \overline{h}^1_B(\sigma|Z_1) = \log 5 = h^0(\sigma|Z_0) = \overline{h}^0_B(\sigma|Z_0)$ and $h^0(\sigma|Z_1) = \overline{h}^0_B(\sigma|Z_1) = 0 = h^1(\sigma|Z_0) = \overline{h}_B(\sigma|Z_0)$. Hence, for $Z = Z_1 \cup Z_0$ we have $h^1(\sigma|Z) = h^0(\sigma|Z) = \overline{h}^1_B(\sigma|Z) = \overline{h}^1_B(\sigma|Z) = \log 5$, and by Corollary 3.7,

$$0 = h^*(\sigma|Z) < \min_{0\leq p\leq 1} h^p(\sigma|Z) = \min_{0\leq p\leq 1} ph^1(\sigma|Z) + (1-p)h^0(\sigma|Z) = \log 5.$$

Example 2. In this example we will show that the upper bound in Theorem 3.5 is sharp.

Let $0 \le p \le 1$ and $Z \in \Sigma$ be the set of sequences $\mathbf{i}$ with $i_k \in \{0,1\}$ if there is a number $n \in \mathbb{N}$ with $[p(2n)!] \le k < [p(2n+1)!]$, $i_k \in \{0,1,2\}$ if $-[(1-p)(2n-1)!] \ge k > -[(1-p)(2n)!]$, and $i_k \in \{0,1,2,3\}$ if $[p(2n-1)!] \le k < [p(2n)!]$. There is a constant $c > 1$ such that

$$\begin{aligned}
A^1_{[p(2n+1)!]}(\sigma, Z, \mathcal{U}) &\le c2^{[p(2n+1)!]},\\
c^{-1}5^{-[(1-p)(2n-1)!]} &\le A^0_{[(1-p)(2n-1)!]}(\sigma, Z, \mathcal{U}),\\
c^{-1}3^{-[(1-p)(2n)!]} &\le A^0_{[(1-p)(2n)!]}(\sigma, Z, \mathcal{U}) \le c3^{-[(1-p)(2n)!]},\\
c^{-1}2^k &\le A^1_k(\sigma, Z, \mathcal{U}) \le c4^k,\\
c^{-1}3^k &\le A^0_k(\sigma, Z, \mathcal{U}) \le c5^k,\\
c^{-1}4^{[p(2n)!]} &\le A^1_{[p(2n)!]}(\sigma, Z, \mathcal{U}) \le c4^{[p(2n)!]}
\end{aligned}$$

for all $n, k \in \mathbb{N}$. This implies $\overline{h}^1_B(\sigma|Z) = \log 4$, $\underline{h}^1_B(\sigma|Z) = \log 2$, $\overline{h}^0_B(\sigma|Z) = \log 5$ and $\underline{h}^0_B(\sigma|Z) = \log 3$.

On the other hand, we can define a measure μ on Z by

$$\mu(C^n_m(\mathbf{U}, \mathbf{V})) := \frac{1}{\operatorname{card}\{C^n_m : C^n_m \cap Z \ne \emptyset\}}.$$

It is easy to check that the set function μ satisfies Kolmogorov's consistency conditions (the number of allowed symbols i_k depends only on k) and hence can be extended to a probability measure on Z. Let $(\mathbf{U}, \mathbf{V}) \in \Omega^p_l$ and $k \in \mathbb{N}$ be such that $[pk!] \le m(\mathbf{U}) < [p(k+1)!]$. Then if $m(\mathbf{U})$ is large enough, $[(1-p)k!] - 1 \le m(\mathbf{V}) \le [(1-p)(k+1)!]$. We get

$$\begin{aligned}
&-\frac{\log \mu(C^{m(\mathbf{U})}_{m(\mathbf{V})}(\mathbf{U}, \mathbf{V}))}{m(\mathbf{U}, \mathbf{V})}\\
&\qquad = \frac{\log \operatorname{card}\{C^{m(\mathbf{U})}_{m(\mathbf{V})}(\mathbf{U}', \mathbf{V}') : C^{m(\mathbf{U})}_{m(\mathbf{V})}(\mathbf{U}', \mathbf{V}') \cap Z \ne \emptyset\}}{m(\mathbf{U}, \mathbf{V})}\\
&\qquad \ge \frac{\log(A^1_{m(\mathbf{U})}(\sigma, Z, \mathcal{U}) \cdot A^0_{m(\mathbf{V})}(\sigma, Z, \mathcal{U}))}{m(\mathbf{U}, \mathbf{V})}.
\end{aligned}$$

Let us assume that k is even. If $n = m(\mathbf{U}, \mathbf{V})$ then $m(\mathbf{U}) = [pn]$ and $m(\mathbf{V}) = [(1-p)n]$ and

$$\begin{aligned}
&A^1_{m(\mathbf{U})}(\sigma, Z, \mathcal{U}) \cdot A^0_{m(\mathbf{V})}(\sigma, Z, \mathcal{U})\\
&\qquad = A^1_{[pk!]}(\sigma, Z, \mathcal{U})2^{m(\mathbf{U})-[pk!]} \cdot A^0_{[(1-p)k!]}(\sigma, Z, \mathcal{U})5^{m(\mathbf{V})-[(1-p)k!]}\\
&\qquad \ge c^{-2}4^{[pk!]}2^{[pn]-[pk!]}3^{[(1-p)k!]}5^{[(1-p)n]-[(1-p)k!]}.
\end{aligned}$$

Hence,

$$
\begin{aligned}
-&\frac{\log \mu(C^{m(\mathbf{U})}_{m(\mathbf{V})}(\mathbf{U},\mathbf{V}))}{m(\mathbf{U},\mathbf{V})}\\
&\geq \frac{b}{n} + \frac{pk!}{n}\log 4 + \frac{p(n-k!)}{n}\log 2\\
&\quad + \frac{(1-p)k!}{n}\log 3 + \frac{(1-p)(n-k!)}{n}\log 5\\
&\geq \frac{b}{n} + \alpha(p\log 4 + (1-p)\log 3) + (1-\alpha)(p\log 2 + (1-p)\log 5)
\end{aligned}
$$

where $b > 0$ and $0 < \alpha = k!/n \leq 1$. Similarly, for k odd one obtains

$$
\begin{aligned}
-&\frac{\log \mu(C^{m(\mathbf{U})}_{m(\mathbf{V})}(\mathbf{U},\mathbf{V}))}{m(\mathbf{U},\mathbf{V})}\\
&\geq \frac{b}{n} + (1-\alpha)(p\log 4 + (1-p)\log 3) + \alpha(p\log 2 + (1-p)\log 5).
\end{aligned}
$$

Now Theorems 3.5 and 3.3 imply

$$
\begin{aligned}
\min\{&p\overline{h}^1_B(\sigma|Z) + (1-p)h^0(\sigma|Z), ph^1(\sigma|Z) + (1-p)\overline{h}^0_B(\sigma|Z)\}\\
&= \min\{p\log 4 + (1-p)\log 3, p\log 2 + (1-p)\log 5\} \geq h^p(\sigma|Z)\\
&\geq \operatorname{ess\,sup} \underline{d}^p_{\mu,\mathfrak{U}}(x)\\
&\geq \min_{0\leq\alpha\leq 1} \alpha(p\log 4 + (1-p)\log 3) + (1-\alpha)(p\log 2 + (1-p)\log 5)\\
&= \min\{p\log 4 + (1-p)\log 3, p\log 2 + (1-p)\log 5\}.
\end{aligned}
$$

This shows that the inequality in Theorem 3.5 is sharp.

Example 3. Here we give an example showing that the lower estimate in Theorem 3.6 is sharp.

Fix $0 \leq p \leq 1/2$. The case $1/2 < p \leq 1$ may be treated by a similar example. Let $Z^+ \subset \Sigma$ be the set of sequences $\mathbf{i}$ with $i_k \in \{0,1\}$ if $[p(2n)!] \leq k < [p(2n+1)!]$, $n \in \mathbb{N}$, and $i_k \in \{0,1,2,3\}$ otherwise. We define $Z^- \subset \Sigma$ as the set of sequences $\{\pi(\mathbf{i}) : \mathbf{i} \in Z^+\}$ where π is defined as follows. For $-[(1-p)l!] \geq k > -[(1-p)l!] - ([p(l+1)!] - [pl!])$ we set $(\pi(\mathbf{i}))_k = i_{k-[(1-p)l!]+[pl!]}$. For all other $k < 0$ we set $i_k = 0$. We note that $p \leq 1/2$ implies $[(1-p)l!] + ([p(l+1)!] - [pl!]) > 0$. Finally, $Z = Z^+ \cap Z^-$. By construction there is a $c > 1$ such that

$$
c^{-1}A^0_{[(1-p)k!]}(\sigma,Z,\mathfrak{U}) \leq A^1_{[pk!]}(\sigma,Z,\mathfrak{U}) \leq cA^0_{[(1-p)k!]}(\sigma,Z,\mathfrak{U})
$$

and for $[(1-p)l!] \leq k < [(1-p)(l+1)!]$ sufficiently large

$$
\frac{A^0_k(\sigma,Z,\mathfrak{U})}{k} \geq \frac{A^0_{[(1-p)l!}(\sigma,Z,\mathfrak{U})}{[(1-p)l!]}, \qquad l = 2m+1,
$$

and

$$\frac{A_k^0(\sigma, Z, \mathcal{U})}{k} \geq \frac{A^0_{[(1-p)(l+1)!]}(\sigma, Z, \mathcal{U})}{[(1-p)(l+1)!]}, \qquad l = 2m.$$

This means that the most effective coverings to calculate $\underline{h}_B^0(\sigma|Z)$ are by cylinders $C^0_{[(1-p)(2m+1)!]}$ and for $\underline{h}_B^1(\sigma|Z)$ by cylinders $C_0^{[p(2m+1)!]}$. Hence, coverings by cylinders of the form $C^{[p(2m+1)!]}_{[(1-p)(2m+1)!]}$ are most effective to calculate $h^p(\sigma|Z)$. Similar calculations to those in the example above give

$$\overline{h}_B^1(\sigma|Z) = \log 4, \qquad \underline{h}_B^1(\sigma|Z) = \log 2,$$

and

$$\overline{h}_B^0(\sigma|Z) = \log 2, \qquad \underline{h}_B^0(\sigma|Z) = \frac{p}{1-p}\log 2.$$

As in the previous example we define a measure on Z by

$$\mu(C_m^n(\mathbf{U}, \mathbf{V})) := \frac{1}{\operatorname{card}\{C_m^n : C_m^n \cap Z \neq \emptyset\}}.$$

As in the previous example a simple calculation gives

$$\operatorname*{ess\,sup}_{\mu} \underline{d}^1_{\mu,\mathcal{U}}(x) = \log 2, \qquad \operatorname*{ess\,sup}_{\mu} \underline{d}^0_{\mu,\mathcal{U}}(x) = \frac{p}{1-p}\log 2.$$

For $(\mathbf{U}, \mathbf{V}) \in \Omega_n^p$, $X(\mathbf{U}) \cap X^-(\mathbf{V}) \cap Z \neq \emptyset$, the word $\mathbf{V}$ is completely determined by $\mathbf{U}$. This yields $A_k^p(\sigma, Z, \mathcal{U}) = A^0_{[pk]}(\sigma, Z, \mathcal{U})$ and

$$\mu(X(\mathbf{U}) \cap X^-(\mathbf{V})) = \mu(X(\mathbf{U})).$$

Therefore

$$\begin{aligned}\operatorname*{ess\,sup}_{\mu} \underline{d}^p_{\mu,\mathcal{U}}(x) &= \operatorname*{ess\,sup}_{\mu} \liminf_{k\to\infty} -\frac{[pk]}{k}\frac{\log \mu(C_0^{[pk]})}{[pk]} \\ &= p \operatorname*{ess\,sup}_{\mu} \underline{d}^1_{\mu,\mathcal{U}}(x) = p\log 2.\end{aligned}$$

In view of Theorems 3.3 and 3.6 we get

$$\begin{aligned}h^p(\sigma|Z) &\geq \max\{ph^1(\sigma|Z), (1-p)ph^0(\sigma|Z)\} \\ &= p\log 2 = \operatorname*{ess\,sup}_{mu} \underline{d}^p_{\mu,\mathcal{U}}(x) \geq h^p(\sigma|Z).\end{aligned}$$

This shows that the estimate in Theorem 3.6 is sharp.

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FB Mathematik und Informatik
Freie Universität Berlin
Arnimallee 2–6
D-14195 Berlin, Germany
E-mail: shmeling@math.fu-berlin.de

Received 2 August 1999;
revised 18 December 1999 (3802)