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STRONG AND WEAK STABILITY OF SOME MARKOV OPERATORS

BY

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To the memory of Anzelm Iwanik

Abstract. An integral Markov operator P appearing in biomathematics is investigated. This operator acts on the space of probabilistic Borel measures. Let μ and ν be probabilistic Borel measures. Sufficient conditions for weak and strong convergence of the sequence $(P^n\mu - P^n\nu)$ to 0 are given.

1. Introduction. Many biological and physical processes can be modelled by means of randomly perturbed dynamical systems. Such systems are generally of the form

$$X_{n+1} = S(X_n, \xi_{n+1}), \tag{1.1}$$

where $(\xi_n)_{n=1}^{\infty}$ is a sequence of independent random variables (or elements) with the same distribution, and the initial value of the system X_0 is independent of the sequence $(\xi_n)_{n=1}^{\infty}$. Studying systems of the form (1.1) we are often interested in the behaviour of the sequence of measures (μ_n) defined by

$$\mu_n(A) = \mathrm{Prob}(X_n \in A). \tag{1.2}$$

The evolution of these measures can be described by a Markov operator P given by $\mu_{n+1} = P\mu_n$. The operator P is defined on the space of probability measures. If the distribution of the random variables ξ_n is absolutely continuous with respect to the Lebesgue measure and the partial derivative $\frac{\partial S}{\partial \xi}$ exists and $\frac{\partial S}{\partial \xi}(x,\xi) \neq 0$ a.e., then P is given by a stochastic kernel, i.e.

$$P\mu(A) = \int_A \Big(\int_X k(x,y)\,\mu(dy)\Big)\,dx. \tag{1.3}$$

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In this case the measure $P\mu$ is absolutely continuous with respect to the Lebesgue measure and P can be defined on L^1 by

$$Pf(x) = \int_X k(x,y)f(y)\,dy. \tag{1.4}$$

The general theory of such operators is given in [4, 5].

Asymptotic behaviour of the sequences $(P^n\mu)$ has been examined by many authors (see e.g. [1, 7, 9]). Most of the results are devoted to the problem of existence and stability of invariant measures. For example, a conservative Markov operator given by a stochastic kernel always has an invariant absolutely continuous (possibly infinite) measure (see [3, Chap. VI]). But a lot of systems of the form (1.1) have no invariant probability measures, e.g. $X_{n+1} = X_n + \xi_{n+1}$. In this case we can still ask if the system is stable in the following sense: for any probability measures μ and ν the sequence $(P^n\mu - P^n\nu)$ converges to zero. If P is of the form (1.3) then the measures $P^n\mu$ and $P^n\nu$ have densities. Then the strong convergence of all sequences $(P^n\mu - P^n\nu)$ to zero is equivalent to the convergence of the sequences $(P^nf - P^ng)$ to zero in L^1 for all densities f and g. This condition means that the trajectory (P^nf) is asymptotically independent of the initial density f. This property of Markov operators is also called *completely mixing* [12] and some general results concerning this notion are given in [3, 14, 15].

In this paper we study some randomly perturbed dynamical system which plays an important role in mathematical models of the cell cycle ([9, 17, 18, 19, 20]) and in a model of the electrical activity of neurons [11]. We give sufficient conditions for weak and strong stability of this system. The plan of the paper is as follows. In Section 2 we define our system and formulate the main results concerning its asymptotic behaviour. The proofs of the results are given in Section 3.

2. Main results. Our main object is the following randomly perturbed dynamical system:

$$X_{n+1} = \lambda^{-1}\{Q^{-1}[Q(X_n) + \xi_{n+1}]\}, \qquad n \ge 0. \tag{2.1}$$

We assume that $\xi_1, \xi_2, \ldots$ are independent and identically distributed random variables with values in $[0,\infty)$. We also assume that X_0 is a random variable with values in $[0,\infty)$ and X_0 is independent of the sequence (ξ_n). By H we denote the distribution function of ξ_n. We assume that H is absolutely continuous and let $h = H'$. Assume that the functions Q and λ satisfy the following condition: $Q : \mathbb{R}_+ \to \mathbb{R}_+$ and $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ are nondecreasing locally absolutely continuous functions, $Q(0) = \lambda(0) = 0$, and $\lim_{x\to\infty} Q(x) = \lim_{x\to\infty} \lambda(x) = \infty$.

As λ can be a non-invertible function we adhere to the convention that $\lambda^{-1}(y) = \max\{x : \lambda(x) = y\}$. In a similar way we define Q^{-1}. Let $F_n(x) = \mathrm{Prob}(X_n < x)$. Then

$$\begin{aligned} F_{n+1}(x) &= \mathrm{Prob}(X_{n+1} < x) = \mathrm{Prob}(\lambda^{-1}\{Q^{-1}[Q(X_n) + \xi_{n+1}]\} < x) \\ &= \mathrm{Prob}(Q(X_n) + \xi_{n+1} < Q(\lambda(x))) = SF_n(x), \end{aligned}$$

where the operator S is defined on the space $L^\infty[0,\infty)$ by

$$SF(x) = \int_0^{\lambda(x)} Q'(y)h(Q(\lambda(x)) - Q(y))F(y)\,dy. \tag{2.2}$$

If F is an absolutely continuous function and $f = F'$ then SF is also absolutely continuous and $(SF)' = Pf$, where P is the operator defined on $L^1[0,\infty)$ by

$$Pf(x) = \lambda'(x)Q'(\lambda(x)) \int_0^{\lambda(x)} h(Q(\lambda(x)) - Q(y))f(y)\,dy. \tag{2.3}$$

Let $L^1 = L^1[0,\infty)$ and denote by D the set of all densities, i.e.

$$D = \{f \in L^1 : f \geq 0,\ \|f\| = 1\},$$

where $\|\cdot\|$ stands for the norm in L^1. From the definition of P it follows immediately that P is a Markov operator, i.e. $P : L^1 \to L^1$ is linear and $P(D) \subset D$.

Asymptotic properties of the iterates of the operator (2.3) depend on the function $\alpha(x) = Q(\lambda(x)) - Q(x)$. In [2, 6] it was proved that if $h(x) > 0$ and $\alpha(x) > \int_0^\infty th(t)\,dt$ for sufficiently large x, then there exists a density f_* such that

$$\lim_{n\to\infty} \|P^n f - f_*\| = 0 \quad \text{for } f \in D. \tag{2.4}$$

In [13] it was shown that if $h(x) = e^{-x}$ and $\alpha(x) \leq 1$ for sufficiently large x, then P is *sweeping*, i.e.

$$\lim_{n\to\infty} \int_0^c P^n f(x)\,dx = 0 \tag{2.5}$$

for $f \in L^1(\mathbb{R}_+)$ and $c > 0$.

REMARK 1. The property of sweeping is also known as *zero type*. Generally, a Markov operator P on a measure space (X, Σ, μ) is called *sweeping* from a set $A \in \Sigma$ if for every density f we have

$$\lim_{n\to\infty} \int_A P^n f(x)\,\mu(dx) = 0.$$

Some sufficient conditions for sweeping are given in [8, 17]. It is clear that if a Markov operator is sweeping from sets of finite measure then it has no invariant density. But even a Markov operator given by a strictly positive stochastic kernel and which has no invariant density can be non-sweeping from sets of finite measure (see [17, Remark 7]). Also dissipativity does not imply sweeping (see [8, Example 1]). It is interesting that a Markov operator given by (2.3) can be sweeping from bounded sets but can be no sweeping from some set of finite Lebesgue measure (see [17, Remark 3]).

In [16] it was proved that if $h(x) = e^{-x}$ and $\alpha(x) \geq c$ for all $x \geq 0$ and some $c \in \mathbb{R}$, then

$$\lim_{n\to\infty} \|P^n f - P^n g\| = 0 \quad \text{for } f, g \in D. \tag{2.6}$$

Our aim is to prove the following theorems.

THEOREM 1. *Assume that the functions Q, λ and h satisfy the following condition*:

(C) $\int_0^\infty xh(x)\,dx < \infty$ *and* $Q(\lambda(x)) \geq Q(x) + c$ *for all* $x \geq 0$ *and some* $c \in \mathbb{R}$.

Let F and G be the distribution functions of some probability measures on $[0, \infty)$. If the support of h has infinite Lebesgue measure then the sequence $(S^n F - S^n G)$ is uniformly convergent to zero.

The next theorem generalizes the result from [16].

THEOREM 2. *Assume that condition* (C) *holds. Suppose that $h(x) = 0$ for $x \leq \overline{x}$ and $h(x) = \exp(-\varphi(x))$ for $x > \overline{x}$, where $\overline{x} \geq 0$ and φ is a twice differentiable function such that $\varphi''(x) \geq 0$. Then the operator P satisfies* (2.6).

3. Proofs. We split the proofs of Theorems 1 and 2 into six lemmas.

LEMMA 1. *For every $a > 0$ there exists a positive number $\delta(a)$ such that*

$$\sum_{n=0}^{\infty} S^n \mathbf{1}_{[0,a]}(x) \leq \delta(a) \quad \textit{for } x \geq 0. \tag{3.1}$$

Proof. From the definition of S it follows that

$$S\mathbf{1}_{[0,\infty)}(x) = H(Q(\lambda(x))) \leq \mathbf{1}_{[0,\infty)}(x) - (1 - H(Q(\lambda(a))))\mathbf{1}_{[0,a]}(x)$$

and generally

$$S^n\mathbf{1}_{[0,\infty)}(x) \leq \mathbf{1}_{[0,\infty)}(x) - (1 - H(Q(\lambda(a)))) \sum_{k=0}^{n-1} S^k \mathbf{1}_{[0,a]}(x).$$

Since $S^n\mathbf{1}_{[0,\infty)}(x)\ge 0$ we have

$$\sum_{k=0}^{\infty} S^k\mathbf{1}_{[0,a]}(x)\le \delta(a)$$

for $\delta(a)=(1-H(Q(\lambda(a))))^{-1}$. ■

Lemma 2. *Let $\gamma>Q(a)-c$. If $Q(x)\ge\gamma n$ and $n\ge1$ then*

$$S^n\mathbf{1}_{[0,a]}(x)\le(\gamma-Q(a)+c)^{-1}\int_0^\infty y\,dH(y). \tag{3.2}$$

Proof. Let (X_n) be the sequence given by (2.1) such that $X_0=a$. From (2.1) it follows that $Q(\lambda(X_{n+1}))=Q(X_n)+\xi_{n+1}$ for $n\ge0$. Since $Q(\lambda(x))\ge Q(x)+c$ for $x\ge0$, we have

$$Q(X_{n+1})\le Q(X_n)+\xi_{n+1}-c\quad \text{for } n\ge 0. \tag{3.3}$$

Consequently,

$$Q(X_n)\le Q(a)-cn+\xi_1+\ldots+\xi_n\quad \text{for } n\ge1. \tag{3.4}$$

Let $g_n(x)=Q(x)-Q(a)+cn$. As Q is a non-decreasing function from (3.4) we obtain

$$\mathrm{Prob}(X_n<x)\ge\mathrm{Prob}(Q(X_n)<Q(x))\ge\mathrm{Prob}(\xi_1+\ldots+\xi_n<g_n(x)).$$

Since $\mathbf{1}_{(a,\infty)}(x)$ is the distribution function of the random variable X_0, the function $S^n\mathbf{1}_{(a,\infty)}(x)$ is the distribution function of X_n. This implies that

$$\begin{aligned}S^n\mathbf{1}_{[0,a]}(x)&=S^n\mathbf{1}_{[0,\infty]}(x)-S^n\mathbf{1}_{(a,\infty)}(x)\\&\le 1-\mathrm{Prob}(X_n<x)\le\mathrm{Prob}(\xi_1+\ldots+\xi_n\ge g_n(x)).\end{aligned}$$

Using the Chebyshev inequality we obtain

$$S^n\mathbf{1}_{[0,a]}(x)\le\frac{nE\xi_1}{g_n(x)}. \tag{3.5}$$

If $\gamma>Q(a)-c$ and $Q(x)>\gamma n$ from (3.5) it follows that

$$S^n\mathbf{1}_{[0,a]}(x)\le\frac{nE\xi_1}{Q(x)-Q(a)+cn}\le\frac{E\xi_1}{\gamma-Q(a)+c}. \ ■$$

Lemma 2 immediately yields

Corollary 1. *For every $a>0$ and $b>0$ there exists $\gamma>0$ such that*

$$S^n\mathbf{1}_{[0,a]}(x)<b\quad \textit{if } Q(x)\ge\gamma n \textit{ and } n\ge1. \tag{3.6}$$

Let m denote the Lebesgue measure on $[0,\infty)$.

Lemma 3. *If $m(\operatorname{supp} h)=\infty$, then for every $a>0$ the sequence $(S^n\mathbf{1}_{[0,a]})$ is uniformly convergent to 0.*

Proof. Let $F_n(x) = S^n \mathbf{1}_{[0,a]}(x)$ and $\beta_n = \sup\{F_n(x) : x \ge 0\}$. Since $F_n(x) \le \beta_n$ and

$$S^{n+1}\mathbf{1}_{[0,a]}(x) = SF_n(x) \le \beta_n S\mathbf{1}_{[0,\infty)}(x) \le \beta_n,$$

the sequence (β_n) is decreasing. Let $\beta = \lim_{n\to\infty} \beta_n$. We show that $\beta = 0$. Suppose, by contradiction, that $\beta > 0$. Let

$$\eta(y) = \sup\Big\{ \int_A h(x)\,dx : m(A) \le y,\ A \text{ measurable}\Big\}$$

and

$$A_n = \{ x \in [0,\infty) : F_n(x) \ge \beta/2\}, \qquad A'_n = [0,\infty) \setminus A_n.$$

Then

$$\begin{aligned} F_{n+1}(x) \le \beta_n & \int_0^{\lambda(x)} Q'(y)h(Q(\lambda(x)) - Q(y))\mathbf{1}_{A_n}(y)\,dy \\ &+ \frac{\beta}{2}\int_0^{\lambda(x)} Q'(y)h(Q(\lambda(x)) - Q(y))\mathbf{1}_{A'_n}(y)\,dy \\ &\le (\beta_n - \beta/2)\eta(m(Q(A_n))) + \beta/2. \end{aligned}$$

Hence

$$\beta_{n+1} \le (\beta_n - \beta/2)\eta(m(Q(A_n))) + \beta/2$$

and consequently

$$\eta(m(Q(A_n))) \ge \frac{\beta_{n+1} - \beta/2}{\beta_n - \beta/2}.$$

Letting $n \to \infty$ we obtain

$$\lim_{n\to\infty} \eta(m(Q(A_n))) = 1. \tag{3.7}$$

Since $m(\operatorname{supp} h) = \infty$, we have $\eta(y) < 1$ for every $y > 0$. From (3.7) it follows that

$$\lim_{n\to\infty} m(Q(A_n)) = \infty. \tag{3.8}$$

Now, according to Corollary 1, there exists $\gamma > 0$ such that

$$F_n(x) < \beta/2 \quad \text{ if } Q(x) \ge \gamma n,\ n \ge 1.$$

Let x_n be a positive constant such that $Q(x_n) = \gamma n$. Then

$$F_k(x) < \beta/2 \quad \text{ if } x \ge x_n \text{ and } k = 1,\dots,n.$$

Thus $A_k \subset [0, x_n]$ for $k = 1,\dots,n$. Since $\mathbf{1}_{A_k} \le (2/\beta)F_k\mathbf{1}_{[0,x_n]}$ for $k = 1,\dots,n$, from (3.1) it follows that

$$\sum_{k=1}^n \int_{A_k} Q'(t)\,dt \le \frac{2}{\beta}\int_0^{x_n} Q'(t)\Big(\sum_{k=1}^n F_k(t)\Big)\,dt = \frac{2\delta(a)}{\beta}Q(x_n) = \frac{2\gamma\delta(a)n}{\beta}.$$

This implies that

$$\frac{1}{n}\sum_{k=1}^{n} m(Q(A_k)) \le \frac{2\gamma\delta(a)}{\beta},$$

which contradicts (3.8). ■

Now observe that Theorem 1 is a simple consequence of the following lemma.

LEMMA 4. *Assume that* $m(\operatorname{supp} h) = \infty$. *If* $F : [0,\infty) \to \mathbb{R}$ *is a continuous function such that* $\lim_{x\to\infty} F(x) = 0$, *then* $(S^n F)$ *is uniformly convergent to* 0.

Proof. Fix $\varepsilon > 0$. Since $\lim_{x\to\infty} F(x) = 0$, there exist $m > 0$ and $a_\varepsilon > 0$ such that $|F(x)| < \varepsilon$ for $x \ge a_\varepsilon$ and $|F(x)| \le m$ for $x \ge 0$. From (2.2) it follows that S is a positive operator such that $S\mathbf{1}_{[0,\infty)} \le \mathbf{1}_{[0,\infty)}$. Since

$$|F(x)| \le m\mathbf{1}_{[0,a_\varepsilon]}(x) + \varepsilon\mathbf{1}_{[a_\varepsilon,\infty)}(x),$$

we have

$$|S^n F(x)| \le S^n\mathbf{1}_{[0,a_\varepsilon]}(x) + \varepsilon.$$

Lemma 3 implies that $(S^n F)$ is uniformly convergent on $[0,\infty)$. ■

Now we give the proof of Theorem 2. Let $L^1_0 = \{f \in L^1 : \int_0^\infty f(x)\,dx = 0\}$. Since P^n is a linear operator, condition (2.6) is equivalent to $\lim_{n\to\infty} \|P^n f\| = 0$ for $f \in L^1_0$. Denote by M the subset of L^1_0 which contains all functions satisfying the following condition:

- *there exists* $x_0 > 0$ *such that* $f(x) \ge 0$ *for* $x \le x_0$ *and* $f(x) \le 0$ *for* $x > x_0$.

LEMMA 5. *The set* M *is linearly dense in* L^1_0.

Proof. It is sufficient to show that each $f \in L^1_0$ is a difference of two functions from M. Let $f_+ = \max(f,0)$, $f_- = \max(-f,0)$ and $x_0 > 0$ be a constant such that $\int_0^{x_0} |f(x)|\,dx = \|f\|/2$. Then the functions $f_1 = f_+\mathbf{1}_{[0,x_0]} - f_-\mathbf{1}_{(x_0,\infty)}$ and $f_2 = f_-\mathbf{1}_{[0,x_0]} - f_+\mathbf{1}_{(x_0,\infty)}$ satisfy $f_1 \in M$, $f_2 \in M$ and $f = f_1 - f_2$. ■

LEMMA 6. *We have* $P(M) \subset M$.

Proof. Let $f \in M$. Let $x_0 > 0$ be such that $f(x) \ge 0$ for $x \le x_0$ and $f(x) \le 0$ for $x > x_0$. Let y_0 be such that $\lambda(y_0) = x_0$. Then $Pf(x) \ge 0$ for $x \le y_0$. Let $z_0 > y_0$ be such that $Pf(z_0) = 0$ and $Pf(x) \ge 0$ for $x \le y_0$. Let $a = Q^{-1}(Q(\lambda(z_0)) - \overline{x})$. Since

$$Pf(x) \le \lambda'(x)Q'(\lambda(x))\int_0^a e^{-\varphi(Q(\lambda(x))-Q(y))} f(y)\,dy \tag{3.9}$$

for $x \ge z_0$ it is sufficient to check that

$$\int_0^a e^{-\varphi(Q(\lambda(x))-Q(y))} f(y)\,dy \le 0 \tag{3.10}$$

for $x \ge z_0$. Define an auxiliary function

$$g(t) = \int_0^a e^{-\varphi(\bar{x}+Q(a+t)-Q(y))} f(y)\,dy. \tag{3.11}$$

Then

$$g'(t) = -Q'(a+t) \int_0^a \varphi'(\bar{x}+Q(a+t)-Q(y)) e^{-\varphi(\bar{x}+Q(a+t)-Q(y))} f(y)\,dy. \tag{3.12}$$

Since φ' is non-decreasing and $f(x) \ge 0$ for $x \le x_0$ and $f(x) \le 0$ for $x > x_0$ from (3.12) it follows that

$$g'(t) \le -Q'(\bar{x}+a+t) \int_0^a \varphi'(\bar{x}+Q(a+t)-Q(x_0)) e^{-\varphi(\bar{x}+Q(a+t)-Q(y))} f(y)\,dy. \tag{3.13}$$

Set $\psi(t) = -Q'(a+t)\varphi'(\bar{x}+Q(a+t)-Q(x_0))$. Then $g(t)$ satisfies the differential inequality

$$g'(t) \le \psi(t) g(t)$$

and $g(0) = 0$. This implies that $g(t) \le 0$ for $t \ge 0$. Consequently, inequality (3.10) holds. ■

Proof of Theorem 2. According to Lemma 5 it is sufficient to check that the sequence $(P^n f)$ converges to zero in L^1 for $f \in M$. Let $f \in M$. From Lemma 6 we have $P^n f \in M$ for $n \ge 1$ and, consequently, there exists a sequence (x_n) such that $P^n f(x) \ge 0$ for $x \le x_n$ and $P^n f(x) \le 0$ for $x > x_n$. This implies that

$$\|P^n f\| = 2 \int_0^{x_n} P^n f(t)\,dt = S^n F(x_n),$$

where $F(x) = \int_0^x f(t)\,dt$. From Lemma 4 it follows that the sequence $S^n F$ converges uniformly to zero. ■

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