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AN EXOTIC FLOW ON A COMPACT SURFACE

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Abstract. In 1988 Anosov [1] published the construction of an example of a flow (continuous real action) on a cylinder or annulus with a phase portrait strikingly different from our normal experience. It contains orbits whose ω-limit sets contain a non-periodic orbit along with a simple closed curve of fixed points, but these orbits do not wrap down on this simple closed curve in the usual way. In this paper we modify some of Anosov's methods to construct a flow on a surface of genus 2 with equally striking behavior that does not occur on a surface of genus 1 or a cylinder. Moreover, our construction is relatively simple and can easily be modified to produce a variety of examples exhibiting similar types of behavior.

The key idea that we use from Anosov's paper can be described in the following way. A flow on a cylinder can always be slowed down near one of the boundary circles so that it becomes fixed. If you slow a flow down very rapidly as you approach a bounding circle, then you can also spin the orbits further and further around the axis of the cylinder as you approach the boundary without destroying the flow. In particular, the boundary circle remains fixed, but orbits that approach even a single point on it now spiral toward it.

1. Introduction. Before we describe the properties of the examples mentioned in the Abstract more fully, we need to state more explicitly the mathematical context. Let M be a compact connected surface and let $\widetilde{M}$ be its universal cover. A *flow* or *continuous real action* on M is a continuous mapping $\phi : M \times \mathbb{R} \to M$, where $\mathbb{R}$ is the reals, such that $\phi(\phi(x,t),s) = \phi(x,t+s)$ and $\phi(x,0) = x$ for all $x \in M$ and $s,t \in \mathbb{R}$. For convenience we will often follow the convention of writing xt for $\phi(x,t)$. As is often the case, our flow will be constructed from an autonomous differential equation $\dot{x} = f(x)$ by taking the solution which is x at time 0 and evaluating it at time t to obtain $\phi(x,t)$.

The set of *fixed points* of ϕ is $F = \{x \in M : xt = x \text{ for all } t \in \mathbb{R}\}$. If $x \notin F$, then we say x is a *moving point*. The *orbit* of x is defined by $\mathcal{O}(x) = \{xt : t \in \mathbb{R}\}$. The *positive semiorbit* of x is defined by $\mathcal{O}^+(x) = \{xt : t \geq 0\}$, and the *negative semiorbit* of x is defined by $\mathcal{O}^-(x) = \{xt : t \leq 0\}$. The

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ω-limit set of x is defined by $\omega(x) = \bigcap_{t\geq 0} \overline{\mathcal{O}^+(xt)}$, and the *$\alpha$-limit set* of x is defined by $\alpha(x) = \bigcap_{t\leq 0} \overline{\mathcal{O}^-(xt)}$.

A *local cross section* Σ of ϕ at a point $x \in M$ is a closed subset Σ of M containing x such that the map $(x,t) \mapsto xt$ is a homeomorphism of $\Sigma \times [-\varepsilon, \varepsilon]$ onto the closure on an open neighborhood V of x for some $\varepsilon > 0$. If x is a moving point, then there exists a local cross section at x.

The flow on M lifts to a unique flow $\widetilde{\phi}$ on $\widetilde{M}$ such that the covering projection $\pi : \widetilde{M} \to M$ is a homomorphism of flows, i.e., $\pi(\widetilde{\phi}(\widetilde{x},t)) = \phi(\pi(\widetilde{x}),t)$, and every covering transformation T of $\widetilde{M}$ is an automorphism of the flow $\widetilde{\phi}$, i.e., $T\widetilde{\phi}(\widetilde{x},t) = \widetilde{\phi}(T\widetilde{x},t)$. Moreover, $\pi(\widetilde{x}) \in F$ if and only if $\widetilde{x} \in \widetilde{F}$, where $\widetilde{F}$ denotes the fixed points of $\widetilde{\phi}$. These results are a consequence of the homotopy lifting theorem and can be found in [3]. For Anosov's example $\widetilde{M}$ is an infinite planar strip of the form $\mathbb{R} \times [0,1]$, and for our example $\widetilde{M}$ will be the interior of the unit disk.

Anosov's example, which can easily be imbedded in a flow on a torus, has orbits in $\widetilde{M}$ which are neither bounded nor go to infinity. In other words, there exists $\widetilde{x}$ in $\widetilde{M}$ such that $\widetilde{\phi}(\widetilde{x},t)$ neither stays in a closed bounded subset of $\widetilde{M}$ for all positive t nor eventually stays outside of every such set as t goes to infinity. Anosov goes on to prove in Theorem 1 of [1] that this phenomenon cannot occur when the fixed points are contractible.

In contrast to Anosov's example, $\mathcal{O}^+(\widetilde{q})$, a lift of the interesting orbit in our example, will actually approach a point σ on the unit circle bounding $\widetilde{M}$ as t approaches infinity. However, it is the way that it approaches this point at infinity which is of interest. There is a periodic orbit γ_0 on M with a lift $\widetilde{\gamma}_0$ to $\widetilde{M}$ which approaches the same point σ on the unit circle as t goes to infinity. The periodic orbit is not contained in the ω-limit set of our special orbit. But this ω-limit set does contain an orbit γ_2, and a simple closed curve γ_1 of fixed points which is not homotopic to the periodic orbit γ_0. The orbit γ_2 wraps down on γ_1 in such a way that $\mathcal{O}^+(\widetilde{q})$ tends to follow lifts of γ_2 for long periods of time that take it far away from $\widetilde{\gamma}_0$. Geometrically, what we observe on $\widetilde{M}$ is that for positive t the hyperbolic distance between $\widetilde{q}t$ and $\widetilde{\gamma}_0$ is unbounded. Such an example would be impossible on the torus. N. Markley showed that a lifted positive semiorbit $\mathcal{O}^+(\widetilde{x})$ of a continuous flow on the torus that goes to infinity (i.e., $|\widetilde{x}t| \to \infty$ as $t \to \infty$) will lie between two parallel lines if there is a moving point in the ω-limit set of $\pi(\widetilde{x})$ (Theorem 5 of [4]).

We will use the following notation found in [4] for segments of curves and orbits. If C is a simple curve, hence homeomorphic to an interval, and a and b lie on C, then $(a,b)_C$ will denote the open segment of C between a and b. If $s, \tau \in \mathbb{R}$, then $[xs, x\tau]_\phi$ will denote $\{xt : s \leq t \leq \tau\}$ or $\{xt : \tau \leq t \leq s\}$ according as $s < \tau$ or $\tau < s$.

2. Construction of a planar flow. We are now ready to describe the construction. We begin with the following system of differential equations which defines a flow on $\mathbb{R}^2$:

$$\dot{x} = 1, \qquad \dot{y} = 1 - y^2. \tag{1}$$

Denote this system by $\dot{z} = F(z)$, where $z = (x, y)$. Let $z(t, (x_0, y_0))$ denote the solution which is (x_0, y_0) at time 0. Observe that $z(t, (0, 0)) = (t, \tanh t)$, $z(t, (0, 1)) = (t, 1)$, and $z(t, (0, -1)) = (t, -1)$.

Let $f(x) = \frac{1}{4}(1 + \cos \pi x)$. Note that $f(x)$ is a smooth periodic function of period 2; $f(0) = 1/2$; $f(1) = 0$; $f(x)$ is strictly decreasing on the interval $(0, 1)$; and $f(x)$ is strictly increasing on $(1, 2)$. Define $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\Phi(x, y) = (x, (1 + f(x))y + f(x)),$$

where $w = \Phi(x, y) = (u, v)$. Let Φ' denote the matrix of partials of $\Phi(x, y)$. If $z(t)$ is a solution to $\dot{z} = F(z)$, then $\Phi(z(t)) = (u(t), v(t)) = w(t)$ is a solution to

$$\dot{w} = G(w) = \Phi'(\Phi^{-1}(w))F(\Phi^{-1}(w)) = (g_1(u, v), g_2(u, v)). \tag{2}$$

Let $w(t, (u_0, v_0))$ denote the solution to $\dot{w} = G(w)$ which is (u_0, v_0) at time 0. Observe that Φ maps the solution $z(t, (0, 0)) = (t, \tanh t)$ of $\dot{z} = F(z)$ to the solution $w(t, (0, 1/2)) = (t, (1 + f(t)) \tanh t + f(t))$ of $\dot{w} = G(w)$. Also note that Φ maps the solutions $z(t, (0, 1)) = (t, 1)$ and $z(t, (0, -1)) = (t, -1)$ to the solutions $w(t, (0, 2)) = (t, 2f(t)+1)$ and $w(t, (0, -1)) = (t, -1)$, respectively.

There exists t' such that $(3/2)f(t) + 1/2 < w(t, (0, 1/2)) < 2f(t) + 1$ for all $t \geq t'$. Hence for $t \geq t'$, $w(t, (0, 1/2))$ has a minimum between any two consecutive maximums of $2f(t) + 1$, and $w(t, (0, 1/2))$ has a maximum between any two consecutive minimums of $2f(t) + 1$.

Consider the system $\dot{w} = \varrho(v)G(w)$, where $\varrho(v)$ is a C^∞-function for which $\varrho(v) > 0$ for $v < 2$ and which vanishes together with its derivatives when $v \geq 2$. The line $v = 2$ now consists entirely of fixed points. Note that the solution $w(t, (0, 2))$ is now broken into a sequence of fixed points at $(2n, 2)$, $n \in \mathbb{Z}$, and a sequence of orbits for which $\omega(w) = (2n, 2)$ and $\alpha(w) = (2(n - 1), 2)$. Solutions not intersecting the region $v \geq 2$ remain unchanged except for a change in speed. We will now restrict our attention to the strip $\mathbb{R} \times [-1, 2]$.

The following technique is directly from Anosov [1]. Let $h(u, v) = (u + \lambda(v), v)$, where $z = h(u, v) = (x, y)$ and $\lambda(v)$ is a C^∞-function for $v < 2$ such that $\lambda(v) \geq 0$, $\lambda(v) = 0$ for $|v| \leq 1.5$, $\lambda(v)$ is strictly increasing for $v > 1.5$ and $\lambda(v) \to \infty$ as $v \to 2$. If $w(t)$ is a solution to $\dot{w} = \varrho(v)G(w)$, then $h(w(t))$ is a solution to

$$\dot{z} = H(z) = h'(h^{-1}(z))\varrho(y)G(h^{-1}(z)). \tag{3}$$

Setting $H(z) = 0$ for the given function λ, when $v = 2$, gives a C^2-field on $\mathbb{R} \times [-1, 2]$, if ϱ and its derivatives vanish fast enough near $v = 2$.

A trajectory of $\dot{w} = \varrho(v)G(w)$ that passes through $(2n+1, 1)$ and lies on $\{(u, v) : 2n \le u \le 2(n+1),\ v = 2f(u) + 1\}$ is mapped by h onto a curve which lies in the strip $1 < |y| < 2$ and asymptotically approaches (in positive and negative time) the line $y = 2$. For $-1 \le y \le 1.5$, the phase portraits of $\dot{z} = H(z)$ and $\dot{w} = \varrho(v)G(w)$ will be identical. Let $\phi_1(t, \zeta) = (x(t, \zeta), y(t, \zeta)) = \zeta t$ denote the solution to $\dot{z} = H(z)$ such that $\phi_1(0, \zeta) = \zeta$ (which defines a flow on $\mathbb{R} \times [-1, 2]$).

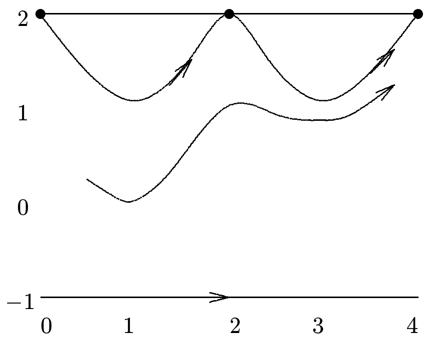

Fig. 1. Phase portrait of $\dot{w} = \varrho(v)G(w)$

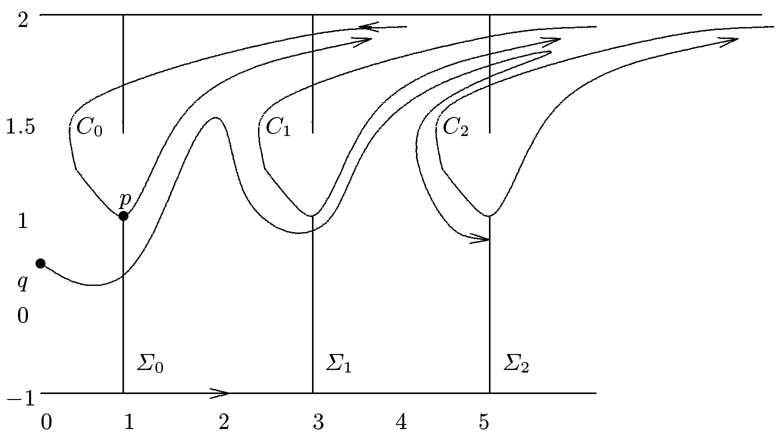

Fig. 2. Phase portrait of (extended) $\dot{z} = H(z)$

For $n \in \mathbb{Z}$ let C_n be the vertical line segment with endpoints $(2n+1, 1.5)$ and $(2n+1, 2)$, and let Σ_n be the vertical line segment with endpoints $(2n+1, -1)$ and $(2n+1, 1)$. Let $p = (1, 1)$ and $q = (0, 1/2)$. Note that for each $n \in \mathbb{Z}$, Σ_n is a local cross section. The following lemmas and corollary describe the behavior of the orbits through p and q. Their proofs are immediate consequences of the following facts: $f(u)$ is strictly decreasing on

$(0,1)$ and strictly increasing on $(1,2)$; $\lambda(v)$ is strictly increasing for $v>1.5$; $H(x+2,y)=H(x,y)$; and $w(t,(0,1/2))$ has a maximum (minimum) between any two successive minimums (maximums) of $f(t)$.

Lemma 2.1. *There exist sequences* $\{t_n\}_{n=1}^{\infty}$ *and* $\{t_{-n}\}_{n=1}^{\infty}$, *with* $t_n \to \infty$ *and* $t_{-n} \to -\infty$, *such that*

(1) pt_n *and* $pt_{-n} \in C_n$,
(2) $p\tau \in C_k$, $k \geq 1 \Leftrightarrow \tau \in \{t_n\}_{n=1}^{\infty}$ *or* $\tau \in \{t_{-n}\}_{n=1}^{\infty}$,
(3) $y(t_n,p) \nearrow 2$ *and* $y(t_{-n},p) \nearrow 2$, *and*
(4) $y(t_{n+1},p) > y(t_{-n},p) > y(t_n,p)$ *for* $n \geq 1$.

Note that there also exists a unique solution to $p\tau \in C_0$, which we will denote t_0, and which satisfies $t_{-1} < t_0 < 0$.

Lemma 2.2. *There exists* $\{\tau_n\}_{n=0}^{\infty}$ *with* $\tau_n \to \infty$ *such that* $q\tau_n \in \Sigma_n$, $y(\tau_n,q) \nearrow 1$, *and* $(q\tau_n, q\tau_{n+1})_{\phi_1} \cap \Sigma_j = \emptyset$ *for all* $j \in \mathbb{Z}$, $n \in \mathbb{Z}^+$. *Moreover, there exists* $\{s_n\}_{n=0}^{\infty}$, $\tau_n < s_n \leq \tau_{n+1}$, *such that* $x(s_n,q) - x(\tau_n,q) \to \infty$, *and* $x(t,q) - x(\tau_n,q) \leq x(s_n,q) - x(\tau_n,q)$ *for* $\tau_n \leq t \leq \tau_{n+1}$.

Corollary 2.3. *There exist* $\mu \in \mathbb{Z}^+$ *and* $s, \widehat{s} \in \mathbb{R}$ *with* $\tau_\mu < s < \widehat{s} < \tau_{\mu+1}$, *such that*

(1) $x(s_\mu,q) - x(\tau_\mu,q) > 6$,
(2) $qs, q\widehat{s} \in C_{\mu+3}$,
(3) $x(t,q) < 2\mu + 7$ *for* $\tau_\mu \leq t < s$ *and* $\widehat{s} < t \leq \tau_{\mu+1}$, *and*
(4) $y(t_3,p) > y(s,q) > y(\widehat{s},q) > y(t_{-2},p)$.

We are now able to construct our example on a compact genus 2 surface.

3. A flow on a compact surface. Since $H(x+2,y) = H(x,y)$, the system $\dot{z} = H(z)$ defines a flow on $S^1 \times [-1,2]$. Let $\gamma_2 = \mathcal{O}(p)$, let $\gamma_0 = S^1 \times \{-1\}$ and let $\gamma_1 = S^1 \times \{2\}$ denote the two bounding circles of the cylinder. Observe that γ_0 is a periodic orbit of period 2, and γ_1 consists entirely of fixed points. Note that $\omega(p) = \alpha(p) = \gamma_1$. Also note that $\omega(q) = \gamma_1 \cup \gamma_2$, which is not locally connected. By the Poincaré–Bendixson Theory, $\alpha(q) = \gamma_0$. Let $C = \pi C_0$ and let $z_1 = C \cap \gamma_1$. Let $\Sigma = \pi \Sigma_0$ and let $z_0 = \Sigma \cap \gamma_0$.

We now describe how to imbed this flow into a surface of genus 2 so that γ_0 and γ_1 have different homotopy types, and neither is null homotopic. (We also orient γ_1 to have the same orientation as γ_0, with the natural orientation of γ_0 given by the flow.)

Let D be the projection of $\{(x,y) : 2f(x)+1 < y < 1.5\}$ into $S^1 \times [-1,2]$. Note that D is open and connected. Let D_0 and D_1 be disjoint closed discs in D. Modify the flow so that ∂D_0 and ∂D_1 are fixed and remove their interiors. Since γ_1 and ∂D_1 are fixed, we can glue γ_1 and ∂D_1 to obtain a torus with two holes: γ_0 and ∂D_0.

We now construct a flow on the cylinder $S^1 \times [0,1]$ using the system

$$\dot{x} = y^2, \quad \dot{y} = 0.$$

Note that $S^1 \times \{0\}$ is fixed, and $S^1 \times \{1\}$ is a periodic orbit. Finally, we attach this cylinder to the torus with two holes by gluing ∂D_0 to $S^1 \times \{0\}$ and γ_0 to $S^1 \times \{1\}$ to obtain a flow ϕ on a surface M of genus 2.

Let μ, s, and $\widehat{s}$ be as described in Corollary 2.3. Henceforth p, q, γ_0, γ_1, z_0, z_1, C and Σ will be in M. We restate Lemma 2.2 and Corollary 2.3 for (M, ϕ).

LEMMA 3.1. *There exists a sequence* $\{t_n\}_{n=-\infty}^{\infty}$, *with* $t_n \to \infty$ *and* $t_{-n} \to -\infty$ *as* $n \to \infty$, *such that*

(1) $p\tau \in C \Leftrightarrow \tau \in \{t_n\}_{n=-\infty}^{\infty}$,
(2) $\{pt_n\}_{n=0}^{\infty}$ *and* $\{pt_{-n}\}_{n=0}^{\infty}$ *converge strictly monotonely to* z_1, *and*
(3) $pt_{-n} \in (pt_n, pt_{n+1})_C$ *for* $n \geq 1$.

LEMMA 3.2. *There exists a sequence* $\{\tau_n\}_{n=0}^{\infty}$, $\tau_n \to \infty$, *such that* $q\tau_n \in \Sigma$, $(q\tau_n, q\tau_{n+1})_\phi \cap \Sigma = \emptyset$, *and* $\{q\tau_n\}$ *converges strictly monotonely to* p. *Moreover*, *there exist* $\mu \in \mathbb{Z}^+$, *and* $s, \widehat{s} \in \mathbb{R}$ *with* $\tau_\mu < s < \widehat{s} < \tau_{\mu+1}$ *such that* $qs, q\widehat{s} \in (pt_{-2}, pt_3)_C$, *and* $qt \notin (pt_{-2}, pt_3)_C$ *for* $\tau_\mu \leq t < s$ *and* $\widehat{s} < t \leq \tau_{\mu+1}$.

Note that for $i \geq 1$, $[q\tau_i, q\tau_{i+1}]_\phi \cup (q\tau_{i+1}, q\tau_i)_\Sigma$ is homotopic to γ_0, and $[pt_i, pt_{i+1}]_\phi \cup (pt_{i+1}, pt_i)_C$ is homotopic to γ_1. Observe that the following three loops are null homotopic: $[p, pt_3]_\phi \cup (pt_3, qs)_C \cup [qs, q\tau_\mu]_\phi \cup (q\tau_\mu, p)_\Sigma$; $[q\tau_{\mu+1}, q\widehat{s}]_\phi \cup (q\widehat{s}, pt_{-2})_C \cup [pt_{-2}, p]_\phi \cup (p, q\tau_{\mu+1})_\Sigma$; and $[qs, q\widehat{s}]_\phi \cup (q\widehat{s}, qs)_C$.

In the next section we will describe the properties of the lifted flow.

4. The lifted flow. Since the genus of M is 2, $\widetilde{M}$ is the Poincaré disc: the open unit disc equipped with the hyperbolic metric d_{h} derived from the differential

$$ds = \frac{2\sqrt{dx^2 + dy^2}}{1 - x^2 - y^2}.$$

The covering transformations are hyperbolic linear fractional transformations and preserve the hyperbolic metric on the interior of the unit disk. Each covering transformation has exactly two fixed points (one is attracting and the other repelling), and these points lie on the unit circle. A covering transformation T is called *primitive* if $T = S^j$, $S \in \Gamma$, implies that $|j| = 1$.

The lifted flow $\widetilde{\phi}$ can be extended to the closed disk by taking the boundary circle of $\widetilde{M}$, which we denote by S_∞, to be fixed points of $\widetilde{\phi}$. The following definitions can be found in [1, 2, 4]. If there exists $\sigma \in S_\infty$ such that

$|\widetilde{x}t-\sigma|\to 0$ as $t\to\infty$, then we say that $\mathcal{O}^+(\widetilde{x})$ *tends to a point of* S_∞. We write this as $\widetilde{x}t\to\sigma$ as $t\to\infty$. Suppose $\widetilde{x}t\to\sigma\in S_\infty$ as $t\to\infty$, and let C be a hyperbolic ray through $\widetilde{x}$ limiting to σ. If there exists $K>0$ such that $d_{\rm h}(\widetilde{x}t,C)<K$ for all $t\ge 0$, then we say that $\mathcal{O}^+(\widetilde{x})$ is the *type of an h-ray*.

We now return to our example. Let $\widetilde{\Sigma}$ be a lift of Σ and let $\widetilde{z}_0,\widetilde{p},\widetilde{q}\tau_\mu$ denote the lifts of $z_0,p,q\tau_\mu$, respectively, contained in $\widetilde{\Sigma}$. Let $\widetilde{C}$ denote the lift of C containing $\widetilde{p}t_1$ and let $\widetilde{z}_1$ denote the lift of z_1 contained in $\widetilde{C}$. Let $\widetilde{\gamma}_0$ and $\widetilde{\gamma}_1$ denote the extended lifts of γ_0 and γ_1 containing $\widetilde{z}_0$ and $\widetilde{z}_1$, respectively.

There exists a primitive transformation $S\subset\Gamma$ with attracting and repelling fixed points α^+ and α^-, respectively, such that $S\widetilde{\gamma}_1=\widetilde{\gamma}_1$. Note that $\widetilde{p}t\to\alpha^+$ as $t\to\infty$. Moreover, $\widetilde{p}t_n\in S^{n-1}\widetilde{C}$ for $n\ge 1$.

There exists a primitive transformation $T\subset\Gamma$ with attracting and repelling fixed points σ^+ and σ^-, respectively, such that $T\widetilde{\gamma}_0=\widetilde{\gamma}_0$ and $\widetilde{z}_0t\to\sigma^+$ as $t\to\infty$. Clearly, $\widetilde{q}t\to\sigma^+$ as $t\to\infty$, and $\widetilde{q}\tau_{\mu+1}\in T^n\widetilde{\Sigma}$ for $n\ge 1$. We will show that for positive t the hyperbolic distance between $\widetilde{q}t$ and $\widetilde{z}_0t$ is unbounded.

LEMMA 4.1. *$\widetilde{p}(-t)\to T^{-1}\alpha^+$ as $t\to\infty$.*

Proof. We first show that the points $\widetilde{q}s$ and $\widetilde{q}\,\widehat{s}$ lie on $S^2\widetilde{C}$. Recall that the loop $[p,pt_3]_\phi\cup(pt_3,qs)_C\cup[qs,q\tau_\mu]_\phi\cup(q\tau_\mu,p)_\Sigma$ is null homotopic. Thus, since $\widetilde{q}\tau_\mu\in\widetilde{\Sigma}$, $\widetilde{p}\in\widetilde{\Sigma}$, and $\widetilde{p}t_3\in S^2\widetilde{C}$, it follows that $\widetilde{q}s\in S^2\widetilde{C}$. Because $[qs,q\widehat{s}]_\phi\cup(q\widehat{s},qs)_C$ is null homotopic and $\widetilde{q}s\in S^2\widetilde{C}$, we also have $\widetilde{q}\,\widehat{s}\in S^2\widetilde{C}$.

There exists a lift of pt_{-2}, say $H\widetilde{p}t_{-2}$, which lies on $S^2\widetilde{C}$. Clearly, $H\neq I$ and $(H\widetilde{p}t_{-2})(-t)\to\alpha^+$ as $t\to\infty$. Since $q\widehat{s}\in S^2\widetilde{C}, H\widetilde{p}t_{-2}\in S^2\widetilde{C}$, $\widetilde{q}\tau_{\mu+1}\in T\widetilde{\Sigma}$, and $[q\tau_{\mu+1},q\widehat{s}]_\phi\cup(q\widehat{s},pt_{-2})_C\cup[pt_{-2},p]_\phi\cup(p,q\tau_{\mu+1})_\Sigma$ is null homotopic, we have $H\widetilde{p}\in T\widetilde{\Sigma}$. Recall that $\widetilde{p}\in\widetilde{\Sigma}$ and thus $T\widetilde{p}\in T\widetilde{\Sigma}$. Hence $H\widetilde{p}=T\widetilde{p}$, and so $(T\widetilde{p})(-t)\to\alpha^+$ as $t\to\infty$. Therefore $\widetilde{p}(-t)\to T^{-1}\alpha^+$ as $t\to\infty$. ∎

LEMMA 4.2. *$\mathcal{O}^+(\widetilde{q})$ is not the type of an h-ray.*

Proof. Since $q\tau_n\to p$ as $n\to\infty$, and $\widetilde{q}\tau_{\mu+n}\in T^n\widetilde{\Sigma}$ for $n\ge 1$, we have $T^{-n}\widetilde{q}\tau_{\mu+n}\to\widetilde{p}$. Since $\alpha^+\neq\sigma^+$, and T is an isometry, the result immediately follows by continuity in initial conditions. ∎

Let $\widetilde{\gamma}_2$ be the extended orbit of $\widetilde{p}$, i.e., $\widetilde{\gamma}_2=\mathcal{O}(\widetilde{p})\cup\alpha^+\cup T^{-1}\alpha^+$. Let J be the Jordan curve determined by the following orbit and section pieces: $\mathcal{O}^+(\widetilde{p})\cup\alpha^+$, $\alpha^+\cup\mathcal{O}^-(T\widetilde{p})$, $(T\widetilde{p},T\widetilde{z}_0)_{T\widetilde{\Sigma}}$, $[T\widetilde{z}_0,\widetilde{z}_0]_{\widetilde{\gamma}_0}$, and $(\widetilde{z}_0,\widetilde{p})_{\widetilde{\Sigma}}$. By applying all powers of T to the curve J we build a region Q whose boundary consists of $\widetilde{\gamma}_0$ and $\bigcup_{n\in\mathbb{Z}}T^n\widetilde{\gamma}_2$. Note that $TQ=Q$, Q is invariant under the flow, and $\mathcal{O}(\widetilde{q}) \subset Q$. Since the α-limit set of q is γ_1, it follows that $\widetilde{q}(-t) \to \sigma^-$ as $t \to \infty$ and $\mathcal{O}^-(\widetilde{q})$ is the type of an h-ray. It is not hard to see that the Hausdorff limit of $\{T^{-n}\mathcal{O}(\widetilde{q})\}$ as $n \to \infty$ is $\bigcup_{n=-\infty}^{\infty} T^n\widetilde{\gamma}_2$.

We summarize the results of this paper with the following theorem.

THEOREM 4.3. *There exists a flow ϕ on a surface M of genus 2 that has the following properties*:

(1) *There exists a periodic point z_0 whose orbit γ_0 is not null homotopic.*

(2) *There exists a simple closed curve of fixed points γ_1 which is not null homotopic and is not homotopic to γ_0.*

(3) *There exists a point p such that $\omega(p) = \alpha(p) = \gamma_1$.*

(4) *There exists an open set Q of points such that for any $q \in Q$ we have $\alpha(q) = \gamma_0$ and $\omega(q) = \mathcal{O}(p) \cup \gamma_1$, and $\omega(q)$ is not locally connected.*

Furthermore, the lift $\widetilde{\phi}$ of ϕ to the Poincaré disk with the hyperbolic metric has the following additional properties:

(5) *If $\widetilde{p}$ is a lift of p, then there exists a point α^+ on the unit circle S_∞ such that $\widetilde{\phi}(\widetilde{p}, t) \to \alpha^+$ as $t \to \infty$, and there exist a primitive covering transformation T and a lift $\widetilde{\gamma}_0$ of the periodic orbit γ_0 such that $T\widetilde{\gamma}_0 = \widetilde{\gamma}_0$ and $\widetilde{\phi}(\widetilde{p}, -t) \to T^{-1}\alpha^+$ as $t \to \infty$.*

(6) *If $\widetilde{q}$ is a lift of $q \in Q$, then there exist points σ^+ and σ^- in S_∞ and a lift $\widetilde{z}_0$ of z such that $\widetilde{\phi}(\widetilde{q}, t) \to \sigma^+$, $\widetilde{\phi}(\widetilde{z}_0, t) \to \sigma^+$, $\widetilde{\phi}(\widetilde{q}, -t) \to \sigma^-$, and $\widetilde{\phi}(\widetilde{z}_0, -t) \to \sigma^-$ as $t \to \infty$. Moreover, $\mathcal{O}^-(\widetilde{q})$ is the type of an h-ray, but $\mathcal{O}^+(\widetilde{q})$ is not the type of an h-ray.*

The basic properties of this example are predicted by the results in [5]. By using multiples of 2 as the period of $H(z)$ to construct a flow on the cylinder, we can construct flows with more points whose behavior is like that of p and have more regions like D to use in attaching handles.

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