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ON SUBRELATIONS OF ERGODIC MEASURED TYPE III EQUIVALENCE RELATIONS

BY

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Dedicated to the memory of Anzelm Iwanik

Abstract. We discuss the classification up to orbit equivalence of inclusions $\mathcal{S} \subset \mathcal{R}$ of measured ergodic discrete hyperfinite equivalence relations. In the case of type III relations, the orbit equivalence classes of such inclusions of finite index are completely classified in terms of triplets consisting of a transitive permutation group G on a finite set (whose cardinality is the index of $\mathcal{S} \subset \mathcal{R}$), an ergodic nonsingular $\mathbb{R}$-flow V and a homomorphism of G to the centralizer of V.

0. Introduction. We consider nonsingular discrete ergodic hyperfinite equivalence relations on a standard measure space. Our concern is to classify pairs $(\mathcal{R}, \mathcal{S})$ where $\mathcal{R}$ is an ergodic equivalence relation and $\mathcal{S} \subset \mathcal{R}$ is a subrelation of finite index (which means that the $\mathcal{R}$-equivalence class of a.e. point consists of finitely many $\mathcal{S}$-classes), up to orbit equivalence. This problem is closely related to the classification of subfactors in von Neumann algebras theory. For a single equivalence relation $\mathcal{R}$ the problem was solved by H. Dye [Dy] and W. Krieger [Kr] in terms of the associated flows. Then, in the case where $\mathcal{R}$ is of type II_1, J. Feldman, C. Sutherland, and R. Zimmer [FSZ] provided a simple classification of ergodic $\mathcal{R}$-subrelations of finite index and normal $\mathcal{R}$-subrelations of arbitrary index. (We remark that in an earlier paper [Ge] M. Gerber classified $\mathcal{R}$-subrelations of finite index in a different—but equivalent—context of finite extensions of ergodic probability preserving transformations.) These results were further extended in [Da1, §4] and [Da2], where quasinormal subrelations of type II_1 were introduced and studied.

Recently, T. Hamachi considered finite index subrelations of a type III_0 equivalence relation $\mathcal{R}$, introduced a system of invariants for orbit equiva-

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lence and claimed that it was complete [Ha]. However, in the present paper we construct orbitally inequivalent subrelations of $\mathcal{R}$ which are not distinguishable by these invariants. Moreover, for an arbitrary type III equivalence relation $\mathcal{R}$, we provide another system of invariants for orbit equivalence of $\mathcal{R}$-subrelations of finite index and show that it is complete. It consists of a transitive subgroup G of permutations on a finite set (whose cardinality equals the index), an ergodic nonsingular $\mathbb{R}$-flow V and a homomorphism l of G to the centralizer of V such that the $l(G)$-quotient of V is conjugate to the associated flow of $\mathcal{R}$. Roughly speaking, Hamachi's invariants "remember" only the range and kernel of l but not l itself and that is why they are not complete. It should be noted that the argument of [Ha] uses a common discrete decomposition for $\mathcal{S}$ and $\mathcal{R}$, a lacunary measure, etc., i.e. modified techniques from [Kr] (see also [HO]). Our approach is different. We apply more recent advances in orbit theory ([FSZ], [GS1], [GS2]), which results in a short argument.

The outline of the paper is as follows. Section 1 contains background on orbit theory. Section 2 begins with the "measurable index theory" and contains our main classification result—Theorem 6. In Section 3 we provide a counterexample to [Ha, Theorem 6.1]. In the final Section 4, the case of type III$_\lambda$ equivalence relations, $0 < \lambda \leq 1$, is considered in more detail. It turns out that our classification invariants have simpler (more explicit) form in this case.

1. Background on orbit theory. Let $(X, \mathfrak{B}, \mu)$ be a standard probability space. Denote by $\mathrm{Aut}(X, \mu)$ the group of its automorphisms, i.e. Borel one-to-one, onto, μ-nonsingular transformations. We do not distinguish between maps which agree on a μ-conull set. Given a Borel discrete μ-nonsingular equivalence relation $\mathcal{R} \subset X \times X$, we endow it with the induced Borel structure and a σ-finite measure $\mu_{\mathcal{R}}$, $d\mu_{\mathcal{R}}(x, y) = d\mu(x)$, $(x, y) \in \mathcal{R}$. Write also

$$[\mathcal{R}] = \{\gamma \in \mathrm{Aut}(X, \mu) \mid (\gamma x, x) \in \mathcal{R} \text{ for } \mu\text{-a.e. } x \in X\},$$
$$N[\mathcal{R}] = \{\theta \in \mathrm{Aut}(X, \mu) \mid (\theta x, \theta y) \in \mathcal{R} \text{ iff } (x, y) \in \mathcal{R}\ \mu_{\mathcal{R}}\text{-a.e.}\}$$

for the *full group* of $\mathcal{R}$ and the *normalizer* of $[\mathcal{R}]$ respectively. For a countable subgroup Γ of $\mathrm{Aut}(X, \mu)$, we denote by $\mathcal{R}_\Gamma$ the Γ-orbital equivalence relation. It is known that each $\mathcal{R}$ is of the form $\mathcal{R}_\Gamma$ (see [FM]). Recall that $\mathcal{R}$ is *hyperfinite* if it can be generated by a single automorphism. We assume from now on that $\mathcal{R}$ is *ergodic*, i.e. every $\mathcal{R}$-saturated Borel subset is either μ-null or μ-conull.

Let G be a locally compact second countable (l.c.s.c.) group, 1_G the identity of G and λ_G a right Haar measure on G. A Borel map $\alpha : \mathcal{R} \to G$

is a (1-)*cocycle* of $\mathcal{R}$ if

$$\alpha(x,y)\alpha(y,z)=\alpha(x,z) \quad \text{for a.e. } (x,y),(y,z)\in\mathcal{R}.$$

Two cocycles, $\alpha,\beta:\mathcal{R}\to G$, are *cohomologous* ($\alpha\approx\beta$) if

$$\alpha(x,y)=\phi(x)^{-1}\beta(x,y)\phi(y) \quad \text{for } \mu_{\mathcal{R}}\text{-a.e. } (x,y),$$

where $\phi: X\to G$ is a Borel function (we call it a *transfer function*). A cocycle is a *coboundary* if it is cohomologous to a trivial one. The set of all $\mathcal{R}$-cocycles with values in G will be denoted by $Z^1(\mathcal{R},G)$. Let $\mathcal{R}=\mathcal{R}_\Gamma$. There is a cocycle $\varrho\in Z^1(\mathcal{R},G)$ such that

$$\varrho(x,\gamma x)=\log\frac{d\mu\circ\gamma}{d\mu}(x)$$

for all $\gamma\in\Gamma$ at a.e. $x\in X$. It is called the *Radon–Nikodym cocycle* of $\mathcal{R}$. Notice that it is independent of the particular choice of Γ. If ϱ is a coboundary then $\mathcal{R}$ is *of type II*. Otherwise $\mathcal{R}$ is *of type III*. Given $\alpha\in Z^1(\mathcal{R},G)$, we denote by α_0 the "double" cocycle $\alpha\times\varrho\in Z^1(\mathcal{R},G\times\mathbb{R})$.

Recall that α and β are *weakly equivalent* if $\alpha\approx\beta\circ\theta$ for a transformation $\theta\in N[\mathcal{R}]$. Clearly, α and β are weakly equivalent if and only if the double cocycles α_0 and β_0 are. Given $\alpha\in Z^1(\mathcal{R},G)$, we define an equivalence relation $\mathcal{R}(\alpha)$ on $(X\times G,\mu\times\lambda_G)$ by setting $(x,g)\sim(y,h)$ if $(x,y)\in\mathcal{R}$ and $h=g\alpha(x,y)$. It is called the α-*skew product extension* of $\mathcal{R}$. If the $\mathcal{R}(\alpha)$-partition is measurable (i.e. admits a measurable cross-section) then α is called *transient*. Otherwise α is *recurrent*. By [Sc], α is recurrent if and only if α_0 is. We say that α *has dense range* in G if $\mathcal{R}(\alpha)$ is ergodic. It then follows that α is recurrent.

Next, we define a Borel action V_α of G on $(X\times G,\mu\times\lambda_G)$ by setting $V_\alpha(h)(x,g)=(x,hg)$. Since $V_\alpha\in N[\mathcal{R}(\alpha)]$, it induces an automorphism, say $W_\alpha(h)$, on the measure space of $\mathcal{R}(\alpha)$-ergodic components. Moreover, $G\ni h\mapsto W_\alpha(h)$ is an ergodic G-action on this space. It is called the *Mackey action* of G associated to α. If two cocycles α and β are weakly equivalent, then they are either both transient or both recurrent and the associated Mackey G-actions W_α and W_β are conjugate. We call $\mathbb{R}$-actions *flows*.

THEOREM 1 (Golodets–Sinel'shchikov, [GS1], [GS2]). (i) *Let $\mathcal{R}$ be an ergodic hyperfinite equivalence relation on (X,μ) and $\alpha,\beta\in Z^1(\mathcal{R},G)$ recurrent cocycles. If the Mackey $G\times\mathbb{R}$-actions W_{α_0} and W_{β_0} are conjugate then α and β are weakly equivalent.*

(ii) *Given an ergodic $G\times\mathbb{R}$-action V, there exist a hyperfinite ergodic equivalence relation $\mathcal{R}$ on (X,μ) and a recurrent cocycle $\alpha\in Z^1(\mathcal{R},G)$ such that V is conjugate to W_{α_0}.*

2. Subrelations of type III equivalence relations. Let $\mathcal{S}$ be an ergodic subrelation of $\mathcal{R}$. Then there exist $N \in \mathbb{N} \cup \{\infty\}$ and Borel functions $\{\phi_j : X \to X \mid 0 \le j < N\}$ such that $\{\mathcal{S}[\phi_j(x)] \mid 0 \le j < N\}$ is a partition of $\mathcal{R}[x]$, where $\mathcal{R}[x]$ (resp. $\mathcal{S}[x]$) stands for the $\mathcal{R}$- (resp. $\mathcal{S}$-) class of x [FSZ]. The N is called the *index* of $\mathcal{S}$ in $\mathcal{R}$ and the $\{\phi_j\}_j$ are *choice functions* for $\mathcal{S}$. From now on we assume that $\operatorname{ind}\mathcal{S} := N$ is finite. Denote by $\Sigma(J)$ the full permutation group on the set $J := \{0, 1, \dots, N-1\}$ and define a cocycle $\sigma \in Z^1(\mathcal{R}, \Sigma(J))$ by setting $\sigma(x, y)(i) = j$ if $\mathcal{S}[\phi_i(y)] = \mathcal{S}[\phi_j(x)]$. Notice that although choice functions are nonunique, the cohomology class of σ is independent of their particular choice and is an invariant of $\mathcal{S}$. According to [FSZ], σ (or its cohomology class) is called the *index cocycle* of $\mathcal{S}$. Given a cocycle $\alpha \in Z^1(\mathcal{R}, \Sigma(J))$, we put

$$\mathcal{R} \times_\alpha J = \{(x, j, y, k) \in X \times J \times X \times J \mid (x, y) \in \mathcal{R} \text{ and } k = \sigma(x, y)[j]\}.$$

Clearly, $\mathcal{R} \times_\alpha J$ is a $(\mu \times \lambda_J)$-nonsingular discrete equivalence relation on $X \times J$, where λ_J is the "counting" measure on J. We set

$$Z^1_{\text{ind}} = \{\alpha \in Z^1(\mathcal{R}, \Sigma(J)) \mid \mathcal{R} \times_\alpha J \text{ is ergodic}\}.$$

Two subrelations $\mathcal{S}_1, \mathcal{S}_2$ of $\mathcal{R}$ are said to be $\mathcal{R}$-*conjugate* if $\mathcal{S}_1 = (\theta \times \theta)\mathcal{S}_2$ for a transformation $\theta \in N[\mathcal{R}]$. We recall some fundamental facts on subrelations from [FSZ]:

THEOREM 2. *Let $\mathcal{R}$ be a discrete ergodic hyperfinite equivalence relation and $\mathcal{S} \subset \mathcal{R}$ an ergodic subrelation with $\operatorname{ind}\mathcal{S} = N$. Then every index cocycle of $\mathcal{S}$ belongs to $Z^1_{\text{ind}}(\mathcal{R}, \Sigma(J))$. Conversely, for each $\sigma \in Z^1_{\text{ind}}(\mathcal{R}, \Sigma(J))$, there is an ergodic subrelation $\mathcal{S} \subset \mathcal{R}$ with $\operatorname{ind}\mathcal{S} = N$ such that σ is an index cocycle of $\mathcal{S}$. Two ergodic subrelations $\mathcal{S}_1, \mathcal{S}_2$ of finite index in $\mathcal{R}$ are $\mathcal{R}$-conjugate if and only if $\operatorname{ind}\mathcal{S}_1 = \operatorname{ind}\mathcal{S}_2$ and their index cocycles are weakly equivalent.*

Thus the classification of ergodic $\mathcal{R}$-subrelations of index N up to $\mathcal{R}$-conjugacy is equivalent to the classification of cocycles from $Z^1_{\text{ind}}(\mathcal{R}, \Sigma(J))$ up to weak equivalence.

THEOREM 3. *Let $\sigma \in Z^1_{\text{ind}}(\mathcal{R}, \Sigma(J)$. Then there exists a transitive subgroup $G \subset \Sigma(J)$ and a cocycle $\sigma' : \mathcal{R} \to G$ with dense range in G such that $\sigma' \approx \sigma$. Two cocycles $\sigma_1 : \mathcal{R} \to G_1$ and $\sigma_2 : \mathcal{R} \to G_2$ with dense ranges in transitive subgroups G_1 and G_2 of $\Sigma(J)$ are weakly equivalent as elements of $Z^1(\mathcal{R}, \Sigma(J))$ if and only if there is $g \in \Sigma(J)$ such that $G_1 = gG_2g^{-1}$ and the cocycles σ_1 and $\operatorname{Ad}_g \circ \sigma_2$ are weakly equivalent as elements of $Z^1(\mathcal{R}, G_1)$, where Ad_g is the inner automorphism of $\Sigma(J)$ generated by g.*

Proof. The existence of G and σ' with the required properties follows from [Zi, Corollary 3.8]. Note that G acts transitively on J because $\mathcal{R} \times_\sigma J$ (and hence $\mathcal{R} \times_{\sigma'} J$) is ergodic. The last statement of the theorem can be easily deduced from [Zi, the argument of Theorem 6.1], where it was proved in a slightly weaker form: with "cohomologous" instead of "weakly equivalent". Observe also that although the theorems from [Zi] to which we refer were stated there only in the type II, i.e. measure preserving, case they also hold for the type III case with the same argument. ∎

Note that every cocycle of $\mathcal{R}$ with values in a finite (or compact) group is recurrent. From Theorems 1 and 3 we deduce

Corollary 4. *Let $\sigma_1 : \mathcal{R} \to G_1$ and $\sigma_2 : \mathcal{R} \to G_2$ be two cocycles with dense ranges in transitive subgroups G_1 and G_2 of $\Sigma(J)$ respectively. Denote by $W_{(\sigma_1)_0}$ and $W_{(\sigma_2)_0}$ the Mackey $G_1 \times \mathbb{R}$- and $G_2 \times \mathbb{R}$-actions associated to the double cocycles $(\sigma_1)_0$ and $(\sigma_2)_0$ respectively. Then σ_1 and σ_2 are weakly equivalent as elements of $Z^1(\mathcal{R}, \Sigma(J))$ if and only if there is $g \in \Sigma(J)$ such that $G_1 = gG_2g^{-1}$ and the $G_2 \times \mathbb{R}$-actions $W_{(\sigma_2)_0}$ and $W_{(\sigma_1)_0} \circ (\mathrm{Ad}_g \times \mathrm{Id})$ are conjugate.*

Every measured $G\times\mathbb{R}$-action W on a space (Ω, ν) determines a measured flow V acting on the same measure space and a group homomorphism l from G to the centralizer $C(W)$ of W as follows: $V(t) = W(1_G, t)$, $l(g) = W(g, 0)$ for all $t \in \mathbb{R}$ and $g \in G$. Recall that

$$C(W) = \{R \in \mathrm{Aut}(\Omega, \nu) \mid RW(g,t) = W(g,t)R \text{ for all } t \in \mathbb{R} \text{ and } g \in G\}.$$

We call (V, l) the *constituents* of W.

Let an $\mathcal{R}$-cocycle σ take values and have dense range in a transitive subgroup $G \subset \Sigma(J)$. Denote by (V_σ, l_σ) the constituents of the Mackey $G \times \mathbb{R}$-action W_{σ_0} associated to the double cocycle σ_0. It is easy to verify (and well known) that the $l_\sigma(G)$-quotient of V_σ, i.e. the restriction of V_σ to the subalgebra of $l(G)$-invariant measured subsets, is conjugate to W_σ. On the other hand, the $V_\sigma(\mathbb{R})$-quotient of l_σ is a singleton, since σ has dense range in G and hence the associated Mackey action is trivial. It follows that V_σ is ergodic. We illustrate these with the commutative diagram

$$\begin{array}{ccccc} X \times G & \longleftarrow & X \times G \times \mathbb{R} & \longrightarrow & X \times \mathbb{R} \\ \downarrow & & \downarrow & & \downarrow \\ \{\bullet\} & \longleftarrow & \Omega_0 & \longrightarrow & \Omega \end{array}$$

where $\{\bullet\}$, Ω_0, and Ω stand for the spaces of the Mackey actions associated to σ, σ_0, and the Radon–Nikodym cocycle of $\mathcal{R}$ respectively; the vertical arrows represent the corresponding ergodic decompositions (see §1); the upper horizontal arrows are natural projections, and the lower arrows are determined by the universality of the "middle" ergodic decomposition.

Definition 5. Let V_i be an ergodic nonsingular flow on a measure space (Ω_i, ν_i), G_i a transitive subgroup of $\Sigma(J)$, and $l_i : G_i \to C(V_i)$ a group homomorphism, $i = 1, 2$. We say that the triplets (V_1, G_1, l_1) and (V_2, G_2, l_2) are *conjugate* if there is a nonsingular isomorphism $\xi : \Omega_2 \to \Omega_1$ and $g \in \Sigma(J)$ such that $G_1 = gG_2g^{-1}$, $V_1(t) = \xi V_2(t)\xi^{-1}$ and $l_1(\mathrm{Ad}_g(g_2)) = \xi l_2(g_2)\xi^{-1}$ for all $t \in \mathbb{R}$ and $g_2 \in G_2$.

Now we are ready to record our main classification result.

THEOREM 6. *Let $\mathcal{R}$ be an ergodic type III hyperfinite equivalence relation on $(X, \mathfrak{B}, \mu)$, and W_ϱ its associated flow (ϱ stands for the Radon–Nikodym cocycle).*

(i) *With every ergodic subrelation of index N, we can associate a triplet (V, G, l) consisting of an ergodic flow V, a transitive subgroup $G \subset \Sigma(J)$ and a homomorphism $l : G \to C(V)$ such that the $l(G)$-quotient flow of V is conjugate to W_ϱ.*

(ii) *Conversely, given such a triplet, there exists an ergodic subrelation $\mathcal{S} \subset \mathcal{R}$, $\operatorname{ind} \mathcal{S} = N$, whose associated triplet is as given.*

(iii) *Two ergodic $\mathcal{R}$-subrelations of index N are $\mathcal{R}$-conjugate if and only if their associated triplets are conjugate.*

Proof. (i) follows from Theorems 2, 3 and the remark before Definition 5.

(ii) Given a triplet (V, G, l), we consider a $G \times \mathbb{R}$-action W whose constituents are (V, l). By Theorem 1 there are an ergodic hyperfinite equivalence relation $\mathcal{R}'$ on $(X, \mathfrak{B}, \mu)$ and a cocycle $\sigma' : \mathcal{R}' \to G$ such that W is conjugate to the Mackey $G \times \mathbb{R}$-action associated to the double cocycle σ'_0. It is clear that the associated flow of $\mathcal{R}'$ is conjugate to the $l(G)$-quotient flow of V. By the assumptions on (V, G, l), this flow is conjugate to W_ϱ. It follows from the Krieger theorem [Kr], [FM] that $\mathcal{R}$ and $\mathcal{R}'$ are orbit equivalent and hence we may identify them. Next, since V is ergodic, σ' has dense range in G. But G is a transitive subgroup of J-permutations and this implies that $\sigma' \in Z^1_{\mathrm{ind}}(\mathcal{R}, \Sigma(J))$. It remains to apply Theorem 2.

(iii) follows from Theorem 2 and Corollary 4. ∎

3. On Hamachi's invariants. Let an $\mathcal{R}$-cocycle σ take values and have dense range in a transitive subgroup G of $\Sigma(J)$. Denote by H the G-stability group of 0, i.e. $H = \{g \in G \mid g[0] = 0\}$. Then $H \subset G$ is irreducible, i.e. H contains no nontrivial G-normal subgroups. If a subgroup $G_1 \subset \Sigma(J)$ is conjugate to G, then there exists $k \in \Sigma(J)$ such that $G_1 = kGk^{-1}$ and $k[0] = 0$ (recall that G is transitive). It follows that $H_1 = kHk^{-1}$, where H_1 is the G_1-stability group of 0. Thus the conjugacy classes of transitive subgroups of $\Sigma(J)$ are in one-to-one correspondence with the isomorphism classes of irreducible pairs of finite groups $H \subset G$ such that the cardinality of G/H is N. (We say that two pairs $H \subset G$ and $H' \subset G'$ are *isomorphic* if there is an isomorphism of G onto G' taking H onto H'.)

Let (V, G, l) be a triplet as in Theorem 6. Denote by G_0 the kernel of l and by (Ω, ν) the measure space of W_ϱ. Then V is a G/G_0-extension of W_ϱ, i.e. we may assume without loss in generality that V is defined on the space $(\Omega_0, \nu_0) := (\Omega \times G/G_0, \nu \times \lambda_{G/G_0})$ as follows:

$$(*) \qquad V(t)(\omega, h) = (W_\varrho(t)\omega, h\alpha(\omega, t)),$$

where λ_{G/G_0} is Haar measure on G/G_0 and $\alpha : \Omega \times \mathbb{R}$ a measurable W-cocycle, i.e.

$$\alpha(\omega, t_1 + t_2) = \alpha(\omega, t_1)\alpha(W_\varrho(t_1)\omega, t_2)$$

at a.e. $\omega \in \Omega$ for all $t_1, t_2 \in \mathbb{R}$. (Do not confuse cocycles of group actions with cocycles of equivalence relations.) Denote by $\pi : \Omega_0 \ni (\omega, h) \mapsto \omega \in \Omega$ the canonical projection. Then $\pi V(t) = W_\varrho(t)\pi$ for all $t \in \mathbb{R}$. It is convenient to use the notation $\pi : V \xrightarrow{G/G_0} W_\varrho$.

Recall that two group extensions $\pi : V \xrightarrow{G} W$ and $\pi' : V \xrightarrow{G'} W'$ are *conjugate* if there are nonsingular isomorphisms $\psi : (\Omega_0, \nu_0) \to (\Omega'_0, \nu'_0)$ and $\phi : (\Omega, \nu) \to (\Omega', \nu')$ such that $\phi W(t)\phi^{-1} = W'(t)$, $\psi V(t)\psi^{-1} = V'(t)$, and $\psi\pi\psi^{-1} = \pi'$. This implies that G and G' are isomorphic.

Thus with a given triplet (V, G, l), we associate a system $(G, H, G_0, \pi : V \xrightarrow{G/G_0} W_\varrho)$ consisting of an irreducible pair of finite groups $H \subset G$, a normal subgroup $G_0 \subset G$ and a G/G_0-extension of W_ϱ. We shall call it an $\mathcal{H}$-*system* (see [Ha]).

Definition 7 (see [Ha, Definition 6.1]). Two $\mathcal{H}$-systems

$$(G, H, G_0, \pi : V \xrightarrow{G/G_0} W_\varrho) \quad \text{and} \quad (G', H', G'_0, \pi' : V' \xrightarrow{G'/G'_0} W_\varrho)$$

are *equivalent* if there is an isomorphism $\varrho : G \to G'$ such that $\varrho(H) = H'$, $\varrho(G_0) = G'_0$ and the extensions π and π' are conjugate.

It is easy to see that if two triplets are conjugate then the associated $\mathcal{H}$-invariants are equivalent. It is claimed in [Ha] that the converse also holds, which implies that $\mathcal{R}$-nonconjugate ergodic $\mathcal{R}$-subrelations of finite index have inequivalent $\mathcal{H}$-invariants. We present a counterexample to this statement.

Example 8. Let Σ_3 be the permutation group of $\{0, 1, 2\}$ and A_5 the group of even permutations of $\{0, 1, \ldots, 5\}$. We put $H := (\Sigma_3)^5 \rtimes A_5$ and $G := H^2$. It is easy to verify that $Z(H)$, the center of H, is trivial but $\operatorname{Out} H$, the group of outer automorphisms of H, is nontrivial. Denote by $\Sigma(H)$ the permutation group of H and define a homomorphism $b : G \to \Sigma(H)$ by setting $b(h_1, h_2)[h] = h_1 h h_2^{-1}$, $h \in H$. Since the kernel of b is isomorphic to $Z(H)$, b is an embedding. It is obvious that G (or, more precisely, $b(G)$) acts transitively on H. Denote by G_0 the G-stability group of 1_H. Clearly, $G_0 = \{(h,h) \mid h \in H\}$. Define an automorphism κ of G by setting $\kappa(h_1,h_2) = (h_1, \tau(h_2))$, where τ is a noninner automorphism of H.

We claim that κ cannot be extended to an automorphism of $\Sigma(H)$. Suppose the contrary: there exists $k \in \Sigma(H)$ such that $\kappa(g) = kgk^{-1}$ for all $g \in G$. (Recall that every automorphism of $\Sigma(H)$ is inner.) Put $h_0 := k[1_H] \in H$. Then $\kappa(g)[h_0] = h_0$ for all $g \in G_0$. Since G acts transitively on H, we deduce that $\kappa(G_0) = g_0 G_0 g_0^{-1}$ for an element $g_0 \in G$ with $g_0[0] = h_0$. Thus $\bigcup_{h\in H}(h, \tau(h)) = \bigcup_{h \in H}(h, h_1 h h_1^{-1})$ for some $h_1 \in H$. It follows that τ is an inner automorphism of H, a contradiction.

Let W be an ergodic properly nontransitive $\mathbb{R}$-flow on (Ω, ν) with trivial centralizer, i.e. $C(W) = W(\mathbb{R})$. Take a cocycle α of W with values in G such that the flow V determined by $(*)$ with G_0 trivial is ergodic. Define a one-to-one homomorphism $l : G \to C(V)$ by setting

$$l(g')(\omega, g) = (\omega, g'g) \quad \text{for all } (\omega, g) \in \Omega \times G,$$

and put $l_1 = l \circ \kappa$. We claim that the triplets (V, G, l) and (V, G, l_1) are not conjugate. Suppose the contrary: there exist $\xi \in C(V)$ and $s \in \Sigma(H)$ such that

$$(**) \qquad l \circ \mathrm{Ad}_s(g) = \xi l(\kappa(g))\xi^{-1} \quad \text{for all } g \in G.$$

Since ξ passes through the natural projection $\Omega \times G \to \Omega$, it is well known (see for example [Da1, Theorem 5.3 and §6]) that ξ is of the form $\xi(\omega, g) = (\zeta\omega, d(g)f(x))$ for a transformation $\zeta \in C(W)$, a G-automorphism d, and a measurable map $f : X \to G$. Hence $\zeta \in W(\mathbb{R})$. It follows from [Da1, Lemma 5.2 and §6] that d is inner. On the other hand, it is easy to verify that $\xi l(g) \xi^{-1} = l(d(g))$ for all $g \in G$. We deduce from $(**)$ that $l \circ \mathrm{Ad}_s(g) = l \circ d \circ \kappa(g)$ and hence $\mathrm{Ad}_s = d \circ \kappa$. This contradicts the fact that κ cannot be extended to a $\Sigma(H)$-automorphism.

Since W is nontransitive, it is the associated flow of a type III_0 ergodic hyperfinite equivalence relation $\mathcal{R}$. By Theorem 6 there are ergodic $\mathcal{R}$-subrelations $\mathcal{S}$ and $\mathcal{S}_1$ of finite index whose associated triplets are (V, G, l) and (V, G, l_1) respectively. It follows that $\mathcal{S}$ and $\mathcal{S}_1$ are not $\mathcal{R}$-conjugate. On the other hand, the $\mathcal{H}$-invariants associated with (V, G, l) and (V, G, l_1) are obviously identical.

4. Case of III_λ equivalence relations, $0 < \lambda \le 1$. If $\mathcal{R}$ is of type III_λ, $0 < \lambda \le 1$, our invariants (see Theorem 6) can be described in a more apparent way.

We first consider the case where $\mathcal{R}$ is of type III_1. Then the associated flow W_ϱ and any ergodic finite group extension V of W_ϱ are trivial. Thus we deduce from Theorem 6

COROLLARY 9. *The $\mathcal{R}$-conjugacy classes of ergodic $\mathcal{R}$-subrelations of index N are in one-to-one correspondence with the conjugacy classes of transitive subgroups of $\Sigma(J)$, $J = \{0, 1, \ldots, N-1\}$.*

Now let $\mathcal{R}$ be of type III_λ, $0 < \lambda < 1$. Then W_ϱ is a transitive periodic flow with period $-\log\lambda$. If V is an ergodic finite group extension of W_ϱ, then there is a nonnegative integer n such that V is a periodic flow with period $-n\log\lambda$ and V is a $\mathbb{Z}/n\mathbb{Z}$-extension of W.

DEFINITION 10. A collection (n, G, l) consisting of a positive integer n, a transitive subgroup $G \subset \Sigma(J)$ and an onto homomorphism $l : G \to \mathbb{Z}/n\mathbb{Z}$ will be called a λ-*triplet*. Two λ-triplets (n, G, l) and (n', G', l') are *conjugate* if $n = n'$ and there is $s \in \Sigma(J)$ with $G = sG's^{-1}$ and $l \circ \mathrm{Ad}_s = l'$.

It is easy to deduce from Theorem 6

COROLLARY 11. *The $\mathcal{R}$-conjugacy classes of ergodic $\mathcal{R}$-subrelations of index N are in one-to-one correspondence with the conjugacy classes of λ-triplets.*

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