COLLOQUIUM MATHEMATICUM

VOL. 84/85 2000 PART 1

A NOTE ON DYNAMICAL ZETA FUNCTIONS FOR S-UNIMODAL MAPS

BY

GERHARD KELLER (ERLANGEN)

Abstract. Let f be a nonrenormalizable S-unimodal map. We prove that f is a Collet–Eckmann map if its dynamical zeta function looks like that of a uniformly hyperbolic map.

1. Introduction. A unimodal map $f:[0,1]\to[0,1]$ is called *S-unimodal* if $f(0)=f(1)=0$ and if it has nonpositive Schwarzian derivative $Sf = f'''/f' - \frac{3}{2}(f''/f')^2$. For such a map set $\varphi(x):=\log|f'(x)|$ and $\varphi_n(x):=\varphi(x)+\varphi(fx)+\ldots+\varphi(f^{n-1}x)$. Let $\mathit{\Pi}_n=\{x\in[0,1]:f^n(x)=x\}$ and define for $t\in\mathbb{R}$ the zeta function

$$\zeta_t(z)=\exp\sum_{n=1}^{\infty}\frac{z^n}{n}\zeta_{n,t}\quad\text{where}\quad\zeta_{n,t}=\sum_{x\in\mathit{\Pi}_n}e^{(t-1)\varphi_n(x)}.$$

Observe that $\zeta_0(z)$ is just the usual dynamical zeta function. Set

$$\lambda_{\mathrm{per}}:=\inf\{|(f^n)'(x)|^{1/n}:n>0,\ x\in\mathit{\Pi}_n\}.$$

Nowicki and Sands [6] proved that $\lambda_{\mathrm{per}}>1$ (i.e. f is *uniformly hyperbolic on periodic orbits*) if and only if f satisfies the *Collet–Eckmann condition* (i.e. there are $C>0$ and $\lambda_{\mathrm{CE}}>1$ such that $|(f^n)'(fc)|\ge C\lambda_{\mathrm{CE}}^n$ for all $n>0$ where c denotes the critical point of f). Extending the transfer operator method used in [1] Keller and Nowicki had previously shown in [4] that the zeta function of a nonrenormalizable S-unimodal map f which satisfies the Collet–Eckmann condition and some additional regularity assumption has the following property:

(1) There are $r>1$ and $t_1>0$ such that $\zeta_t^{-1}(z)$ is analytic in $\{z:|z|<r\}$ if $|t|<t_1$ and for those t the function $\zeta_t^{-1}(z)$ has a unique and simple zero $z(t)\in\{z:|z|<r\}$ with $z(0)=1$.

2000 *Mathematics Subject Classification*: 37C30, 37E05.

Key words and phrases: Collet–Eckmann map, dynamical zeta function.

This work was supported by DFG-grant Ke 514/4-1. The author also gratefully acknowledges the hospitality and financial support of the Mathematics Department at Penn State University, where this research was started.

Both papers, [6] and [4], relied in an essential way on previous work of Nowicki. Using the result of [1] more directly Ruelle proved this theorem in [9] without the additional regularity assumption.

In her review paper [2] Baladi asked whether the converse implication also holds. The main result of the present note is an affirmative answer to this question:

Theorem 1. *Let f be a nonrenormalizable S-unimodal map with at least two periodic orbits. Suppose that there are $r > 1$ and $\tau > 0$ such that $\zeta_t^{-1}(z)$ is analytic in $\{z : |z| < r\}$ for $t = 0$ and $t = -\tau$ and such that for those t the function $\zeta_t^{-1}(z)$ has a unique and simple zero $z(t) \in \{z : |z| < r\}$. Suppose also that $z(0) \geq 1$. Then $\lambda_{\mathrm{per}} > 1$, i.e. f is uniformly hyperbolic on periodic orbits and hence satisfies the Collet–Eckmann condition.*

Therefore we can now say that

- a nonrenormalizable S-unimodal map f with at least two periodic orbits satisfies (1) if and only if it is uniformly hyperbolic on periodic orbits.

Very early statements and conjectures related to this equivalence were made by Takahashi in [7, 10].

Remark 1. 1. Using renormalization theory for unimodal maps (see, e.g., [5]) the result is easily adapted to the renormalizable case.

2. If $f(0) = 0$ is the only periodic point of f and if $f'(0) = 1$ (e.g. $f(x) = x(1-x)$), then $\zeta_t^{-1}(z) = 1 - z$, but f is not uniformly hyperbolic on periodic orbits. Hence the assumption on the existence of a second periodic orbit cannot simply be skipped.

Acknowledgments. I am grateful to Viviane Baladi and Duncan Sands for drawing my attention to the above problem and for many helpful remarks and corrections to an earlier version of this note.

2. Proof of the theorem. We start with some simple remarks. Since all $\zeta_{n,t}$ are real numbers and since $\zeta_{n,t} \leq \zeta_{n,0} \max |f'|^{tn}$, we have $z(t) > 0$ for all t. Indeed, $z(t)$ is the radius of convergence of the series $\sum_{n=1}^{\infty}(z^n/n)\zeta_{n,t}$.

For $t \in \{0, -\tau\}$ let

$$h_t(z) = \sum_{n=1}^{\infty} \frac{z^n}{n}(\zeta_{n,t} - z(t)^{-n}).$$

Then, by assumption,

$$\left(1 - \frac{z}{z(t)}\right)\zeta_t(z) = \exp h_t(z)$$

is analytic and nonzero in $\{z : |z| < r\}$. So the existence theorem for holomorphic logarithms (see e.g. [8, Ch. 9, §3.1]) guarantees the existence of an analytic function $\widetilde{h}_t(z)$ defined on $\{z : |z| < r\}$ such that $(1 - z/z(t))\zeta_t(z) = \exp \widetilde{h}_t(z)$. It follows that $(2\pi\imath)^{-1}(h_t(z) - \widetilde{h}_t(z)) = k \in \mathbb{Z}$ for z in a neighbourhood of 0. Hence $\widetilde{h}_t(z) + 2k\pi\imath$ is an analytic extension of $h_t(z)$ to all of $\{z : |z| < r\}$ so that

$$|\zeta_{n,t} - z(t)^{-n}| = O(r^{-n}) \tag{2}$$

for $t \in \{0, -\tau\}$ (see [8, Ch. 8, §1.5]).

If f had a stable periodic point $\overline{x} = f^p(\overline{x})$ with $\varrho := |(f^p)'(\overline{x})|^{1/p} < 1$, then we would have $\zeta_{p,0} \ge \varrho^{-p}$ and hence $z(0) \le \varrho < 1$, contradicting our assumption. Hence f has no stable periodic orbit and all $\varphi_n(x)$ involved in the definition of $\zeta_{n,t}$ are nonnegative. Therefore $t \mapsto \zeta_{n,t}$ is nondecreasing for each $n > 0$, and since $|z(0)|, |z(-\tau)| < r$ by assumption, it follows from (2) that $t \mapsto z(t)$ is nonincreasing in $[-\tau, 0]$. Hence $z(-\tau) \ge z(0) \ge 1$.

Suppose now that $z(-\tau) > 1$ and fix $\alpha, \beta > 0$ such that $e^{\tau\alpha} = z(-\tau)^\beta$. Then, for any $\overline{x} \in \Pi_n$ with $\varphi_n(\overline{x}) < n\alpha$,

$$\begin{aligned} e^{-\varphi_n(\overline{x})} &\le \sum_{x \in \Pi_n} e^{-\varphi_n(x)} \cdot 1_{\{\varphi_n(x) < n\alpha\}} \le \sum_{x \in \Pi_n} e^{-\varphi_n(x) + \tau(n\alpha - \varphi_n(x))} \\ &= e^{\tau n\alpha} \cdot \zeta_{n,-\tau} = e^{\tau n \alpha} \cdot (z(-\tau)^{-n} + O(r^{-n})) \\ &= O(z(-\tau)^{-n(1-\beta)}). \end{aligned}$$

Hence, for such $\overline{x}$,

$$(1-\beta)\log z(-\tau) \le n^{-1}\varphi_n(\overline{x}) < \alpha = \beta \frac{\log z(-\tau)}{\tau}.$$

Choose $\beta = \tau/(1+\tau)$. Then $\alpha = (1-\beta)\log z(-\tau)$ and the upper and lower bounds in this chain of inequalities coincide so that no $\overline{x} \in \Pi_n$ can satisfy these estimates. It follows that $n^{-1}\varphi_n(\overline{x}) \ge \alpha$, i.e. $\lambda_{\mathrm{per}} \ge e^\alpha = z(-\tau)^{1/(1+\tau)} > 1$.

It remains to consider the case $z(-\tau) = z(0) = 1$. We first exclude the possibility that f has a neutral periodic point $\overline{x} = f^p(\overline{x})$, p the minimal period of $\overline{x}$, $\varphi_p(\overline{x}) = 0$. If such a point existed, then

$$\begin{aligned} \sum_{n=1}^\infty \frac{z^n}{n} \sum_{x \in \Pi_n \cap \mathrm{orbit}(\overline{x})} e^{-\varphi_n(x)} &= \sum_{k=1}^\infty \frac{z^{pk}}{pk} \sum_{x \in \mathrm{orbit}(\overline{x})} e^{-\varphi_{pk}(x)} = \sum_{k=1}^\infty \frac{z^{pk}}{k} e^{-k\varphi_p(\overline{x})} \\ &= -\log(1 - z^p) \end{aligned}$$

so that

$$\zeta_0^{-1}(z) = (1 - z^p) \cdot \exp(-g(z)), \quad \text{where} \quad g(z) = \sum_{n=1}^\infty \frac{z^n}{n} \underbrace{\sum_{x \in \Pi_n \setminus \mathrm{orbit}(\overline{x})} e^{-\varphi_n(x)}}_{=:\zeta'_{n,0}}.$$

As $0 \le \zeta'_{n,0} \le \zeta_{n,0}$ for all n, it follows that $\varrho(g) := (\limsup_{n\to\infty} |n^{-1}\zeta'_{n,0}|^{1/n})^{-1} \ge z(0) = 1$ and $z = \varrho(g)$ is a singular point of $g(z)$ [8, Ch. 8, §1.5]. On the other hand, the function ζ_0^{-1} has a simple zero at $z = 1$ and is analytic in the disk with radius $r > 1$ by assumption so that $\exp(-g(z))$ is analytic and nonzero in a neighbourhood of $z = 1$. Invoking the existence theorem for holomorphic logarithms once more it follows that $z = 1$ is not a singular point of $g(z)$. Hence $\varrho(g) > 1$ so that $|(f^n)'(x)| = e^{\varphi_n(x)} \ge \varrho(g)^n > 1$ for all $x \in \Pi_n \setminus \mathrm{orbit}(\overline{x})$. But this is not true for any nonrenormalizable S-unimodal map with neutral periodic orbit that has at least one other periodic orbit. One way to see this is the following. Using Hofbauer's Markov extension it is easy to construct periodic orbits that follow for a long time the neutral periodic orbit, do something different for a short time interval (e.g. follow a second periodic orbit), return to the neutral periodic orbit etc. In this way one obtains periodic orbits with Lyapunov exponent as close to zero as one wishes.

Now we can assume that f has neither stable nor neutral periodic orbits, and it follows from [5, Ch. IV, Theorem B′] that $\delta := \inf\{\varphi_n(x) : n \ge 1, x \in \Pi_n\} > 0$. We conclude that for $\overline{x} \in \Pi_n$,

$$e^{-\varphi_n(\overline{x})}(1 - e^{-\tau\delta}) \le e^{-\varphi_n(\overline{x})}(1 - e^{-\tau\varphi_n(\overline{x})}) \le \sum_{x\in\Pi_n} e^{-\varphi_n(x)}(1 - e^{-\tau\varphi_n(x)}) = \zeta_{n,0} - \zeta_{n,-\tau}.$$

As $|\zeta_{n,-\tau} - \zeta_{n,0}| = O(r^{-n})$ in view of (2), it follows that $e^{-\varphi_n(\overline{x})} = O(r^{-n})$, i.e. $\lambda_{\mathrm{per}} \ge r > 1$ also in this case. ∎

Remark 2. It seems that the case $z(-\tau) = z(0) = 1$ in the above proof does not really occur. Recall that f is a Collet–Eckmann map if it is uniformly hyperbolic on periodic points [6]. Assume now that f satisfies the additional regularity assumptions from [4], e.g. let f be a polynomial or let $f(x) = a(1 - |2x - 1|^\ell)$ for some real $\ell > 1$. The results of that paper show that $z(t)^{-1}$ is the spectral radius of a suitable transfer operator associated with f and t and that $z(t)$ is a real-analytic function of t in a neighbourhood of $t = 0$. It then follows from [3, Proposition 4.5] that $-\log z(t)$ is the pressure $P((t-1)\varphi)$ of the function $(t-1)\varphi$, where $P((t-1)\varphi) = \sup\{h(\mu) + (t-1)\mu(\varphi) : \mu = \mu\circ f^{-1}\}$. Hence, if $z(-\tau) = z(0) = 1$ for some τ close to zero, then $P((t-1)\varphi) = 0$ for t in a neighbourhood of $t = 0$. By [3, Theorems 5.1, 6.1] the unique absolutely continuous invariant measure μ_1 for f is the unique equilibrium state for $-\varphi$, and the pressure $P((t-1)\varphi)$ can be constant only if $\mu_1(\varphi) = 0$. But absolutely continuous invariant measures have positive exponent, a contradiction.

REFERENCES

[1] V. Baladi and G. Keller, *Zeta-functions and transfer operators for piecewise monotone transformations*, Comm. Math. Phys. 127 (1990), 459–478.

[2] V. Baladi, *Periodic orbits and dynamical spectra*, Ergodic Theory Dynam. Systems 18 (1998), 255–292.

[3] H. Bruin and G. Keller, *Equilibrium states for S-unimodal maps*, ibid. 18 (1998), 765–789.

[4] G. Keller and T. Nowicki, *Fibonacci maps re(aℓ)visited*, ibid. 15 (1995), 99–120.

[5] W. de Melo and S. van Strien, *One-Dimensional Dynamics*, Springer, 1993.

[6] T. Nowicki and D. Sands, *Non-uniform hyperbolicity and universal bounds for S-unimodal maps*, Invent. Math. 132 (1998), 633–680.

[7] Y. Oono and Y. Takahashi, *Chaos, external noise and Fredholm theory*, Progr. Theor. Phys. 63 (1980), 1804–1807.

[8] R. Remmert, *Theory of Complex Functions*, Grad. Texts in Math. 122, Springer, New York, 1991.

[9] D. Ruelle, *Analytic completion for dynamical zeta functions*, Helv. Phys. Acta 66 (1993), 181–191.

[10] Y. Takahashi, *An ergodic-theoretical approach to the chaotic behaviour of dynamical systems*, Publ. R.I.M.S. Kyoto Univ. 19 (1983), 1265–1282.

Mathematisches Institut
Universität Erlangen-Nürnberg
Bismarckstr. $1\frac{1}{2}$
D-91054 Erlangen, Germany
E-mail: keller@mi.uni-erlangen.de

Received 15 July 1999 (3790)