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ORDERED K-THEORY AND MINIMAL SYMBOLIC DYNAMICAL SYSTEMS

BY

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In honor of the late Professor Anzelm Iwanik
for whom we had the greatest respect
both as a mathematician and as a person

Abstract. Recently a new invariant of K-theoretic nature has emerged which is potentially very useful for the study of symbolic systems. We give an outline of the theory behind this invariant. Then we demonstrate the relevance and power of the invariant, focusing on the families of substitution minimal systems and Toeplitz flows.

1. Introduction. For the study of Cantor minimal systems—in particular, minimal symbolic systems—entropy and spectral invariants have been used extensively. Recently, however, a new invariant has emerged which is independent of the two former. This new invariant is of ordered (sic) K-theoretic nature, and is closely related to the orbit structure of the systems. Furthermore, the invariant is effectively computable for important families of symbolic systems—for instance, for the family of substitution minimal systems.

The crucial tool in proving our results—connecting the dynamics with the K-theoretic invariant—is a model theorem for Cantor minimal systems. The key concept is that of a *Bratteli diagram*, which is a special type of an infinite graph. Originally, Bratteli diagrams were introduced to encode the embedding scheme of an ascending sequence of multimatrix algebras (or finite-dimensional C^*-algebras), thereby providing a tool to tell when such inductive limits (called AF-algebras) are isomorphic [2]. This could most conveniently be formulated in terms of the so-called *dimension group* associated with the Bratteli diagram [9]. Subsequently, its K-theoretic underpinning was realized, namely as the K_0-group of the associated AF-algebra, endowed with an ordering. The dynamical interpretation of the Bratteli diagram originated with Vershik [25]. By a careful modification of Vershik's approach, Herman, Putnam and Skau proved the basic model theorem for

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Cantor minimal systems—employing *ordered Bratteli diagrams* with the associated lexicographic map (or *Vershik map*) [16].

In the sequel we first define the key concepts and formulate some relevant results. We then apply this theory to the families of substitution minimal systems and Toeplitz flows. As a general reference for the underlying theory we refer to [16], [11] (cf. also [13]). We refer to [6] and [10] for results on substitution minimal systems, and to [12] for results on Toeplitz flows.

2. Bratteli diagrams and dimension groups

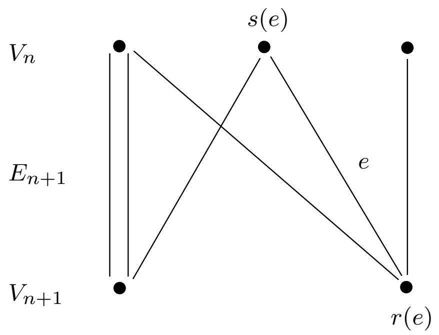
2.1. *Bratteli diagrams*

DEFINITION 1. A *Bratteli dagram* is an infinite directed graph (V, E), where V is the vertex set and E is the edge set. These sets are partitioned into non-empty disjoint finite sets $V = V_0 \cup V_1 \cup \dots$ and $E = E_1 \cup E_2 \cup \dots$, where $V_0 = \{v\}$ is a one-point set. There are two maps $r, s : E \to V$ such that $r(E_n) \subseteq V_n$ and $s(E_n) \subseteq V_{n-1}$ for $n \in \mathbb{N}$. Furthermore, $s^{-1}(v) \neq \emptyset$ for all $v \in V$ and $r^{-1}(v) \neq \emptyset$ for all $v \in V \setminus V_0$. We call r and s the *range map* and the *source map* of (V, E), respectively. We say that $u \in V_n$ is *connected* to $v \in V_{n+1}$ if there is an edge $e \in E_n$ such that $s(e) = u$ and $r(e) = v$.

If $V_n = \{u_1, \dots, u_k\}$ and $V_{n+1} = \{v_1, \dots, v_m\}$, we define an $m \times k$ matrix $A_n = (a_{ij})$, where a_{ij} is the number of edges connecting v_i to u_j. We call A_n the nth *incidence matrix* of (V, E).

It is convenient to give a diagrammatic presentation of the Bratteli diagram with V_n the vertices at (horizontal) level n and E_{n+1} the edges (downward directed) connecting the vertices at level n with those at level $n+1$, as illustrated in Figure 1. The vertices of $V_n = \{u_1, \dots, u_k\}$ are placed in the given order from left to right.

Given $e_k \in E_k, e_{k+1} \in E_{k+1}, \dots, e_{k+m} \in E_{k+m}$ such that $r(e_i) = s(e_{i+1})$ for $i = k, k+1, \dots, k+m-1$, we call the sequence $(e_k, \dots, e_{k+m})$ a *path*



$$A_n = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

Fig. 1. Two levels of a Bratteli diagram and the corresponding incidence matrix

(of length $m+1$) in (V,E) starting at $s(e_k) \in V_{k-1}$ and terminating at $r(e_{k+m}) \in V_{k+m}$.

Let $\{n_k\}_k^\infty = 0$ be a subsequence of $\{0,1,2,\ldots\}$, where we assume that $n_0 = 0$. We telescope (V,E) into a new Bratteli diagram (V',E') by letting $V'_k = V_{n_k}$ and letting $A'_k = A_{n_{k+1}-1}A_{n_{k+1}-2}\ldots A_{n_k}$ be the new incidence matrices. Notice that E', r', and s' are obtained in a natural way from the incidence matrices $\{A'_k\}$. Furthermore, we see that the edges E'_k from V'_{k-1} to V'_k correspond to the paths from $V_{n_{k-1}}$ to V_{n_k} in (V,E).

A Bratteli diagram is *simple* if it can be telescoped so that all the incidence matrices have strictly positive entries. Equivalently, a Bratteli diagram (V,E) is simple if and only if for each $n \in \mathbb{N} \cup \{0\}$ there exists an $l_n \in \mathbb{N}$ such that there is at least one path from each vertex in V_n to each vertex in V_{n+l_n}.

There is an obvious notion of isomorphism between Bratteli diagrams (V,E) and (V',E'): namely, there exists a pair of bijections between V and V' and between E and E' preserving the gradings and intertwining the respective source and range maps. We let $\sim$ denote the equivalence relation on ordered Bratteli diagrams generated by isomorphism and telescoping. One can show that $(V,E) \sim (V',E')$ if and only if there exists a Bratteli dagram $(\widetilde{V},\widetilde{E})$ so that telescoping $(\widetilde{V},\widetilde{E})$ to odd levels $0 < 1 < 3 < \ldots$ yields a telescoping of either (V,E) or (V',E'), and telescoping $(\widetilde{V},\widetilde{E})$ to even levels $0 < 2 < 4 < \ldots$ yields a telescoping of the other.

2.2. *Dimension groups.* With the Bratteli diagram (V,E) is associated a dimension group which we denote by $K_0(V,E)$; the notation is motivated by the connection to K-theory. In fact, with the Bratteli diagram (V,E) is associated a system of ordered groups

$$\mathbb{Z}^{|V_0|} \xrightarrow{A_0} \mathbb{Z}^{|V_1|} \xrightarrow{A_1} \mathbb{Z}^{|V_2|} \xrightarrow{A_2} \mathbb{Z}^{|V_3|} \to \ldots$$

where the positive homomorphism A_n is given by matrix multiplication with the incidence matrix between levels $n-1$ and n. By definition, $K_0(V,E)$ is the inductive limit of the system above endowed with the induced order, $K_0(V,E)^+$ denoting the positive cone. $K_0(V,E)$ has a distinguished *order unit*, namely the element of $K_0(V,E)^+$ corresponding to the element $1 \in \mathbb{Z}^{|V_0|} = \mathbb{Z}$. One shows easily that $(V,E) \sim (V',E')$ if and only if $K_0(V,E)$ is order isomorphic to $K_0(V',E')$ by a map sending the distinguished order unit of $K_0(V,E)$ to the distinguished order unit of $K_0(V',E')$.

Example. The two Bratteli diagrams (V,E) and (V',E') exhibited in Figure 2 are $\sim$-equivalent. In fact, if one telescopes the diagram $(\widetilde{V},\widetilde{E})$ to even levels one gets (V,E), and if one telescopes $(\widetilde{V},\widetilde{E})$ to odd levels one gets (V',E'). The dimension group associated with all these three diagrams

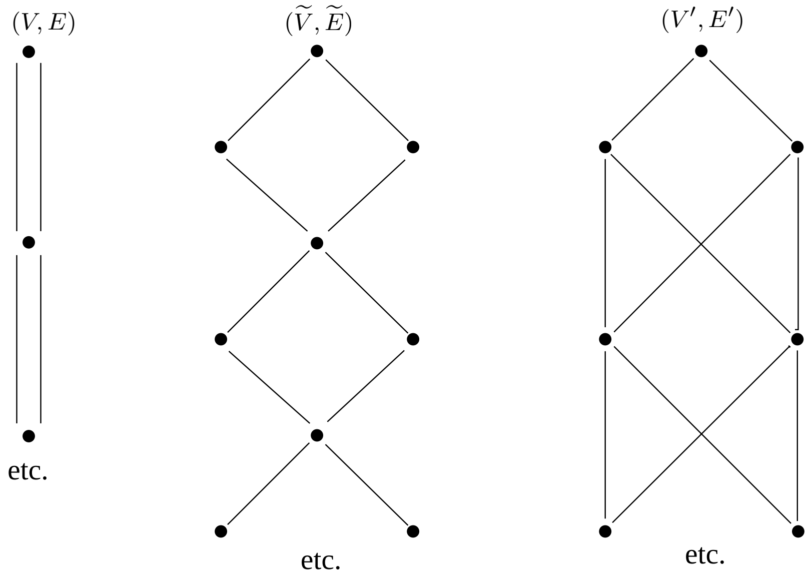


Fig. 2. $\sim$-equivalent Bratteli diagrams with associated dimension group $\mathbb{Z}[1/2]$

is the dyadic rationals $\mathbb{Z}[1/2]$ with the natural ordering, with distinguished order unit 1.

It is a theorem by Effros, Handelman, and Chen [8] that dimension groups $(K_0(V,E), K_0(V,E)^+)$ may be abstractly characterized as follows:

A dimension group is a pair (G, G^+), *where* G *is a countable abelian group and* $G^+ = \{a \in G : a \geq 0\}$ *is the positive cone*, *with the properties*

(i) (G, G^+) *is* unperforated, *i.e.*, *if* $a \in G$ *and* $na \in G^+$ *for some* $n \in \mathbb{N}$, *then* $a \in G^+$. (*Note that this implies that* G *is torsion free.*)

(ii) (G, G^+) *satisfies the* Riesz interpolation property, *i.e.*, *given* $a_1, a_2, b_1, b_2 \in G$ *such that* $a_i \leq b_j$, *there exists a* $c \in G$ *such that* $a_i \leq c \leq b_j$ $(i, j = 1, 2)$.

A dimension group (G, G^+) is *simple* if G contains no non-trivial *order ideal*. (A subgroup J of G is an order ideal if $J = J^+ - J^+$ (where $J^+ = J \cap G^+$) and $0 \leq a \leq b \in J$ implies that $a \in J$.) It is a fact that the dimension group $K_0(V,E)$ associated with the Bratteli diagram (V,E) is simple if and only if (V,E) is a simple Bratteli diagram. Since we want to avoid trivial cases, we will tacitly assume that all the simple dimension groups we encounter are acyclic, i.e. not isomorphic to $\mathbb{Z}$. One can show that the *linearly ordered* dimension groups (G, G^+) (i.e. if $a, b \in G$, then either $a \leq b$ or $b \leq a$) coincide up to order isomorphism with the countable subgroups of the additive group $\mathbb{R}$, with the inherited order. These dimension groups are necessarily simple.

If (G, G^+) is a simple dimension group, then any $u \in G^+ \backslash \{0\}$ is an *order unit*, i.e., $G^+ = \{a \in G : 0 \le a \le nu$ for some $n \in \mathbb{N}\}$. Fixing an order unit u, we say that a homomorphism $p : G \to \mathbb{R}$ is a *state* if p is positive (i.e. $p(g) \ge 0$ whenever $g \in G^+$) and $p(u) = 1$. Denote the collection of all states by $S_u(G)$. Clearly, $S_u(G)$ is a convex set. By a Hahn–Banach like argument one proves that $S_u(G)$ determines the order on G. In fact, we have

$$G^+ = \{a \in G : p(a) > 0 \text{ for all } p \in S_u(G)\} \cup \{0\}.$$

If $\varepsilon = p/q$, where $p, q \in \mathbb{N}$, then $a \le \varepsilon u$ means that $qa \le pu$. The *infinitesimal subgroup* $\operatorname{Inf} G$ of G consists of those $a \in G$ such that $-\varepsilon u \le a \le \varepsilon u$ for all $0 < \varepsilon \in \mathbb{Q}^+$. By the above,

$$\operatorname{Inf} G = \{a \in G : p(a) = 0 \text{ for all } p \in S_u(G)\}.$$

It is easy to see that $\operatorname{Inf} G$ does not depend on which order unit u we choose. The quotient $G/\operatorname{Inf} G$ with the induced ordering is again a simple dimension group. (For details, cf. [7].)

2.3. *Simple dimension groups and Choquet simplices.* The relation between simple dimension groups and *Choquet simplices* is rather intimate, and we refer to [7] for a detailed discussion. Here we just state the one result that we need in the sequel. First of all, if (G, G^+, u) is a simple dimension group with order unit u, then the state space $S_u(G)$ is a Choquet simplex when $S_u(G) \subseteq \mathbb{R}^G$ is given the relative topology from $\mathbb{R}^G$. The following holds:

Let K be a Choquet simplex and let G be a countable, dense (in the uniform topology) subgroup of $\operatorname{Aff}(K)$, *the additive group of real, affine, and continuous functions on K. Then (G, G^+) is a simple dimension group without infinitesimal elements, where $G^+ = \{a \in G : a(k) > 0$ for $k \in K\} \cup \{0\}$. Furthermore, if G contains the constant function $u = 1$, then K is affinely homeomorphic to $S_u(G)$ by the evaluation map $k \mapsto \widehat{k}$, where $\widehat{k}(a) = a(k)$, $a \in G$, $k \in K$.*

In Section 4 we will give yet another characterization of simple dimension groups in terms of purely dynamical concepts.

3. Ordered Bratteli diagrams and Cantor minimal system

3.1. *Dynamical concepts.* Recall that a topological dynamical system is a pair (X, T), where X is a compact metric space and $T : X \to X$ is a homeomorphism. We say that (X, T) is *minimal* if $TA = A$, where A is a closed subset of X, implies that $A = X$ or $A = \emptyset$. Observe that this is equivalent to all T-orbits being dense, i.e.

$$\operatorname{orbit}_T(x)^- := \{T^n x \mid n \in \mathbb{Z}\}^- = X \qquad \text{for all } x \in X.$$

The minimal system (X, T) is *Cantor minimal* if X is a Cantor set, i.e. X is totally disconnected without isolated points. (Recall that all Cantor sets are homeomorphic.) In particular, the Cantor minimal system (X, T) is a minimal *symbolic system* if T is *expansive*, i.e. there exists an $a > 0$ so that for every $x \neq y$, $\sup_n d(T^n x, T^n y) > a$, where d is a metric yielding the topology of X. For Cantor minimal systems, expansiveness is equivalent to (X, T) being conjugate to a (minimal) subshift on a finite alphabet (cf. [26, Thm. 5.24]).

We say that two dynamical systems (X, T) and (Y, S) are *conjugate* (respectively, *flip conjugate*) if there exists a homeomorphism $F : X \to Y$ so that $F \circ T = S \circ F$ (respectively, $F \circ T = S \circ F$ or $F \circ T = S^{-1} \circ F$).

We say that (X, T) is an *extension* of (Y, S), or that (Y, S) is a *factor* of (X, T), if there exists a continuous surjection $F : X \to Y$, called a *factor map*, so that $F \circ T = S \circ F$.

3.2. *Ordered Bratteli diagrams*

DEFINITION 2. An *ordered Bratteli diagram* $B = (V, E, \geq)$ is a Bratteli diagram (V, E) together with a partial order $\geq$ on E, so that edges e, e' in E are comparable if and only if $r(e) = r(e')$; in other words, we have a linear order on each set $r^{-1}(\{v\})$, where v belongs to $V \backslash V_0$.

Note that if $(V, E, \geq)$ is an ordered Bratteli diagram and $k < l$ in $\mathbb{Z}^+$, then the set $E_{k+1} \circ E_{k+2} \circ \ldots \circ E_l$ of paths from V_k to V_l may be given an induced (lexicographic) order: $(e_{k+1}, e_{k+2}, \ldots, e_l) > (f_{k+1}, f_{k+2}, \ldots, f_l)$ if and only if for some i with $k + 1 \leq i \leq l$, $e_j = f_j$ for $i < j \leq l$ and $e_i > f_i$. It is a simple observation that if $(V, E, \geq)$ is an ordered Bratteli diagram and (V', E') is a telescoping of (V, E), then with the induced order $\geq'$, $(V', E', \geq')$ is again an ordered Bratteli diagram. We say that $(V', E', \geq')$ is a *telescoping* of $(V, E, \geq)$.

Again there is an obvious notion of isomorphism between ordered Bratteli diagrams. We let $\approx$ denote the equivalence relation on ordered Bratteli diagrams generated by isomorphism and by telescoping. One can show that $B \approx B'$, where $B = (V, E, \geq)$, $B' = (V', E', \geq')$, if and only if there exists an ordered Bratteli diagram $\widetilde{B} = (\widetilde{V}, \widetilde{E}, \widetilde{\geq})$ so that telescoping $\widetilde{B}$ to odd levels $0 < 1 < 3 < \ldots$ yields a telescoping of either B or B', and telescoping $\widetilde{B}$ to even levels $1 < 2 < 4 < \ldots$ yields a telescoping of the other. (This is analogous to the equivalence relation $\sim$ above.)

Let $B = (V, E, \geq)$ be an ordered Bratteli diagram. Let X_B denote the associated infinite path space, i.e.

$$X_B = \{(e_1, e_2, \ldots) \mid e_i \in E_i,\ r(e_i) = s(e_{i+1});\ i = 1, 2, \ldots\}.$$

We will exclude trivial cases and assume henceforth that X_B is an infinite set. Two paths in X_B are said to be *cofinal* if they have the same tails, i.e.

the edges agree from a certain stage. We topologize X_B by postulating a basis of open sets, namely the family of *cylinder sets*

$$U(e_1, e_2, \ldots, e_k) = \{(f_1, f_2, \ldots) \in X_B \mid f_i = e_i,\ 1 \leq i \leq k\}.$$

Each $U(e_1, \ldots, e_k)$ is also closed, as is easily seen, and so X_B becomes a compact Hausdorff space with a countable basis of clopen sets, i.e. a zero-dimensional space. We call X_B with this topology the *Bratteli compactum* associated with $B = (V, E, \geq)$. If (V, E) is a simple Bratteli diagram, then X_B has no isolated points, and so it is a Cantor set. We let $X_B^{\max}$ (respectively $X_B^{\min}$) denote those elements $x = (e_1, e_2, \ldots)$ of X_B so that e_n is a maximal edge (respectively minimal edge) for each $n = 1, 2, \ldots$ An easy argument shows that $X_B^{\max}$ (respectively $X_B^{\min}$) is non-empty.

DEFINITION 3. The ordered Bratteli diagram $B = (V, E, \geq)$ is *properly ordered* (called simple ordered in [16], [11]) if

(i) (V, E) is a simple Bratteli diagram,
(ii) $X_B^{\max}$, resp. $X_B^{\min}$, consists of only one point $x_{\max}$, resp. $x_{\min}$.

We can now define a minimal homeomorphism $V_B : X_B \to X_B$, called the *Vershik map* (or the *lexicographic map*), associated with the properly ordered Bratteli diagram $B = (V, E, \leq)$. (We call the resulting Cantor minimal system (X_B, V_B) a *Bratteli–Vershik system.*) We let $V_B(x_{\max}) = x_{\min}$. If $x = (e_1, e_2, \ldots) \neq x_{\max}$, let k be the smallest number so that e_k is not a maximal edge. Let f_k be the successor of e_k (and so $r(e_k) = r(f_k)$). Define $V_B(x) = y = (f_1, \ldots, f_{k-1}, f_k, e_{k+1}, e_{k+2}, \ldots)$, where $(f_1, \ldots, f_{k-1})$ is the minimal edge in $E_1 \circ E_2 \circ \ldots \circ E_{k-1}$ with range equal to $s(f_k)$.

3.3. *The model theorem*

THEOREM 4 [16, Thm. 4.7]. *Let (X, T, x) be a (pointed) Cantor minimal system. Then there exists a properly ordered Bratteli diagram $B = (V, E, \geq)$ so that (X, T, x) is* pointedly conjugate *to $(X_B, V_B, x_{\min})$, where $x_{\min}$ is the unique minimal path of X_B, i.e. the conjugating map $F : X \to X_B$ maps x to $x_{\min}$. Moreover, if (X_i, T_i, x_i) is associated with $B^i = (V^i, E^i, \geq^i)$, $i = 1, 2$, then (X_1, T_1, x_1) is pointedly conjugate to (X_2, T_2, x_2) if and only if $B^1 \approx B^2$.*

Proof (sketch). The proof is based on constructing Kakutani–Rokhlin towers—where the heights of the various towers are determined by the return times to the base set—over the clopen sets A_n, where $\{A_n\}$ is a sequence of clopen sets shrinking to the one-point set $\{x\}$. The vertices V_n at level n correspond to the distinct towers over A_n. We have an edge in E_n each time a tower over A_n (which corresponds to a vertex at level n) passes through a tower over A_{n-1} (which corresponds to a vertex at level $n-1$). The edges are ordered according to the order a given tower over A_n traverses the various towers over A_{n-1}. ∎

Let (X,T) denote the Bratteli–Vershik system (X_B,V_B) associated with the properly ordered Bratteli diagram $B=(V,E,\geq)$. For $k\geq 1$ let Σ_k be the set of paths from V_0 to V_k, i.e. the set of paths starting at $v_0\in V_0$ and terminating at some $v\in V_k$. There is an obvious *truncation map* $\pi_k : X\to\Sigma_k$, obtained by restriction of paths to their initial segments of length k. With each $x\in X$, one associates a bisequence $\widetilde{\pi}_k(x)=(\pi_k(T^n(x)))_{n\in\mathbb{Z}}$ in $\Sigma_k^{\mathbb{Z}}$. The map $\widetilde{\pi}_k$ is obviously continuous. Clearly, $\widetilde{\pi}_k(T(x))=S_k(\widetilde{\pi}_k(x))$, where S_k denotes the shift. Hence the subshift (Y_k,S_k), where Y_k is the compact space $\widetilde{\pi}_k(X)$, is a *factor* of (X,T), the factor map being $\widetilde{\pi}_k$. One verifies that (X,T) is the inverse limit of $\{(Y_k,S_k)\}_{k\in\mathbb{N}}$.

If (X,T) is expansive, it is easy to see that there exists a k such that the map $\widetilde{\pi}_k : X\to Y_k$ is one-to-one. (The converse is also true.) Hence (X,T) and (Y_k,S_k) are conjugate. By telescoping the diagram between level 0 and k, we may assume that $k=1$.

It is important that the Bratteli–Vershik system we get is the same up to conjugacy regardless of the choice of base point and the choice of the sets shrinking down to the base point. This fact yields a great deal of freedom when we are going to do the construction for concrete examples. A judicious choice of the shrinking sequence may hugely simplify the construction and yield a "nice" Bratteli–Vershik model.

Example. We exhibit two examples of Bratteli–Vershik models, with ordering of the edges indicated (Figure 3).

(i) *Odometers*, i.e. minimal rotations on **a**-adic groups (cf. [17] for details about **a**-adic groups). (It is a fact that these systems coincide with the family of *distal* Cantor minimal systems (X,T), i.e. for every $x\neq y$ in X there exists $a>0$ so that $\inf_n d(T^nx,T^ny)>a$, where d is a metric yielding the topology.) If $\mathbf{a}=(a_1,a_2,\ldots)$, then $X_{\mathbf{a}}=\prod_{n=1}^{\infty}\{0,1,\ldots,a_n-1\}$ is endowed with the product topology, and $T_{\mathbf{a}} : X_{\mathbf{a}}\to X_{\mathbf{a}}$ is defined by "add $(1,0,0,\ldots)$". The ("nicest") Bratteli–Vershik model has one vertex at each level; see Figure 3(a). The dimension group associated with the Bratteli diagram (strip the ordering) is the rational numbers $\{m/(a_1a_2\ldots a_k)\mid m\in\mathbb{Z},\ k=1,2,\ldots\}$ with the natural order, the distinguished order unit being 1.

We say that the odometer associated with $\mathbf{a}=(a_1,a_2,\ldots)$ is *stationary* if $a_k=a_{k+1}=a_{k+2}=\ldots$ from some k on.

(ii) *Sturmian flows*. Let $0<\alpha<1$ be an irrational number (we may assume without loss of generality that $0<\alpha<1/2$), and let

$$x=\frac{1}{a_1+}\,\frac{1}{a_2+}\,\frac{1}{a_3+\ldots}$$

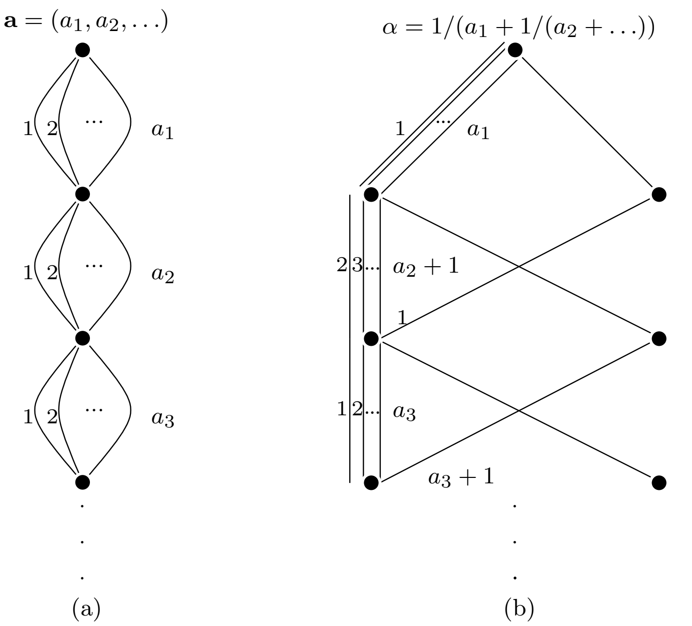


Fig. 3. Bratteli–Vershik models for odometers (a) and Sturmian flows (b)

be the continued fraction expansion of α. Let X be the unit circle "Cantorized" by doubling points at $\mathbb{Z}\alpha$ (mod 1), and let $T : X \to X$ be rotation by α. The ("nicest") Bratteli–Vershik model is shown in Figure 3(b). The dimension group associated with the Bratteli diagram (strip the ordering) is $\mathbb{Z} + \mathbb{Z}\alpha$ with the natural order, the distinguished order unit being 1.

Recall that two Cantor minimal systems (X, T) and (Y, S) are *Kakutani equivalent* if they have (up to conjugacy) a common *derivative*, i.e. there exist clopen sets U (in X) and V (in Y) so that the induced systems on U and V, respectively, are conjugate. There is a simple procedure—given the model theorem—to relate Kakutani equivalent systems. Indeed, the relevant fact is changing the order unit.

Observe first that if (V, E) is a Bratteli diagram with associated dimension group $G = K_0(V, E)$, then any finite change of (V, E), i.e. adding and/or removing a finite number of edges (vertices), thus changing (V, E) into a new Bratteli diagram (V', E'), does not change the isomorphism class of G, but does change the order unit. In fact, $G' = K_0(V', E')$ is order isomorphic to G, but the distinguished order units are not necessarily preserved by the isomorphism. Clearly, any change of order unit of G may be obtained by such a procedure.

Likewise, if $B = (V, E, \geq)$ is a properly ordered Bratteli diagram we may change B into a new properly ordered Bratteli diagram $B' = (V', E', \geq')$ by making a finite change, i.e. adding and/or removing any finite number of edges (vertices), and then making an arbitrary choice of linear orderings of the edges meeting at the same vertex (for a finite number of vertices). So B and B' are cofinally identical, i.e. they only differ on finite initial portions. (Observe that this defines an equivalence relation on the family of properly ordered Bratteli diagrams.)

We have the following theorem about Kakutani equivalent systems.

THEOREM 5 [11, Thm. 3.8]. *Let (X_B, V_B) be the Bratteli–Vershik system associated with the properly ordered Bratteli diagram $B = (V, E, \geq)$. Then the Cantor minimal system (Z, ψ) is Kakutani equivalent to (X_B, V_B) if and only if (Z, ψ) is conjugate to $(X_{B'}, V_{B'})$, where $B' = (V', E', \geq')$ is obtained from B by a finite change as described above.*

4. Ordered K-theory and orbit equivalence. In this section we relate the K-theoretic invariant—which for a Cantor minimal system turns out to be a simple dimension group—to the orbit structure of Cantor minimal systems. For details, cf. [11] (see also [13]).

4.1. *The K-theoretic invariant*

DEFINITION 6. Let (X, T) be a Cantor minimal system. Let $C(X, \mathbb{Z})$ denote the set of continuous functions on X with values in $\mathbb{Z}$. Let

$$K^0(X, T) = C(X, \mathbb{Z})/\partial_T C(X, \mathbb{Z})$$

where $\partial_T : C(X, \mathbb{Z}) \to C(X, \mathbb{Z})$ denotes the *coboundary operator* $\partial_T(f) = f - f \circ T$, and $f - f \circ T$ is a *coboundary*. Define the positive cone

$$K^0(X, T)^+ = \{[f] \mid f \in C(X, \mathbb{Z}^+)\}$$

where $[\cdot]$ denotes the quotient map. $K^0(X, T)$ has a distinguished order unit, namely $[1] = \mathbf{1}$, where 1 denotes the constant function one.

REMARK. The group $C(X, \mathbb{Z})/\partial_T C(X, \mathbb{Z})$, as an abstract group without order, has appeared before in the theory of dynamical systems. It is the first Čech cohomology group $H^1(\widehat{X}, \mathbb{Z})$ of the suspension $\widehat{X}$ of (X, T), where $\widehat{X}$ is obtained from $X \times [0, 1]$ by identifying $(x, 1)$ and $(Tx, 0)$; cf. [21].

REMARK. The notation $K^0(X, T)$ is used because of the K-theoretic underpinning of this notion. In fact, one can show that the K_0-group of the associated C^*-crossed product $C(X) \times_T \mathbb{Z}$ is isomorphic to $K^0(X, T)$. Furthermore, the natural order that the K_0-group is endowed with makes this an order isomorphism by a map preserving the canonical order units. (One can show that the K_1-group of $C(X) \times_T \mathbb{Z}$ is isomorphic to $\mathbb{Z}$.)

The following theorem is an immediate consequence of Theorem 4. It shows that simple dimension groups may be defined by purely dynamical concepts.

Theorem 7 [16, Thm. 5.4 and Cor. 6.3]. *Let (X,T) be a Cantor minimal system. Let $B=(V,E,\geq)$ be the associated properly ordered Bratteli diagram (having chosen a base point in X, cf. Theorem 4). Then*

$$K^0(X,T)\cong K_0(V,E)$$

as ordered groups with distinguished order units. Furthermore, every simple dimension group G ($G\neq\mathbb{Z}$) arises in this manner.

Let $(K^0(X,T),K^0(X,T)^+,\mathbf{1})$ be the dimension group with distinguished order unit associated with the Cantor minimal system (X,T). It is fairly routine to show that there is a natural correspondence between the set of T-invariant probability measures $M(X,T)$ on X, and the set of states (cf. Section 2.2) on $(K^0(X,T),K^0(X,T)^+,\mathbf{1})$. In fact, the map

$$[f]\mapsto\int_X f\,d\mu,\qquad f\in C(X,\mathbb{Z}),\ \mu\in M(X,T),$$

implements this correspondence. This implies that $\operatorname{Inf}K^0(X,T)$ (cf. Section 2.2) is equal to the quotient

$$N(C(X,\mathbb{Z}))/\partial_T C(X,\mathbb{Z})$$

where $N(C(X,\mathbb{Z}))=\{f\in C(X,\mathbb{Z})\mid\int_X f\,d\mu=0\text{ for all }\mu\in M(X,T)\}$. Also, we get

$$\widehat{K}^0(X,T):=K^0(X,T)/\operatorname{Inf}K^0(X,T)\cong C(X,\mathbb{Z})/N(C(X,\mathbb{Z}))$$

with the induced ordering, and with the distinguished order unit corresponding to the constant function equal to 1 (which we again will denote by $\mathbf{1}$). $\widehat{K}^0(X,T)$ is a simple dimension group (cf. Section 2.2).

4.2. *Results on orbit equivalence.* We first give the necessary definitions.

Definition 8 (Orbit equivalence). The dynamical systems (X,T) and (Y,S) are (*topologically*) *orbit equivalent* if there exists a homeomorphism $F:X\to Y$ so that $F(\mathrm{orbit}_T(x))=\mathrm{orbit}_S(F(x))$ for all $x\in X$. We use the generic term *orbit map* for a map like F.

Obviously, orbit equivalence is an equivalence relation. It is easily seen that flip conjugacy (and hence conjugacy) implies orbit equivalence. Furthermore, one shows by a simple argument that $F(M(X,T))=M(Y,S)$, where F is an orbit map as in Definition 8.

Let (X,T), (Y,S) and F be as in Definition 8. For each point x in X there exists an integer $n(x)$ so that $F\circ T(x)=S^{n(x)}\circ F(x)$. Likewise, there exists an integer $m(x)$ so that $F\circ T^{m(x)}(x)=S\circ F(x)$. If (X,T) (and hence (Y,S)) is minimal, m and n are uniquely defined integer-valued functions on X. We call m and n the *orbit cocycles* associated with the orbit map F.

Definition 9 (Strong orbit equivalence). Let (X,T) and (Y,S) be minimal systems that are (topologically) orbit equivalent. We say that (X,T)

and (Y,S) are (*topologically*) *strong orbit equivalent* if there exists an orbit map $F : X \to Y$ so that the associated orbit cocycles $m, n : X \to \mathbb{Z}$ each have at most one point of discontinuity.

Remark. That strong orbit equivalence really is an equivalence relation for Cantor minimal systems is a consequence of Theorem 11 below. Obviously flip conjugacy implies strong orbit equivalence. It is worth pointing out that between strong orbit equivalent systems (in fact, even between conjugate systems) one may find orbit maps so that the associated orbit cocycles each have more than one point of discontinuity.

To put Definition 9 in perspective we cite the following theorem of M. Boyle [1].

Theorem 10 [11, Thm. 2.4]. *Let (X,T) and (X,S) be two dynamical systems on the compact metric space X having the same orbits, one orbit being dense* (*i.e. the systems are transitive*). *Assume that one of the orbit cocycles m and n is continuous everywhere. Then the two systems are flip conjugate.*

The next two theorems show that complete invariants of ordered K-theoretic nature exist for the orbit structure of Cantor minimal systems.

Theorem 11 [11, Thm. 2.1]. *Let (X,T) and (X,S) be two Cantor minimal systems. The following are equivalent*:

(i) *(X,T) and (Y,S) are strong orbit equivalent.*

(ii) *$K^0(X,T)$ is order isomorphic to $K^0(Y,S)$ by a map preserving the distinguished order units.*

Theorem 12 [11, Thm. 2.2]. *Let (X,T) and (Y,S) be two Cantor minimal systems. The following are equivalent*:

(i) *(X,T) and (Y,S) are orbit equivalent.*

(ii) *$\widehat{K}^0(X,T)$ is order isomorphic to $\widehat{K}^0(Y,S)$ by a map preserving the distinguished order units* (cf. Section 4.1 for notation).

(iii) *There exists a homeomorphism $F : X \to Y$ which maps the set of T-invariant probability measures onto the S-invariant probability measures.*

We have already (in Section 3.3) mentioned odometer systems, which in particular are uniquely ergodic. Recall that a *Denjoy homeomorphism* is an aperiodic homeomorphism of the unit circle $\mathbb{T}$ which is not conjugate to a pure rotation. (Denjoy proved that such homeomorphisms cannot be of class C^2.) By a *Denjoy system* we mean a Denjoy homeomorphism restricted to its unique invariant Cantor set (which is the support of the unique invariant measure); cf. [22] for details. (The simplest examples of Denjoy systems are the Sturmian flows exemplified in Section 3.3.) We have the following remarkable corollary of Theorem 12.

Corollary 13. *Let* (X, T) *be a uniquely ergodic Cantor system. Then* (X, T) *is orbit equivalent to either an odometer system or a Denjoy system.*

Remark. There is an abundance of Cantor minimal systems that are strong orbit equivalent but not flip conjugate. Likewise, within each orbit equivalence class there are uncountably many strong orbit equivalence classes. A truly remarkable result proved by N. Ormes [20] says that given any Cantor minimal system (X, T) and an ergodic measure-preserving system $(Y, \mathcal{C}, \mu, S)$, $\mu(Y) = 1$, there exists a system (X', T'), orbit equivalent to (X, T), and a T'-invariant (ergodic) measure ν', $\nu'(X') = 1$, so that $(Y, \mathcal{C}, \mu, S)$ is metrically isomorphic to $(X', \mathcal{B}', \nu', T')$, where $\mathcal{B}'$ is the set of Borel sets in X'. This result implies both the Dye theorem on orbit equivalence of ergodic measure-preserving systems and the Jewett–Krieger realization theorem for ergodic systems. (Ormes also proved a similar result in the strong orbit equivalent case.) A direct consequence of Ormes' result (and also of a result by Sugisaki [24]) is that within each strong orbit equivalence class all (topological) entropies occur. In particular, the K-theoretic and entropy invariants are independent of each other.

5. Dimension groups associated with substitution minimal systems. In this section we present an algorithmic and explicit construction which renders an effective method to compute the (ordered) K-theoretic invariant associated with substitution minimal systems. First we state some relevant definitions and basic results, referring to [6] for further details. For a general reference on substitutions, cf. [23].

5.1. *Basic facts about substitutions.* An *alphabet* is a finite set of symbols called *letters*. If A is an alphabet, a *word* in A is a finite (non-empty) sequence of letters. A^+ denotes the set of words. For $u = u_1 u_2 \ldots u_n \in A^+$, $|u| = n$ is the *length* of u. Given a word $u = u_1 \ldots u_n$, we say that $v = u_i \ldots u_j$, where $1 \leq i \leq j \leq n$, is a *subword* (or a *factor*) of u. We extend this definition in an obvious way to (infinite) sequences. We will use the notation $v \prec u$. We say that v *occurs* in u. Elements of $A^{\mathbb{Z}}$ are called *sequences* over the alphabet A. The *language* $\mathcal{L}(x)$ of the sequence x is the set of words which are factors of x.

A *substitution* on the alphabet A is a map $\sigma : A \to A^+$. Using the extension to words by concatenation, σ can be iterated: for each integer $n > 0$, $\sigma^n : A \to A^+$ is again a substitution. In this paper we only consider *primitive* substitutions, i.e. substitutions σ on A such that there exists $n \in \mathbb{N}$ so that for every $a, b \in A$, b occurs in $\sigma^n(a)$, i.e. $b \prec \sigma^n(a)$. We also require that $\lim_n |\sigma^n(a)| = \infty$ for every $a \in A$.

Let $A = \{a_1, \ldots, a_k\}$ and let σ be a substitution on A. The matrix $M(\sigma)$ is defined by $M(\sigma)_{ij} =$ the number of occurrences of a_j in $\sigma(a_i)$. One observes that $M(\sigma^n) = M(\sigma)^n$, and that σ is primitive if and only if there exists n so that $M(\sigma)^n$ has only non-zero entries, i.e. $M(\sigma)$ is a *primitive* matrix.

We denote by $\mathcal{L}(\sigma)$ the language of σ, i.e. the set of words on A which are factors of $\sigma^n(a)$ for some $a \in A$ and some $n \geq 1$, and by X_σ the subset of $A^{\mathbb{Z}}$ associated with this language, i.e. the set of $x \in A^{\mathbb{Z}}$ whose every finite factor belongs to $\mathcal{L}(\sigma)$. X_σ is closed in $A^{\mathbb{Z}}$ and invariant under the shift. We denote by T_σ the restriction of the shift with X_σ. The dynamical system (X_σ, T_σ) is called the *substitution dynamical system* associated with σ. The following is well known:

Every substitution dynamical system is minimal and uniquely ergodic, *and has* (*topological*) *entropy equal to zero.*

In the literature, substitution dynamical systems are often defined by a different (but equivalent) method, using fixed points:

For every integer $p > 0$, the substitution σ^p defines the same language, thus the same system, as σ does. Substituting σ^p for σ if needed, we can assume that there exist two letters $r, l \in A$ such that:

(i) r is the last letter of $\sigma(r)$,
(ii) l is the first letter of $\sigma(l)$,
(iii) $rl \in \mathcal{L}(\sigma)$.

Whenever r and l satisfy the conditions (i) and (ii), it is easy to check that there exists a unique $\omega = (\omega_n)_{n\in\mathbb{Z}} \in A^{\mathbb{Z}}$ such that

$$\omega_{-1} = r, \quad \omega_0 = l \quad \text{and} \quad \sigma(\omega) = \omega$$

where it is obvious how to extend σ to sequences. Such an ω is called a *fixed point* of σ. If r and l also satisfy (iii), we say that ω is an *admissible fixed point* of σ.

If ω is an admissible fixed point of σ, then X_σ is the closure of the orbit of ω for the shift. This property is often taken as the definition of X_σ in the literature.

In the sequel, (X_σ, T_σ) is the system associated with the primitive substitution σ on the alphabet A, ω is an admissible fixed point of σ, $r = \omega_{-1}$ and $l = \omega_0$.

To avoid trivial cases, we consider henceforth only *aperiodic* substitutions, i.e. substitutions giving rise to infinite systems, in fact, minimal symbolic systems. (There is an algorithm which decides whether a given substitution is aperiodic or not [15].) For substitution dynamical systems associated with primitive and aperiodic substitutions we use the term *substitution minimal systems.*

We now introduce a class of substitutions which are easier to study.

DEFINITION 14. A substitution σ on the alphabet A is *proper* if there exists an integer $p > 0$ and two letters $r, l \in A$ such that:

(i) For every $a \in A$, r is the last letter of $\sigma^p(a)$.
(ii) For every $a \in A$, l is the first letter of $\sigma^p(a)$.

A proper substitution has only one fixed point. A crucial result that we shall prove is that every substitution minimal system is conjugate to the system associated with some proper substitution, which can be explicitly constructed from the former.

5.2. *Stationary Bratteli diagrams and associated dimension groups*

DEFINITION 15. A Bratteli diagram (V, E) is *stationary* if $k = |V_1| = |V_2| = \dots$ and if (by an appropriate labeling of the vertices) the incidence matrices between levels n and $n+1$ are the same $k \times k$ matrix C for all $n = 1, 2, \dots$ In other words, beyond level 1 the diagram repeats. (Clearly, we may label the vertices in V_n as $V(n, a_1), \dots, V(n, a_k)$, where $A = \{a_1, \dots, a_k\}$ is a set of k distinct symbols.)

$B = (V, E, \geq)$ is a *stationary ordered Bratteli diagram* if (V, E) is stationary, and the ordering on the edges with range $V(n, a_i)$ is the same as the ordering on the edges with range $V(m, a_i)$ for $m, n = 2, 3, \dots$ and $i = 1, \dots, k$. In other words, beyond level 1 the diagram with the ordering repeats. (For each a_i in $A = \{a_1, \dots, a_k\}$ and each $n = 2, 3, \dots$, we thus get an ordered list of edges whose range is $V(n, a_i)$. By the stationarity of the ordering of B we get a well-defined map from A to A^+ (by taking the sources of the edges in the given order).

(G, G^+) is a *stationary dimension group* if G is order isomorphic to $K_0(V, E)$, where (V, E) is a stationary Bratteli diagram. $K_0(V, E)$ is completely determined by the $k \times k$ incidence matrix C of (V, E) (we disregard the distinguished order unit). Also, $K_0(V, E)$ is simple if and only if C is a primitive matrix. In this case $K_0(V, E)$ has a unique state (with respect to a given order unit), and this state is determined by a left Perron–Frobenius eigenvector of C (appropriately normalized); cf. [7, Chapter 6].

Define

$$\begin{aligned} H(C) &= \{\vec{h} \in \mathbb{Q}^k \mid \exists n \in \mathbb{N} \text{ so that } C^n \vec{h} \in \mathbb{Z}^k\}, \\ H(C)^+ &= \{\vec{h} \in \mathbb{Q}^k \mid \exists n \in \mathbb{N} \text{ so that } C^n \vec{h} \in \mathbb{Z}^k_+\}, \\ K(C) &= \{\vec{h} \in \mathbb{Q}^k \mid \exists n \in \mathbb{N} \text{ so that } C^n \vec{h} = \vec{0}\}. \end{aligned}$$

Then $G = H(C)/K(C)$ and G^+ is the image of $H(C)^+$ in G. (Alternatively, G^+ is determined by the unique state (if C is primitive); cf. Section 2.2.)

We refer to [7] and [14] for further details on how to compute (G, G^+) when C is given.

5.3. *Main results*

THEOREM 16 [6, Thm. 1]. *The family Φ of Bratteli–Vershik systems associated with stationary, properly ordered Bratteli diagrams is (up to conjugacy) the disjoint union of the family of substitution minimal systems and the family of stationary odometer systems. Furthermore, the correspondence in question is given by an explicit and algorithmic effective construction. The same is true for the computation of the (stationary) dimension group associated with a substitution minimal system.*

REMARK. The main part of Theorem 16 is also proved in [10]. However, the proofs given there are mostly of existential nature and do not state a feasible method to compute effectively.

THEOREM 17 [6, Cor. 2 and Thm. 3]. *The family Φ in the above theorem is stable under Kakutani equivalence. Furthermore, a Cantor factor of a system (X, T) belonging to Φ is again in Φ.*

Before we present concrete examples to demonstrate the algorithm referred to in Theorem 16, we give an idea of the proof.

Sketch of proof of Theorem 16. (i) We first consider *proper* substitutions (cf. Definition 14). So let $\sigma : A \to A^+$ be a (primitive, aperiodic) proper substitution. Denote by $B = (V, E, \geq)$ the stationary ordered Bratteli diagram associated with σ in an obvious way (cf. Definition 15), with the proviso that there is a single edge from the top vertex (at level 0) to each of the vertices at level 1. It is easily seen that $B = (V, E, \geq)$ is properly ordered, and that the incidence matrix C of (V, E) is $M(\sigma)$. Furthermore, the substitution minimal system (X_σ, T_σ) is conjugate to the Bratteli–Vershik system (X_B, V_B) via the truncation map $\widetilde{\pi}_1 : X_B \to Y_1$; cf. Section 3.3. In fact, (X_σ, T_σ) may be naturally identified with (Y_1, S_1), and the factor map $\widetilde{\pi}_1$ from (X_B, V_B) onto (Y_1, S_1) is actually one-to-one (thus a conjugation). The last fact depends upon the highly non-trivial result on "recognizability" for primitive, aperiodic substitution by Mossé [18], [19].

(ii) In general, let $\sigma : A \to A^+$ be a (primitive, aperiodic) substitution (with associated substitution minimal system (X_σ, T_σ)) satisfying conditions (i), (ii) and (iii) of Section 5.1, and let

$$\sigma(\omega) = \omega = (\omega_n)_{n\in\mathbb{Z}}$$

be an admissible fixed point for σ. To reduce to the proper substitution case above, it is a remarkable fact that by inducing (X_σ, T_σ) to the clopen set

$$[\omega_{-1\bullet}\omega_0] := \{x = (x_n)_{n\in\mathbb{Z}} \in X_\sigma \mid x_{-1} = \omega_{-1},\ x_0 = \omega_0\}$$

this may be achieved. In fact, by introducing a new alphabet consisting of so-called *return words* to $\omega_{-1}\omega_0$ in ω (i.e. the subwords of ω lying between two consecutive occurrences of $\omega_{-1}\omega_0$ in ω), σ generates a *proper* (primitive, aperiodic) substitution τ. Then (X_τ, T_τ) is conjugate to the induced system. By Theorem 5 the properly ordered Bratteli diagram associated with (X_σ, T_σ) is obtained form the stationary diagram associated with (X_τ, T_τ) by a finite change, and the latter is easily determined. The details are lengthy to explain precisely, but two concrete examples should be instructive and demonstrate the effectiveness of the algorithm to compute the associated dimension groups.

(iii) Conversely, let $B = (V, E, \geq)$ be a stationary properly ordered Bratteli diagram. One shows first that there exists a $\approx$-equivalent stationary (properly ordered) diagram with the same arrangement of edges between levels 0 and 1 as in (i). (Cf. [6, Lemma 9].) By an appropriate telescoping of the diagram we may assume that all the minimal edges in E_n have the same source in V_{n-1} for $n = 2, 3, \ldots$ Similarly for the maximal edges. The substitution σ read off the diagram (cf. Definition 15) is clearly proper. There are two alternatives: Either the Bratteli–Vershik system is a stationary odometer (the associated substitution is then periodic), or it is a substitution minimal system conjugate to (X_σ, S_σ).

5.4. *Examples*

EXAMPLE 1. The substitution σ on the alphabet $A = \{a, b\}$ defined by

$$\sigma(a) = aab, \quad \sigma(b) = abab$$

is proper (and primitive, aperiodic). Clearly, $M(\sigma) = \left(\begin{smallmatrix} 2 & 1 \\ 2 & 2 \end{smallmatrix}\right)$.

So $K^0(X_\sigma, T_\sigma)$ is the stationary dimension group associated with the 2×2 matrix $C = \left(\begin{smallmatrix} 2 & 1 \\ 2 & 2 \end{smallmatrix}\right)$; cf. Section 5.2. The Perron–Frobenius eigenvalue of C is $\lambda = 2 + \sqrt{2}$ and a left Perron–Frobenius eigenvector is $(\sqrt{2}, 1)$, which determines the unique state. Since $\det C = 2$, $K(C) = \{0\}$ and so $K^0(X_\sigma, T_\sigma) = H(C)$ (cf. Section 5.2 for notation). A simple computation shows that

$$K^0(X_\sigma, T_\sigma) = \{(2^{-k}a, 2^{-k}b) \mid a, b \in \mathbb{Z},\ k \geq 0\} \subset \mathbb{Q}^2,$$
$$K^0(X_\sigma, T_\sigma)_+ = \{(r, s) \in K^0(X_\sigma, T_\sigma) \mid r\sqrt{2} + s \geq 0\}.$$

The distinguished order unit is $(1, 1)$.

EXAMPLE 2. Let $\omega = (\omega_n)_{n \in \mathbb{Z}} = \ldots \omega_{-2}\omega_{-1\bullet}\omega_0\omega_1\omega_2 \ldots$ be a sequence over the alphabet A, where $\bullet$ indicates the position of the zeroth coordinate. We denote by $\omega_{[i,j)}$, where $i < j$, the subword $\omega_i\omega_{i+1} \ldots \omega_{j-1}$ of ω. An *occurrence* of the subword $\omega_{-1}\omega_0$ in ω is by definition an integer n so that $\omega_{[n-1,n+1)} = \omega_{-1}\omega_0$. A subword u of ω is a *return word* with respect to $\omega_{-1}\omega_0$ if there exist two consecutive occurrences i, j $(i < j)$ of $\omega_{-1}\omega_0$ in ω so that $u = \omega_{[i,j)}$. If ω is *uniformly recurrent*, which is the case when we are dealing with minimal subshifts, the set of return words is finite. (The notion of return word is intimately tied to the notion of a *tower* over the clopen set $[\omega_{-1\bullet}\omega_0]$ with respect to a minimal subshift.)

Let ϱ be the substitution over the alphabet $A = \{a, b\}$ defined by

$$\varrho(a) = aba, \qquad \varrho(b) = ab.$$

One verifies that ϱ is primitive, aperiodic and satisfies the conditions (i), (ii) and (iii) of Section 5.1. (ϱ is the square of the Fibonacci substitution $a \mapsto ab$, $b \mapsto a$, and so the two dynamical systems are the same.) Let

$$\begin{aligned}\omega &= \lim_n \overleftarrow{\varrho^n(a)}_\bullet \lim_n \overrightarrow{\varrho^n(a)} \\ &= \dots aba|\overbrace{ababa}^{w_2}|_\bullet \overbrace{aba}^{w_1}|\overbrace{ababa}^{w_2}|\overbrace{aba}^{w_1}|\overbrace{ababa}^{w_2}|\overbrace{ababa}^{w_2}|\overbrace{aba}^{w_1}|\overbrace{ababa}^{w_2}|\overbrace{aba}^{w_1}|ab\dots\end{aligned}$$

be an admissible fixed point for ϱ. We have indicated the occurrences and return words of $\omega_{-1}\omega_0 = aa$ by $|$ and $\overbrace{\quad}$, respectively. We find two return words, namely $w_1 = aba$, $w_2 = ababa$. (One of the reasons these are the only ones is that the new (proper) substitution we construct on the alphabet consisting of return words must be primitive, according to the theory. See below.)

Now we construct a proper (primitive, aperiodic) substitution on the new alphabet $B = \{w_1, w_2\}$ in the following way:

Apply ϱ to w_1 and w_2:

$$\varrho(w_1) = \overbrace{aba}^{w_1}\overbrace{ababa}^{w_2}, \qquad \varrho(w_2) = \overbrace{aba}^{w_1}\overbrace{ababa}^{w_2}\overbrace{ababa}^{w_2}.$$

(By the theory the set of return words forms a *circular code*, and we get a unique decomposition of $\varrho(w_1)$ and $\varrho(w_2)$ as concatenations of return words.) So we get a proper substitution τ on $B = \{w_1, w_2\}$ by

$$\tau(w_1) = w_1w_2, \qquad \tau(w_2) = w_1w_2w_2.$$

Now the theory says that the Bratteli–Vershik model for (X_ϱ, T_ϱ) is obtained from the one for (X_τ, T_τ) by adding edges between level 0 and level 1 corresponding to the lengths of the return words (which in this case are 3 and 5, respectively). The diagrams in Figure 4 illustrate this.

The two diagrams have matrix $C = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right)$, which has determinant equal to 1, so $K^0(X_\varrho, T_\varrho) \cong \mathbb{Z}^2$ as abstract groups. The Perron–Frobenius eigenvalue is $\lambda = (3 + \sqrt{5})/2$, and a left Perron–Frobenius eigenvector is $(1, (1 + \sqrt{5})/2) = (1, \theta)$, where $\theta = (1 + \sqrt{5})/2$ is the golden mean. One gets from this immediately

$$\begin{gathered} K^0(X_\varrho, T_\varrho) = \{(a, b) \mid a, b \in \mathbb{Z}\}, \\ K^0(X_\varrho, T_\varrho)^+ = \{(a, b) \mid a + \theta b \geq 0,\ a, b \in \mathbb{Z}\}. \\ \text{Distinguished order unit} = (3, 5). \end{gathered}$$

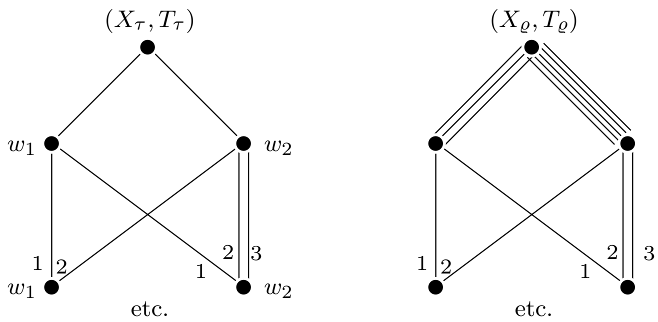


Fig. 4

Remark. The substitution δ on $A = \{a, b\}$ defined by

$$\delta(a) = aab, \qquad \delta(b) = ba$$

has the same matrix $M(\delta) = \left(\begin{smallmatrix} 2 & 2 \\ 2 & 1 \end{smallmatrix}\right)$ as the substitution ϱ in Example 2. One can also show that the two associated (measure-preserving) ergodic systems are metrically isomorphic—hence the spectral properties are the same. However, by a similar procedure one readily shows that $K^0(X_\delta, T_\delta) \cong \mathbb{Z}^3$ as abstract groups. In particular, (X_δ, T_δ) is not flip conjugate to (X_ϱ, T_ϱ).

6. Bratteli–Vershik models for Toeplitz flows and their implications. We now give a brief survey of some results about Toeplitz flows that can be obtained rather easily using the ordered K-theoretic invariant we have introduced above. The details can be found in [12]. We assume the reader has some familiarity with the basic facts about Toeplitz flows—[27] is a good reference.

6.1. *Basic definitions*

Definition 18. A *Toeplitz sequence* is a non-periodic sequence $\eta = (\eta_n)_{n\in\mathbb{Z}}$ in $A^{\mathbb{Z}}$, where A is a finite alphabet, so that for each $m \in \mathbb{Z}$ there exists $n \in \mathbb{N}$ so that $\eta_m = \eta_{m+kn}$ for all $k \in \mathbb{Z}$.

The associated Toeplitz flow—which is easily seen to be minimal (thus it is a symbolic system)—is the dynamical system (X_η, T_η), where X_η is the closure of the orbit of η under the shift, and T_η denotes the restriction of the shift to X_η.

If $x \in X_\eta$, $p \in \mathbb{N}$ and $a \in A$, we let

$$\begin{aligned} \mathrm{Per}_p(x, a) &= \{n \in \mathbb{Z} : x(m) = a \text{ for all } m \equiv n \pmod{p}\}, \\ \mathrm{Per}_p(x) &= \bigcup_{a\in A} \mathrm{Per}_p(x, a). \end{aligned}$$

The *p-skeleton* of x is the part of x which is periodic with period p. We say that p is an *essential period* of x if the p-skeleton of x is not periodic with any smaller period. The least common multiple of two essential periods is again an essential period, a fact which is easily verified. A *periodic structure* for a Toeplitz sequence η is a strictly increasing sequence $(p_i)_{i\in\mathbb{N}}$ such that p_i is an essential period of η for all i, $p_i \,|\, p_{i+1}$ (i.e. p_i is a divisor of p_{i+1}), and $\bigcup_{i=1}^{\infty} \mathrm{Per}_{p_i}(\eta) = \mathbb{Z}$. A periodic structure always exists for a Toeplitz sequence. In fact, order the essential periods, $r_1 < r_2 < r_3 < \ldots$, and let p_i be the least common multiple of $\{r_1, r_2, \ldots, r_i\}$. A periodic structure is obtained by deleting repeated terms from the sequence $(p_i)_{i\in\mathbb{N}}$.

Now let $(p_i)_{i\in\mathbb{N}}$ be a periodic structure for the Toeplitz sequence η. Then the odometer $(X_{\mathbf{a}}, T_{\mathbf{a}})$ associated with $\mathbf{a} = (p_1, p_2/p_1, p_3/p_2, \ldots)$ (cf. Section 3.3) is the *maximal equicontinuous factor* of the Toeplitz flow (X_η, T_η). Furthermore, (X_η, T_η) is an almost one-to-one extension of $(X_{\mathbf{a}}, T_{\mathbf{a}})$, i.e. there exists a point in $X_{\mathbf{a}}$ whose preimage with respect to the factor map is a one-point set. Conversely, if (X, T) is a minimal symbolic system which is an almost one-to-one extension of an odometer $(X_{\mathbf{a}}, T_{\mathbf{a}})$, then (X, T) is conjugate to a Toeplitz flow, whose maximal equicontinuous factor is $(X_{\mathbf{a}}, T_{\mathbf{a}})$ (cf. [5]).

For each $i \in \mathbb{N}$, $n \in \mathbb{Z}/p_i\mathbb{Z}$, let $A_n^i = \{T_\eta^m \eta : m \equiv n \pmod{p_i}\}$. Williams [27] showed that:

(i) $\overline{A_n^i}$ *is exactly the set of all* $\omega \in X_\eta$ *with the same* p_i*-skeleton as* $T_\eta^n \eta$.

(ii) $\{\overline{A_n^i} : n \in \mathbb{Z}/p_i\mathbb{Z}\}$ *is a partition of* X_η *into clopen sets.*

(iii) $\overline{A_m^i} \supset \overline{A_n^j}$ *for* $i < j$ *and* $m \equiv n \pmod{p_i}$.

(iv) $T_\eta \overline{A_n^i} = \overline{A_{n+1}^i}$.

(v) *If* $(p_n)_{n\in\mathbb{N}}$ *is a periodic structure for* η *and* $\bigcap_{i=1}^{\infty} \overline{A_{n_i}^i}$ (*where* $n_i \equiv n_j \pmod{p_i}$ *for* $j \geq i$) *contains a Toeplitz sequence* ω, *then* $\bigcap_{i=1}^{\infty} \overline{A_{n_i}^i} = \{\omega\}$. *In particular*, $\bigcap_{i=1}^{\infty} \overline{A_0^i} = \{\eta\}$. *This implies that the factor map onto* $X_{\mathbf{a}}$ *is* 1-1 *on the set of Toeplitz sequences in* X_η. (*In general, we have* $x, y \in \bigcap_{i=1}^{\infty} \overline{A_{n_i}^i}$ *if and only if* x *and* y *have the same* p_i*-skeleton for all* $i \in \mathbb{N}$.)

We next give the definition of Bratteli diagrams that are relevant for Toeplitz flows.

Definition 19. Let (V, E) be a simple Bratteli diagram. We say that (V, E) has the *equal path number property* if for each $n \in \mathbb{N}$, the number of paths from the top vertex (i.e. the one vertex in V_0) to each of the vertices at level n (i.e. the vertices in V_n) is the same. An equivalent definition would be to say that for each $n \in \mathbb{N}$, $|r^{-1}(u)| = |r^{-1}(w)|$ for every $u, w \in V_n$ (cf. Definition 1 for notation). In other words, if A_n is the nth incidence matrix of (V, E), the row sums of A_n are the same.

Observe that the equal path number property is preserved under telescoping.

6.2. *Results on Toeplitz flows using Bratteli diagrams*

THEOREM 20 [12, Thm. 8]. *The family of expansive Bratteli–Vershik systems associated with Bratteli diagrams with the equal path number property coincides with the family of Toeplitz flows up to conjugacy.*

Proof (sketch). (i) Assume first that $B = (V, E, \geq)$ is a properly ordered Bratteli diagram with the equal path number property, so that the associated Bratteli–Vershik system is expansive. By telescoping we may assume that B has the property that at each level the sources of the minimal edges coincide. Likewise with the maximal edges. (Cf. [16, Prop. 2.8].) Also, we may assume that $\widetilde{\pi}_1 : X_B \to Y_1$ is a conjugacy between (X_B, V_B) and (Y_1, S_1) (cf. Section 3.3 for notation). Now it is easily seen that $\widetilde{\pi}_1(x_{\min})$, where $x_{\min}$ is the unique min path in X_B, is a Toeplitz sequence over the alphabet Σ_1. Hence (Y_1, S_1) is conjugate to a Toeplitz flow.

(ii) Conversely, if (X_η, T_η) is a Toeplitz flow associated with the Toeplitz sequence η, we modify the clopen partitions $\{\overline{A_n^i} \mid n \in \mathbb{Z}/p_i\mathbb{Z}\}$ above for each i to get a *nested* sequence of clopen partitions $(\Omega_i)_{i\in\mathbb{N}}$ (where Ω_{i+1} is a refinement of Ω_i) associated with a new periodic structure $(q_i)_{i\in\mathbb{N}}$ for η. For each i the resulting Kakutani–Rokhlin towers have the same height. This gives rise to a properly ordered Bratteli diagram that has the equal path number property. ■

COROLLARY 21. *Substitution minimal systems associated with* (*primitive*, *aperiodic*) *proper substitutions of constant length are Toeplits flows* (*up to conjugacy*).

Proof. By part (i) of the proof of Theorem 16, the properly ordered Bratteli diagram associated with a proper substitution of constant length has the equal path number property. ■

The Bratteli–Vershik models for Toeplitz flows yield the maximal equicontinuous odometer factors and the associated factor maps straightforwardly by "collapsing" the vertices at each level to one vertex. Figure 5 illustrates how this works: An edge $e \in E_n$ with ordinal number k is mapped to the edge $f \in E'_n$ with ordinal number k (as illustrated with E_3 and E'_3 in the figure). This induces an obvious map on infinite paths, which, by an easy verification, is a factor map. Observe that the pre-image of the unique minimal path of the odometer is the unique minimal path of the "Toeplitz diagram", which corresponds to the Toeplitz sequence η.

REMARK. It is noteworthy that two strong orbit equivalent Toeplitz flows have the same maximal equcontinuous factor. This follows from Theorem 11, and the fact that the rational (continuous) eigenvalues of a Cantor minimal system (X, T) can be detected from $K^0(X, T)$ (cf. [12, Prop. 13]).

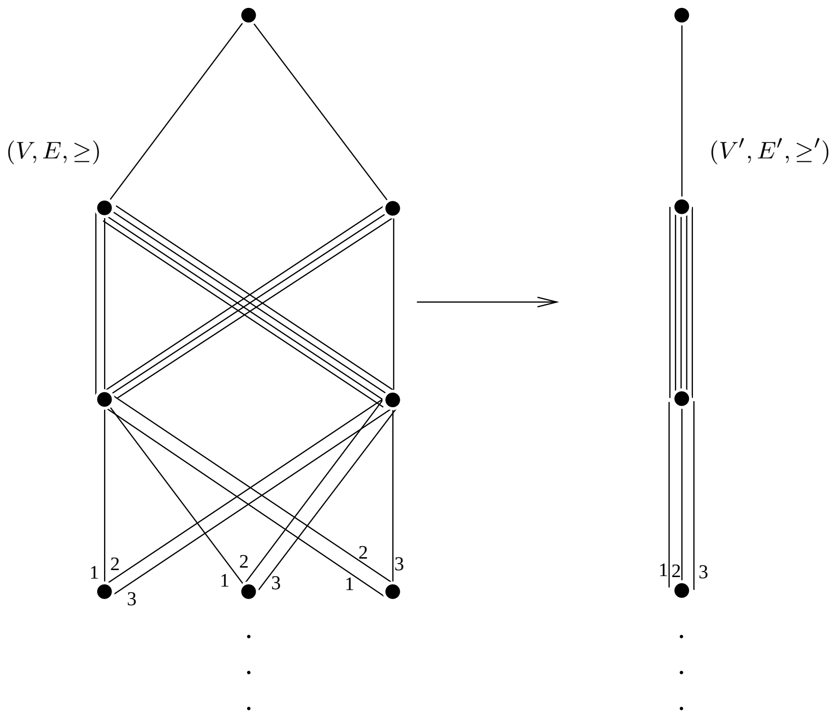


Fig. 5. "Collapsing" the Bratteli–Vershik model of a Toeplitz system to get the maximal equicontinuous factor

Downarowicz proved that any Choquet simplex may be obtained as the set of invariant probability measures of a 0-1 Toeplitz flow [3]. Using Theorem 20 and the result cited in Section 2.3, it is fairly straightforward to prove the following strengthening of Downarowicz' result.

THEOREM 22 [12, Thm. 11]. *Let K be a Choquet simplex. There exists a Toeplitz flow* (X_η, T_η) *of zero entropy over a two-symbol alphabet* (*i.e. a* 0-1 *Toeplitz flow*) *such that K is affinely homeomorphic to the set* $M(X_\eta, T_\eta)$ *of* T_η*-invariant probability measures. Furthermore, we may choose* (X_η, T_η) *so that the* (*unique*) *maximal equicontinuous factor is the odometer* $(X_{\mathbf{a}}, T_{\mathbf{a}})$ *associated with* $\mathbf{a} = (2, 3, 4, 5, \ldots)$. *Hence* $K^0(X_{\mathbf{a}}, T_{\mathbf{a}}) \cong \mathbb{Q}$ *as ordered groups with distinguished order units*, *and the set of* (*topological*) *eigenvalues of* (X_η, T_η) *is* $\{e^{2\pi i r} : r \in \mathbb{Q}\}$.

Proof (sketch). Choose a countable, dense subgroup G of $\mathrm{Aff}(K)$, the additive group of affine continuous functions on K, such that G is a vector space over $\mathbb{Q}$ containing the constant function $u = 1$. So G is a *divisible group*. By Section 2.3, K is affinely homeomorphic to $S_u(G)$.

It is now fairly routine to construct a simple Bratteli diagram (V, E) with the equal path number property (using the divisibility of G) so that

$K_0(V, E) \cong G$ as ordered groups with distinguished order units. Furthermore, the Bratteli diagram (V, E) admits a proper ordering so that the associated Bratteli–Vershik model is expansive, thus getting a Toeplitz flow (X_η, T_η) by Theorem 20. By Theorem 7 and the ensuing remarks we deduce that K is affinely homeomorphic to $M(X_\eta, T_\eta)$.

Finally, we obtain the other properties stated in the theorem by adjusting (V, E) slightly. ∎

To conclude this section we state a theorem—the first assertion of which is well known—whose proof is easily obtained using the Bratteli diagram approach (similar to the "collapsing" procedure in Figure 5).

THEOREM 23 [12, Thm. 14]. *Let $(X_{\mathbf{a}}, T_{\mathbf{a}})$ be the odometer associated with $\mathbf{a} = (q_i)_{i=1}^{\infty}$. There exists a uniquely ergodic* 0-1 *Toeplitz flow (X_η, T_η) of zero entropy such that $(X_{\mathbf{a}}, T_{\mathbf{a}})$ is the maximal equicontinuous factor of (X_η, T_η). Furthermore, we may choose (X_η, T_η) such that $K^0(X_\eta, T_\eta) \cong K^0(X_{\mathbf{a}}, T_{\mathbf{a}})$ as ordered groups with distinguished order units—hence (X_η, T_η) and $(X_{\mathbf{a}}, T_{\mathbf{a}})$ are strong orbit equivalent.*

If $(X_{\mathbf{a}}, T_{\mathbf{a}})$ is associated with a stationary odometer (i.e. $q_k = q_{k+1} = q_{k+2} = \ldots$ from a certain k on), we may choose (X_η, T_η) so that it is also a substitution minimal system.

6.3. *Kakutani equivalence—an example.* Theorem 17 implies that a system that is Kakutani equivalent to a substitution minimal system is again substitution minimal. Furthermore, Theorem 17 says that a Cantor factor of a substitution minimal system is either again a substitution minimal system or a stationary odometer. One may ask: do analogous results hold for Toeplitz flows? The answer is no. In [4] there is an example of a Toeplitz flow that has a Cantor factor that is neither Toeplitz nor an odometer. As for the Kakutani equivalence case, we exhibit in Figure 6 an example (which can be found in [12]) of a Toeplitz flow which has a Kakutani equivalent system that is *prime* (i.e. has no non-trivial factors). In fact, the Toeplitz flow (which incidentally is also a substitution minimal system associated with a proper substitution of constant length) associated with the stationary, properly ordered Bratteli diagram $B = (V, E, \geq)$ in Figure 6 is Kakutani equivalent, according to Theorem 5, to the system associated with the stationary, properly ordered Bratteli diagram $B' = (V', E', \geq')$ in Figure 6. The latter turns out to be conjugate with the *Chacon system.* (Recall that the Chacon system is the minimal symbolic system associated with the *Chacon substitution* $0 \mapsto 0010$, $1 \mapsto 1$.) It is well known that the Chacon system is prime, hence it cannot be conjugate to a Toeplitz flow or to an odometer system. For further details we refer to [12, Section 4.2].

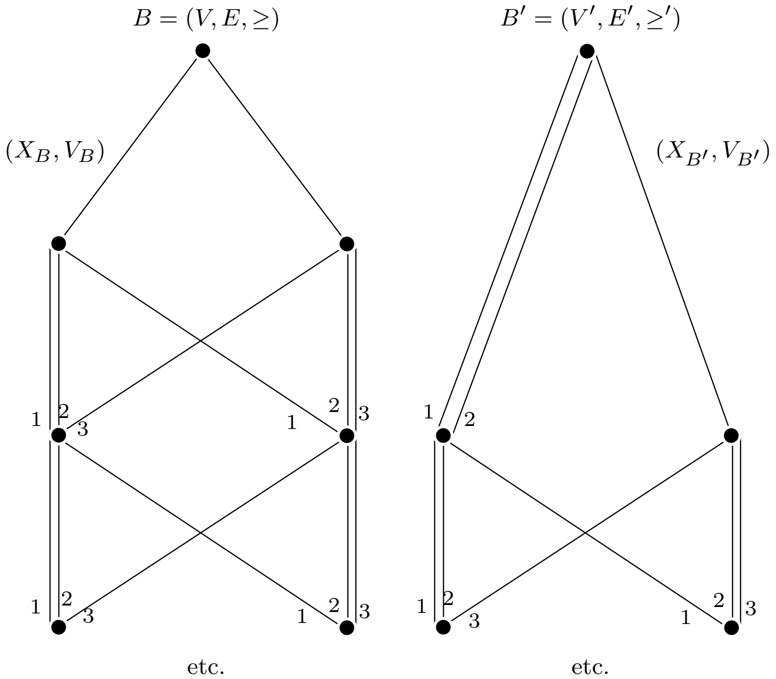


Fig. 6. A Toeplitz system (X_B, V_B) and the Kakutani equivalent Chacon system $(X_{B'}, V_{B'})$

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