COLLOQUIUM MATHEMATICUM

VOL. 84/85 2000 PART 1

GENERICITY OF NONSINGULAR TRANSFORMATIONS WITH INFINITE ERGODIC INDEX

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In memoriam: Anzelm Iwanik

Abstract. It is shown that in the group of invertible measurable nonsingular transformations on a Lebesgue probability space, endowed with the coarse topology, the transformations with infinite ergodic index are generic; they actually form a dense G_δ set. (A transformation has infinite ergodic index if all its finite Cartesian powers are ergodic.) This answers a question asked by C. Silva. A similar result was proved by U. Sachdeva in 1971, for the group of transformations preserving an infinite measure. Exploring other possible (more restrictive) definitions of infinite ergodic index, we find, somewhat surprisingly, that if a nonsingular transformation on a Lebesgue probability space has an *infinite* Cartesian power which is nonsingular with respect to the power measure, then it has to be measure preserving.

Let $\mathcal{G} = \mathcal{G}(X, \mathbf{M}, \mu)$ be the group of invertible measurable nonsingular transformations on a Lebesgue (probability) space, endowed with the coarse topology, and $\mathcal{M}(\lambda)$ the subgroup preserving a finite or σ-finite measure $\lambda \sim \mu$. Let $T \in \mathcal{G}$, $k \in \mathbb{N}$. Then T is said to have *ergodic index* k if the Cartesian product $T^{(k)} = T \times \ldots \times T$ (k factors) is ergodic, but $T^{(k+1)}$ is not ergodic. We say that T has *ergodic index* 0 if T is not ergodic, and that T has *ergodic index* ∞ if $T^{(k)}$ is ergodic for all $k \in \mathbb{N}$. Note that if $\mathcal{E}$ is the set of ergodics in $\mathcal{G}$ and $\mathcal{L}_k = \{T \in \mathcal{G} : T \text{ has ergodic index } k\}$ then $\mathcal{E} = \bigcup_{1 \le k \le \infty} \mathcal{L}_k$. For $T \in \mathcal{M}(\mu)$, T is weakly mixing iff $T \times T$ is ergodic and this implies $T^{(k)}$ ergodic for all $k \in \mathbb{N}$; thus in $\mathcal{M}(\mu)$ the ergodic index can only be 0, 1 or ∞. However, if ν is an infinite σ-finite measure $\sim \mu$, then Kakutani and Parry [K-P] have shown that $\mathcal{L}_k \neq \emptyset$ for all $k \in \mathbb{N}$, and also for $k = 0$ or ∞. Sachdeva [S] has shown that for such infinite, σ-finite ν, $\mathcal{L}_\infty$ is a dense G_δ in $\mathcal{M}(\nu)$ (coarse topology), and since $\mathcal{E} = \bigcup_{1 \le k \le \infty} \mathcal{L}_k$, it follows that $\mathcal{L}_k$ is meagre in $\mathcal{E}$ for all $k \in \mathbb{N}$. Recently, Cesar Silva asked us if a similar result holds in $\mathcal{G}$. Combining the methods of Sachdeva with those of Choksi and Kakutani [C-K], we show that indeed it does.

2000 *Mathematics Subject Classification*: Primary 28D99; Secondary 28D05.

Notation, etc. We follow that of [C-K, pp. 453–457]. Put

$$\omega_n(x) = \frac{d\mu \circ T^{-n}}{d\mu}(x), \qquad n = 0, 1, 2, \ldots,$$

and

$$\omega_n^{(k)}(\underline{x}) = \frac{d\mu^{(k)} \circ (T^{(k)})^{-n}}{d\mu^{(k)}}(\underline{x}), \qquad k \in \mathbb{N},\ n = 0, 1, 2, \ldots,$$

where $\underline{x} = (x_1, \ldots, x_k)$ and $\mu^{(k)}$ is the kth power measure of μ on $X^{(k)} = X \times \ldots \times X$ (k factors). Note that by Fubini's Theorem, $\omega_n^{(k)}(x_1, \ldots, x_k) = \omega_n(x_1) \ldots \omega_n(x_k)$. For each $T \in \mathcal{G}$, we denote by U_T the invertible L^1 isometry given by

$$U_T f(x) = f(T^{-1}x) \frac{d\mu \circ T^{-1}}{d\mu}(x)$$

for $f \in L^1(\mu)$. [Note that in [C-K] elements of $\mathcal{G}$ are denoted by τ etc. and U_T by T_τ.] Note that $U_T^n 1 = \omega_n(x)$. For $f \in L^1$ put

$$\begin{aligned} R_n(T, f) &= \sum_{j=0}^{n-1} f(T^{-j}x)\omega_j(x) \Big/ \sum_{j=0}^{n-1} \omega_j(x) \\ &= \sum_{j=0}^{n-1} (U_T^j f)(x) \Big/ \sum_{j=0}^{n-1} U_T^j 1(x). \end{aligned}$$

The Hopf decomposition theorem says that given $T \in \mathcal{G}$, $X = \Omega_i \cup \Omega_w$ where Ω_i, Ω_w are disjoint and $T|\Omega_i$ is incompressible, $T|\Omega_w$ completely dissipative. Further

$$\Omega_i = \Big\{x : \sum_{n=0}^{\infty} \omega_n(x) = \infty\Big\}, \qquad \Omega_w = \Big\{x : \sum_{n=0}^{\infty} \omega_n(x) < \infty\Big\}.$$

We will also need the Hurewicz–Halmos–Oxtoby ergodic theorem for $T \in \mathcal{G}$ [C-K, pp. 456–457], and for $T^{(k)}$ on $(X^{(k)}, \mu^{(k)})$.

Ergodic Theorem. *Let $T \in \mathcal{G}$. Then on Ω_i, for all $f \in L^1(\mu)$, $R_n(T, f)$ converges a.e. μ to a limit function $\widetilde{f} \in L^1(\mu)$; $\widetilde{f}(Tx) = \widetilde{f}(x)$ a.e. μ and*

$$\int_X \widetilde{f}\, d\mu = \int_X f\, d\mu.$$

If T is ergodic then $\widetilde{f}$ is a constant, $\int_X f\, d\mu$. If in addition $f = \chi_E$, then $\widetilde{f} = \mu(E)$.

On Ω_w, for all $f \in L^1(\mu)$, $R_n(T, f)$ converges to the ratio of the two convergent series $\sum_{n=0}^{\infty} f(T^{-n}x)\omega_n(x)$ and $\sum_{n=0}^{\infty} \omega_n(x)$.

Theorem 1. *$\mathcal{L}_\infty$ is a dense G_δ in $\mathcal{G}$ endowed with the coarse topology.*

Proof. The proof is analogous to that of Theorem 1 in [C-K, pp. 457–458]. Let $\mathcal{A} = \{A_j : j \in \mathbb{N}\}$ be a countable dense sequence in the measure algebra of μ, which is also an algebra of sets. For each $k \in \mathbb{N}$, let $\mathcal{A}^{(k)}$ be the algebra of finite disjoint unions of sets of the form $A_{j_1} \times \ldots \times A_{j_k}$, $A_{j_i} \in \mathcal{A}$. Note that in addition to being an algebra, $\mathcal{A}^{(k)}$ is countable and dense in the measure algebra of $\mu^{(k)}$. Write $\mathcal{A}^{(k)} = \{B_1^{(k)}, B_2^{(k)}, \ldots\}$. For each $j, k, m, n, p \in \mathbb{N}$, let

$$H(j,k,m,n,p) = \{T \in \mathcal{G} :$$
$$\mu^{(k)}(\{(x_1, \ldots, x_k) : |R_n(T^{(k)}, \chi_{B_j^{(k)}}(x_1 \ldots x_k) - \mu^{(k)}(B_j^{(k)})| \geq 1/m\}) < 1/p\}$$

(where R_n is as in the ergodic theorem). Then each $H(j,k,m,n,p)$ is open in $\mathcal{G}$ in the coarse topology. Hence

$$H = \bigcap_{j=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcap_{p=1}^{\infty} H(j,k,m,n,p)$$

is a G_δ in $\mathcal{G}$ in the coarse topology. We claim that $H = \mathcal{L}_\infty$.

For each $k \in \mathbb{N}$, put

$$H_k = \bigcap_{j=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcap_{p=1}^{\infty} H(j,k,m,n,p).$$

First note that if $k = 1$, then the proof of Theorem 1 of [C-K, pp. 457–458] shows that $H_1 = \mathcal{E}$, the set of ergodics in $\mathcal{G}$. [In [C-K], $H(j,1,m,n,p)$ is denoted by $A(j,m,n,p)$ and H_1 by A.] Since $H = \bigcap_{k=1}^{\infty} H_k$, it follows that $H \subseteq \mathcal{E}$, i.e. every $T \in H$ is ergodic, and so incompressible. If $T^{(k)}$ is ergodic, then by the ergodic theorem, for any $\mu^{(k)}$-measurable set E, $R_n(T^{(k)}, \chi_E) \to \mu^{(k)}(E)$ a.e. $\mu^{(k)}$ and hence in $\mu^{(k)}$ measure; in particular this happens for $E = B_j^{(k)}$, and so $T \in H_k$. Thus $\mathcal{L}_\infty \subseteq \bigcap_{k=1}^{\infty} H_k = H$.

To prove that $H \subseteq \mathcal{L}_\infty$ suppose that $T \in H \setminus \mathcal{L}_\infty$. Then $T^{(k)}$ is not ergodic for some $k \in \mathbb{N}$. Then $T^{(k)}$ has a nontrivial invariant set F, clearly $R_n(T^{(k)}, \chi_F) = \chi_F$ for all n, and so $\lim_{n\to\infty} R_n(T^{(k)}, \chi_F) = \chi_F \neq \mu^{(k)}(F)$. However since $T \in H \subseteq H_k$, $R_n(T^{(k)}, \chi_{B_j}^{(k)}) \to \mu^{(k)}(B_j^{(k)})$ in $\mu^{(k)}$ measure for each $j \in \mathbb{N}$. Since $R_n(T^{(k)}, \chi_{B_j}^{(k)})$ *does* converge a.e. $\mu^{(k)}$ by the ergodic theorem, it follows that $R_n(T^{(k)}, \chi_{B_j}^{(k)}) \to \mu^{(k)}(B_j^{(k)})$ a.e. $\mu^{(k)}$ for each $j \in \mathbb{N}$, i.e. for all sets in the algebra $\mathcal{A}^{(k)}$. Now every $\mu^{(k)}$-measurable set is equal, up to a set of zero $\mu^{(k)}$ measure, to a set in $\mathcal{A}_{\sigma\delta}^{(k)}$.

Case (i): $T^{(k)}$ *is incompressible*. We obtain a contradiction by showing that $R_n(T^{(k)}, \chi_E) \to \mu^{(k)}(E)$ for all $\mu^{(k)}$-measurable E.

Since this is trivially true if $\mu^{(k)}(E) = 0$, it is sufficient to show this for all $E \in \mathcal{A}_{\sigma\delta}^{(k)}$. Suppose first that $E \in \mathcal{A}_{\sigma}^{(k)}$. Then since $\mathcal{A}^{(k)}$ is an algebra there exist $B_{j_r}^{(k)} \uparrow E$ as $r \to \infty$, so that $\chi_{B_{j_r}^{(k)}} \uparrow \chi_E$. Since $U_{T^{(k)}}$ and so $R_n(T^{(k)}, \cdot)$ are positive operators, it follows that

$$R_n(T^{(k)}, \chi_{B_{j_r}}^{(k)}) \le R_n(T^{(k)}, \chi_E).$$

Hence

$$\mu^{(k)}(B_{j_r}^{(k)}) \le \lim_{n\to\infty} R_n(T^{(k)}, \chi_E)$$

for all $r \in \mathbb{N}$, and so

$$\mu^{(k)}(E) \le \lim_{n\to\infty} R_n(T^{(k)}, \chi_E).$$

But

$$\begin{aligned}
\mu^{(k)}&(E) \\
&= \int_{X^{(k)}} \mu^{(k)}(E)\,\mu^{(k)}(d(x_1,\ldots,x_k)) \\
&\le \int_{X^{(k)}} \lim_{n\to\infty} R_n(T^{(k)}, \chi_E)(x_1,\ldots,x_k)\,\mu^{(k)}(d(x_1,\ldots,x_k)) \\
&= \int_{X^{(k)}} \chi_E(x_1,\ldots,x_k)\,\mu^{(k)}(d(x_1,\ldots,x_k)) \quad \text{(by the ergodic theorem)} \\
&= \mu^{(k)}(E).
\end{aligned}$$

Hence $\mu^{(k)}(E) = \lim_{n\to\infty} R_n(T^{(k)}, \chi_E)$. A similar argument with decreasing sequences in $\mathcal{A}_\sigma^{(k)}$ proves that for every set $E \in \mathcal{A}_{\sigma\delta}^{(k)}$,

$$\lim_{n\to\infty} R_n(T^{(k)}, \chi_E) = \mu^{(k)}(E) \quad \text{a.e. } \mu^{(k)}.$$

This gives the desired contradiction by taking a set in $\mathcal{A}_{\sigma\delta}^{(k)}$ $\mu^{(k)}$-equivalent to the $T^{(k)}$-invariant set F.

CASE (ii): *$T^{(k)}$ is completely dissipative.* We first show that in this case if for any $\mu^{(k)}$-measurable set E the limit function $\phi = \lim_{n\to\infty} R_n(T^{(k)}, \chi_E)$ is $T^{(k)}$-invariant, then $T^{(k)}(E) = E$. For, by the ergodic theorem, this limit function ϕ is the ratio of the two convergent series

$$\begin{aligned}
\phi&(x_1,\ldots,x_k) \\
&= \sum_{j=0}^{\infty} \chi_E((T^{(k)})^{-j}(x_1,\ldots,x_k))\omega_j^{(k)}(x_1,\ldots,x_k) \Big/ \sum_{j=0}^{\infty} \omega_j^{(k)}(x_1,\ldots,x_k).
\end{aligned}$$

Hence since

$$\omega_j^{(k)}((T^{(k)})^{-1}(x_1,\ldots,x_k)) = \omega_{j+1}^{(k)}(x_1,\ldots,x_k)/\omega_1^{(k)}(x_1,\ldots,x_k),$$

if ϕ is $T^{(k)}$-invariant then

$$\begin{aligned}\phi(x_1,\dots,x_k) & \\ &= \phi((T^{(k)})^{-1}(x_1,\dots,x_k)) \\ &= \sum_{j=1}^{\infty}\chi_E((T^{(k)})^{-j}(x_1,\dots,x_k))\omega_j^{(k)}(x_1,\dots,x_k)\Big/\sum_{j=1}^{\infty}\omega_j^{(k)}(x_1,\dots,x_k).\end{aligned}$$

This implies that

$$\phi(x_1,\dots,x_k)=\frac{\chi_E(x_1,\dots,x_k)\omega_0(x_1,\dots,x_k)}{\omega_0(x_1,\dots,x_k)}=\chi_E(x_1,\dots,x_k),$$

since for any four positive numbers a, a', b, b', $(a+a')/(b+b')=a/b$ implies $a/b=a'/b'$. Since each $\mu^{(k)}(B_j^{(k)})$ is a $T^{(k)}$-invariant constant function it follows that each $B_j^{(k)}$ is $T^{(k)}$-invariant and so $T^{(k)}$ is the identity, contradicting the assumption that it is completely dissipative.

CASE (iii): *Neither* $\Omega_i^{(k)}$ *nor* $\Omega_w^{(k)}$ (the incompressible and dissipative components of $T^{(k)}$) *is empty.* We shall see that if $\Omega_w^{(k)}$ is nonempty, the argument of Case (ii) can be applied. Since $T^{(k)}\Omega_w^{(k)}=\Omega_w^{(k)}$, the argument of Case (ii) applies to any $E\subseteq\Omega_w^{(k)}$. For each $j\in\mathbb{N}$, let $F_j=\Omega_w^{(k)}\cap B_j^{(k)}$. Then $\{F_j : j\in\mathbb{N}\}$ is dense in the measure algebra of $\mu^{(k)}|\Omega_w^{(k)}$. Now

$$R_n(T^{(k)},\chi_{F_j})=\chi_{\Omega_w^{(k)}}R_n(T^{(k)},\chi_{B_j^{(k)}})\to\chi_{\Omega_w^{(k)}}\mu^{(k)}(B_j^{(k)})$$

as $n\to\infty$, which is invariant under $T^{(k)}$. By the argument of Case (ii), $T^{(k)}F_j=F_j$ for all $j\in\mathbb{N}$ and so $T^{(k)}|\Omega_w^{(k)}$ is the identity, which contradicts the fact that $T^{(k)}|\Omega_w$ is completely dissipative.

We thus have a contradiction to the assumption that $H\setminus\mathcal{L}_\infty$ is nonempty and so $H=\mathcal{L}_\infty$ and $\mathcal{L}_\infty$ is a G_δ in $\mathcal{G}$ for the coarse topology.

Now if T is weakly mixing and μ measure preserving then $T\in\mathcal{L}_\infty$, so $\mathcal{L}_\infty$ is nonempty. Since such a T is antiperiodic, by the conjugacy lemma in $\mathcal{G}$ [C-K, Theorem 2] its conjugates, which also belong to $\mathcal{L}_\infty$, are dense in $\mathcal{G}$; thus $\mathcal{L}_\infty$ is dense in $\mathcal{G}$. This completes the proof of the theorem.

COROLLARY 1. *For each* $k\in\mathbb{N}$, $\mathcal{L}_k$ *is meagre in* $\mathcal{G}$.

Proof. $\mathcal{E}=\bigcup_{1\le k\le\infty}\mathcal{L}_k$, the $\mathcal{L}_k$ are disjoint and both $\mathcal{E}$ and $\mathcal{L}_\infty$ are dense G_δ sets. Hence each $\mathcal{L}_k$, $k\in\mathbb{N}$, is meagre in $\mathcal{G}$. (Of course the Kakutani–Parry results show that they are all nonempty.)

COROLLARY 2. $C=\{T\in\mathcal{G} : T$ *has no* L^∞ *eigenvalue*$\}$ *is residual in* $\mathcal{G}$.

Proof. $C\supseteq\{T\in\mathcal{G} : T\times T$ is ergodic$\}\supseteq\mathcal{L}_\infty$.

REMARK 1. It is an easy corollary of Theorems 1 and 3 of [C-K] that the set $\{T \in \mathcal{G} : T \text{ totally ergodic}\} = \{T \in \mathcal{G} : T^i \text{ ergodic for all } i \in \mathbb{N}\}$ is a dense G_δ in $\mathcal{G}$.

REMARK 2. Since $(X, \mathbf{M}, \mu)$ is a probability measure space, we can form the infinite product measure space

$$(X^{(\omega)}, \mathbf{M}^{(\omega)}, \mu^{(\omega)}) = (X \times X \times \ldots, \mathbf{M} \otimes \mathbf{M} \otimes \ldots, \mu \times \mu \times \ldots)$$

and for $T \in \mathcal{G}(X, \mathbf{M}, \mu)$, $T^{(\omega)} = T \times T \times \ldots$, the obvious bijection on $X^{(\omega)}$. One would naturally tend to assume that $T^{(\omega)} \in \mathcal{G}(X^{(\omega)}, \mathbf{M}^{(\omega)}, \mu^{(\omega)})$. However we have (somewhat surprisingly)

PROPOSITION 1. *$T^{(\omega)}$ is nonsingular on $(X^{(\omega)}, \mathbf{M}^{(\omega)}, \mu^{(\omega)})$ if and only if T is measure preserving.*

Proof. The "if" part is obvious. To prove the "only if" part note that T is measure preserving if and only if $\mu \circ T^{-1} = \mu$ or $d\mu \circ T^{-1}/d\mu = 1$ a.e. μ.

We claim that if $\mu \circ T^{-1} \neq \mu$, then the power measures $(\mu \circ T^{-1})^{(\omega)} = (\mu \circ T^{-1}) \times (\mu \circ T^{-1}) \times \ldots$ and $\mu^{(\omega)} = \mu \times \mu \times \ldots$ are mutually singular. This will prove the "only if" part of Proposition 1. It follows from

PROPOSITION 2. *If μ, ν are two equivalent probability measures on $(X, \mathbf{M})$, then the power measures $\mu^{(\omega)}$ and $\nu^{(\omega)}$ are mutually singular unless $\mu = \nu$.*

First proof of Proposition 2. This follows easily from the main result of Kakutani's by now classical paper [K] on equivalence of infinite product measures. By hypothesis $\mu \sim \nu$, put $w(x) = \frac{d\nu}{d\mu}(x)$, and put $\varrho(\mu, \nu) = \int_X \sqrt{w(x)}\, \mu(dx)$. Then the Cauchy–Schwarz inequality shows that $0 < \varrho(\mu, \nu) \le 1$, and that $\varrho(\mu, \nu) = 1$ if and only if $\mu = \nu$. So if $\mu \neq \nu$, then $(\varrho(\mu, \nu))^n \to 0$ as $n \to \infty$. Hence by the Main Theorem of [K, p. 218], the power measures $\mu^{(\omega)}$ and $\nu^{(\omega)}$ are mutually singular, proving Proposition 2, and so completing the proof of Proposition 1.

Second proof of Proposition 2 (not using the Kakutani criterion). First observe that if we have a product measure on the countable product $\{0, 1\}^\omega$ of the two-point space $\{0, 1\}$ with $P(\{0\}) = p$, $P(\{1\}) = 1 - p = q$, then different probability vectors (p, q) give mutually singular product measures. This is an easy consequence of the strong law of large numbers. We apply this to the situation on hand. Assume that $\mu \neq \nu$ and let $A \subseteq X$ be a measurable set with $\mu(A) \neq \nu(A)$. Let $\mathcal{B}$ be the σ-algebra $\{X, \emptyset, A, X \setminus A\}$ and consider the sub-σ-algebra of $\mathbf{M}^{(\omega)}$ given by $\mathcal{B} \times \mathcal{B} \times \ldots$ The restrictions of $\mu^{(\omega)}$ and $\nu^{(\omega)}$ to this sub-σ-algebra are mutually singular and so supported on disjoint subsets of $\mathbf{M}^{(\omega)}$. The proposition follows.

NOTE. Even if T preserves a finite measure $m \sim \mu$, $m \neq \mu$, it is still false that $T^{(\omega)} \in \mathcal{G}(X^{(\omega)}, \mathbf{M}^{(\omega)}, \mu^{(\omega)})$ though $T^{(\omega)} \in \mathcal{M}(X^{(\omega)}, \mathbf{M}^{(\omega)}, m^{(\omega)})$.

Stochastic operators. In 1979–80, Anzelm Iwanik [I] extended the results of Choksi–Kakutani [C-K, Theorems 1 and 3] to the set $\mathcal{S}$ of *stochastic* operators, i.e. positive, integral preserving contractions on $L^1(X, \mathbf{M}, \mu)$ endowed with the strong operator topology, which induces the coarse topology when restricted to $\mathcal{G}$. He showed [I, Theorem 3] that in $\mathcal{S}$, the set of ergodic, conservative operators forms a dense G_δ in the strong operator topology. In the absence of a natural definition of ergodic index for stochastic operators, we are unable to extend the results of the present paper to the situation of stochastic operators.

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Received 1 July 1999;
revised 17 August 1999 (3785)