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CONJUGACIES BETWEEN ERGODIC TRANSFORMATIONS AND THEIR INVERSES

BY

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Dedicated to the memory of Anzelm Iwanik

Abstract. We study certain symmetries that arise when automorphisms S and T defined on a Lebesgue probability space $(X,\mathcal{F},\mu)$ satisfy the equation $ST=T^{-1}S$. In an earlier paper [6] it was shown that this puts certain constraints on the spectrum of T. Here we show that it also forces constraints on the spectrum of S^2. In particular, S^2 has to have a multiplicity function which only takes even values on the orthogonal complement of the subspace $\{f\in L^2(X,\mathcal{F},\mu): f(T^2x)=f(x)\}$. For S and T ergodic satisfying this equation further constraints arise, which we illustrate with examples. As an application of these results we give a general method for constructing weakly mixing rank one maps T for which T^2 has non-simple spectrum.

0. Introduction. Let S and T be invertible measure preserving transformations (automorphisms) defined on a Lebesgue probability space $(X,\mathcal{F},\mu)$. It was shown in [6] that if $ST=T^{-1}S$, where S and T are automorphisms, then T has an even multiplicity function on the orthogonal complement of the subspace

$$H=\{f\in L^2(X,\mu): f(S^2)=f\}.$$

In this paper we are interested in the form a conjugating map S between an ergodic transformation T and its inverse T^{-1} can take. It is known that if T has simple spectrum and $ST=T^{-1}S$, then $S^2=I$, the identity automorphism [5]. Similar results for finite rank maps are known [7]. Generally, it is known that S can take any even order, it can be aperiodic or weakly mixing. We give examples with S and T ergodic but not weakly mixing and we observe that in this case -1 is necessarily the unique eigenvalue of both S and T. Our main theorem, which restricts considerably the form the conjugating map S can take, is:

THEOREM 1. *If S and T are automorphisms with $ST=T^{-1}S$, then S^2 has an even multiplicity function on the orthogonal complement of the subspace*

$$\{f\in L^2(X,\mu): f(T^2)=f\}.$$

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This result is actually a corollary of a more general result concerning unitary operators on L^2-spaces preserving real-valued functions, which we prove in Section 3. Immediate consequences are that if S^2 has simple spectrum, then $T^2 = I$, and if S has simple spectrum and S and T are weakly mixing, then S^2 has a homogeneous spectrum of multiplicity two. Examples of these different situations are given in Section 4. In particular, we answer the question (asked in [4]) whether a rank one transformation T can have the property that its square T^2 has a non-simple spectrum. El Abdalaoui [2], [3] has shown that almost surely, the powers of the Ornstein rank one maps have simple spectrum.

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1. Preliminaries. For a unitary operator $U : H \to H$ defined on a separable Hilbert space H, U is completely determined up to unitary equivalence by a measure σ defined on the unit circle S^1, called the *maximal spectral type* of U, and a function $\varrho : S^1 \to \mathbb{Z}^+ \cup \{\infty\}$, called the *multiplicity function*. The *essential values* of this function are the values it takes almost everywhere with respect to σ (see [8]). An operator $U : H \to H$ is said to have a *homogeneous spectrum of multiplicity* n if its multiplicity function takes the constant value n (a.e. σ). U is said to have *simple spectrum* if its multiplicity function takes the constant value 1 (a.e. σ). Automorphisms give rise to unitary operators in the following way: If we define

$$\widehat{T} : L^2(X,\mu) \to L^2(X,\mu) \quad \text{by} \quad \widehat{T}f(x) = f(Tx), \quad x \in X, \ f \in L^2(X,\mu),$$

then $\widehat{T}$ is a unitary operator. If for an automorphism T, $\widehat{T}$ has simple spectrum, then T is necessarily ergodic.

2. Basic results. Our initial aim is to show that there are severe restrictions on the type of automorphism S which can conjugate an ergodic transformation to its inverse, i.e., we see that the equation $ST = T^{-1}S$ cannot hold for ergodic T, for certain classes of transformations S. The first two parts of the following proposition were given in [5] and [6] respectively, so their proofs are only sketched here.

PROPOSITION 1. *Suppose S and T are automorphisms for which $ST = T^{-1}S$. Then*

(i) *If T is ergodic then either $S^{2n} = I$ for some $n > 0$, or S is aperiodic.*
(ii) *If S^2 and T are ergodic, then S and T are weakly mixing.*
(iii) *If S is ergodic, then either $T^n = I$ for some $n \geq 0$, or T is aperiodic.*

Proof. (i) This proof is similar to that of part (iii).

(ii) Since $ST = T^{-1}S$ where S^2 and T are ergodic, we have $S^2T = TS^2$. If $\mathcal{K}(T)$ denotes the Kronecker factor of T, then $\mathcal{K}(T) = \mathcal{K}(S^2)$. However,

T has an even multiplicity function (from [6]). But this is impossible if T has any eigenvalues as T ergodic implies that every eigenvalue is simple, so $\mathcal{K}(T)$ is trivial and the result follows.

(iii) Suppose that T is not aperiodic and $T \neq I$. Then there exists $n > 0$ such that

$$\text{if} \quad A_n = \{x \in X : T^n x = x\} \quad \text{then} \quad \mu(A_n) > 0.$$

Clearly, A_n is S-invariant, so that since S is ergodic, $\mu(A_n) = 1$, or $T^n = I$. ∎

The following new result shows a type of duality between the properties of S and T appearing throughout this paper. This result is analogous to Proposition 1(ii). Later we give examples of ergodic S and T with $ST = T^{-1}S$, but without S^2 and T^2 being ergodic.

THEOREM 2. *If $ST = T^{-1}S$ where S and T^2 are ergodic, then S and T are weakly mixing.*

Proof. If we can show that S^2 has to be ergodic, then Proposition 1(ii) will give us our result. Consequently, it suffices to show that -1 is not an eigenvalue of S. Suppose then that there exists $f_0 \in \mathbf{C}^\perp$, the orthogonal complement of the constant functions, satisfying $\widehat{S} f_0 = -f_0$. Then $S^2 f_0 = f_0$.

From Theorem 1, we know that S^2 has an even multiplicity function on the subspace

$$H^\perp = \{f \in L^2(X,\mu) : \widehat{T}^2 f = f\}^\perp = \mathbf{C}^\perp,$$

since T^2 is ergodic.

It follows that there exists $g_0 \in H^\perp$ for which $\widehat{S}^2 g_0 = g_0$, with $\langle f_0, g_0\rangle = 0$.

Set $h_0 = g_0 + \widehat{S} g_0$. Then $\widehat{S} h_0 = h_0$, and S ergodic implies that $h_0 =$ constant, which must be zero as h_0 is in the orthogonal complement of the constant functions. Consequently, $\widehat{S} g_0 = -g_0$ and we have two orthogonal eigenfunctions corresponding to the eigenvalue -1, contradicting the ergodicity of S. ∎

COROLLARY 1. *If S and T are ergodic with $ST = T^{-1}S$ and either S^2 or T^2 is ergodic, then both S and T are weakly mixing.*

Proof. Apply Proposition 1(ii) and Theorem 2. ∎

We now show that for S and T ergodic the only possible eigenvalue is -1.

LEMMA 1. *Suppose that S and T are ergodic, $ST = T^{-1}S$ and $f(Tx) = \lambda f(x)$ for $f \neq$ constant. Then $f \circ S^2 = f$ and $f \circ S(x) = -f(x)$.*

Proof. Suppose that $\lambda \neq -1$. Then

$$f(Tx) = \lambda f(x) \Rightarrow f(TSx) = \lambda f(Sx) \Rightarrow f(ST^{-1}x) = \lambda f(Sx)$$
$$\Rightarrow \lambda f(STx) = f(Sx) \Rightarrow \overline{f} \circ S(Tx) = \lambda \overline{f} \circ S(x),$$

and since T is ergodic, $\bar{f} \circ S = c \cdot f$ for some constant c with $|c| = 1$, so that $f(S^2x) = f(x)$. As in the proof of Theorem 2, this implies that $f(Sx) = -f(x)$.

If $\lambda = -1$, a similar proof works on noting that for f with $|f| = 1$, f has to be real-valued taking only the values ± 1. ■

PROPOSITION 2. *If $ST = T^{-1}S$ where S and T are ergodic but not weakly mixing, then -1 is an eigenvalue of both S and T and they have no other eigenvalues.*

Proof. By Corollary 1, S^2 and T^2 are non-ergodic, so -1 is an eigenvalue of both of them. Suppose that T has an eigenvalue $\lambda \neq 1$ and that $f(Tx) = \lambda f(x)$, $f \in \mathbf{C}^{\perp}$. By Lemma 1, $f(Sx) = -f(x)$. But $\widehat{S}\widehat{T}f = \widehat{T}^{-1}\widehat{S}f$, and this implies $\lambda\widehat{S}f = -\widehat{T}^{-1}f$, or $-\lambda f = -\overline{\lambda} f$, so that $\lambda = \overline{\lambda}$. Consequently, λ is real and must be equal to -1.

This shows that -1 is the only eigenvalue of $\widehat{T}$ and that the 1-dimensional subspaces

$$\{g \in L^2(X, \mathcal{F}, \mu) : g(Tx) = -g(x)\}, \qquad \{g \in L^2(X, \mathcal{F}, \mu) : g(Sx) = -g(x)\}$$

are equal. Let ζ be the partition of X corresponding to the Kronecker factor of T ($\zeta = \{A, A^{\mathrm{c}}\}$ where $A = f^{-1}(1)$ and f is the normalized eigenfunction corresponding to the eigenvalue -1). If T_ζ is the factor map, then $S_\zeta T_\zeta = T_\zeta^{-1} S_\zeta$ where S_ζ is the corresponding factor of S. Since S_ζ is ergodic and T_ζ is weakly mixing, it follows from Theorem 2 that S_ζ is weakly mixing, hence the result. ■

We shall give an example of the situation where S is *prime* (i.e., has no non-trivial factors) and T is weakly mixing. It is known [5] that if T is prime and $ST = T^{-1}S$ then $S^2 = I$ or S is weakly mixing, but no examples of the latter situation are known and may in fact be impossible. (See Rudolph [11] or del Junco and Rudolph [9] for definitions and properties of prime transformations and the notion of minimal self-joinings.) The automorphism S has the *weak closure property* if the set $\{S^n : n \in \mathbb{Z}\}$ is dense in $C(S)$ with respect to the weak topology on the set of all automorphisms ($T_n \to T$ if $\mu(T_n A \triangle TA) + \mu(T_n^{-1} A \triangle T^{-1} A) \to 0$ as $n \to \infty$, for each $A \in \mathcal{F}$). We write $C(S) = WC(S)$ when S has the weak closure property.

Note that if S and T are finite rotations with $ST = T^{-1}S$, then we must have $S^2 = T^2 = I$, each acting on a two-point space. This is a consequence of S and T being ergodic but not weakly mixing, so -1 is the only eigenvalue of each.

PROPOSITION 3. *Suppose that S and T are automorphisms satisfying $ST = T^{-1}S$. Then*

(i) *If S is weakly mixing and prime, then $T = I$ or T is weakly mixing.*
(ii) *If S is weakly mixing and has minimal self-joinings, then $T = I$.*
(iii) *If S is ergodic and S^2 has the weak closure property, then S is rigid and $T^2 = I$.*

In case (i), *if S is not weakly mixing, then S is a finite rotation on a 2-point space and $S^2 = T^2 = I$. In case* (ii), *if S is not weakly mixing, then S is a finite rotation, $S^{2n} = I$ for some $n > 0$ and $T^2 = I$.*

Proof. (i) The σ-algebra

$$\mathcal{A} = \{A \in \mathcal{F} : TA = A\}$$

is clearly S-invariant since if $A \in \mathcal{A}$, then $TSA = ST^{-1}A = SA$, so $SA \in \mathcal{A}$. Since S is prime, we have either $\mathcal{A} = \mathcal{F}$, or $\mathcal{A} = \mathcal{N}$, the trivial σ-algebra. In the former case $T = I$, and in the latter case, T is ergodic, and since S is weakly mixing, T is also weakly mixing.

If S were not weakly mixing it would have to be a finite rotation on a two-point space (since -1 is the only eigenvalue), so $S^2 = T^2 = I$.

(ii) S has minimal self-joinings and $S^2T = TS^2$, so $T \in C(S^2) = C(S) = \{S^k : k \in \mathbb{Z}\}$ (del Junco and Rudolph [9]). Thus $T = S^n$ for some $n \in \mathbb{Z}$ and

$$ST = T^{-1}S \Rightarrow S^{n+1} = S^{-n+1} \Rightarrow S^{2n} = I.$$

Now it is known [9] that if S has minimal self-joinings, then S is either weakly mixing or a finite rotation. If $n = 0$, then $T = I$ (which is always a possibility). If $n \neq 0$, then $S^{2n} = I$ implies that S is a finite rotation with $T^2 = S^{2n} = I$.

(iii) As S^2 has the weak closure property, $C(S^2)$ either is uncountable or consists only of the powers of S^2. But $S \in C(S^2)$, so the latter cannot happen. Since $T \in C(S^2) = WC(S^2)$, there is an increasing sequence $n_i \to \infty$ such that

$$T = \lim_{i\to\infty} (S^2)^{n_i} = \lim_{i\to\infty} S^{2n_i}.$$

Then

$$ST = T^{-1}S \Rightarrow \lim_{i\to\infty} S^{2n_i+1} = \lim_{i\to\infty} S^{-2n_i+1} \Rightarrow \lim_{i\to\infty} S^{4n_i} = I \Rightarrow T^2 = I. \blacksquare$$

Part (iii) of the above proposition is applicable when both S and S^2 are rank one (for example, as in the following corollary, if S is ergodic with discrete spectrum and -1 is not an eigenvalue of S).

COROLLARY 2. *Suppose that $ST = T^{-1}S$, where S has discrete spectrum and S^2 is ergodic. Then $T^2 = I$.*

Proof. If S has discrete spectrum with S^2 ergodic, then S^2 has discrete spectrum, so has rank one and hence has the weak closure property. The result follows. ■

3. Main theorems. Suppose that $ST = T^{-1}S$, then notice that $T, S \in C(S^2)$, the centralizer of S^2. However, $ST = T^{-1}S \neq TS$ (unless $T^2 = I$), so that $C(S^2)$ is non-abelian. It follows that S^2 has a non-simple spectrum in this case. Our main theorem, from which Theorem 1 and Corollary 3 immediately follow, shows that we can deduce much more in this case. We consider ∞ as an even number.

THEOREM 3. *Suppose that $S, T : L^2(X,\mu) \to L^2(X,\mu)$ are unitary operators that preserve real-valued functions and satisfy $ST = T^{-1}S$. Then in the orthogonal complement of the subspace*

$$H = \{f \in L^2(X,\mu) : T^2(f) = f\},$$

the essential values of the multiplicity function of S^2 are even.

Proof. In this proof, $\mathrm{supp}(\sigma_f)$ denotes the support of a spectral measure σ_f with respect to T. It is a (not necessarily closed) subset of the unit circle S^1 in the complex plane.

Let S^+ and S^- denote the subsets of S^1 in the upper and lower half-planes respectively (excluding ± 1), i.e., $S^1 = S^+ \cup S^- \cup \{\pm 1\}$, disjointly. Write

$$\begin{aligned} H_1 &= \{f \in L^2(X,\mu) : Tf = f\} = \{f \in L^2(X,\mu) : \mathrm{supp}(\sigma_f) \subseteq \{1\}\}, \\ H_{-1} &= \{f \in L^2(X,\mu) : Tf = -f\} = \{f \in L^2(X,\mu) : \mathrm{supp}(\sigma_f) \subseteq \{-1\}\}, \\ \mathcal{P}_1 &= \{f \in L^2(X,\mu) : \mathrm{supp}(\sigma_f) \subseteq S^+\}, \\ \mathcal{P}_2 &= \{f \in L^2(X,\mu) : \mathrm{supp}(\sigma_f) \subseteq S^-\}. \end{aligned}$$

(In each case the spectral measure is with respect to T.) Clearly,

$$L^2(X,\mu) = \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus H_1 \oplus H_{-1} = \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus H.$$

Let $f \in \mathcal{P}_1$. Then $\mathrm{supp}(\sigma_f) \subseteq S^+$ and

$$\langle T^n f, f\rangle = \langle ST^n f, Sf\rangle = \langle T^{-n}Sf, Sf\rangle = \langle Sf, T^n Sf\rangle = \overline{\langle T^n Sf, Sf\rangle}.$$

We have shown that

$$\int_{S^1} z^n \, d\sigma_f(z) = \overline{\int_{S^1} z^n \, d\sigma_{Sf}(z)} = \int_{S^1} z^n \, d\sigma_{Sf}(\overline{z}),$$

and this shows that $\sigma_{Sf}(A) = \sigma_f(\overline{A})$, or $Sf \in \mathcal{P}_2$.

We now see that $\mathcal{P}_1$ and $\mathcal{P}_2$ are orthogonal invariant subspaces (with respect to both T and S^2) since if $f \in \mathcal{P}_1$ then $Sf \in \mathcal{P}_2$ so that $S^2 f \in \mathcal{P}_1$ and $\sigma_{Tf} = \sigma_f$ (with respect to T), hence $f \in \mathcal{P}_1 \Rightarrow Tf \in \mathcal{P}_1$. Furthermore,

$$\langle S^{2n}(Sf), Sf\rangle = \langle S^{2n} f, f\rangle \Rightarrow \sigma_{Sf} = \sigma_f$$

(spectral measure now with respect to S^2).

The above shows that $Z(Sf) \perp Z(f)$ with $\sigma_{Sf} = \sigma_f$ (with respect to S^2). It is now clear that S^2 has an even multiplicity on $\mathcal{P}_1 \oplus \mathcal{P}_2 = H^\perp$ and the result follows. ∎

COROLLARY 3. *If S and T are automorphisms satisfying $ST = T^{-1}S$, then*

(i) *If S^2 has simple spectrum, then $T^2 = I$.*

(ii) *If S has simple spectrum and S and T are weakly mixing, then S^2 has a homogeneous spectrum of multiplicity two.*

(iii) *If S is of rank one with S^2 ergodic and $T^2 \neq I$, then S^2 has rank two and maximal spectral multiplicity equal to two.*

COROLLARY 4. *For T weakly mixing, $T \times T$ is an ergodic transformation having an even multiplicity function.*

Proof. Suppose that T is weakly mixing. Then the maps $T \times T$, $T \times T^{-1}$ and $R(x, y) = (y, Tx)$ are each weakly mixing. Furthermore, $R \circ (T \times T^{-1}) = (T^{-1} \times T) \circ R$, so that by Theorem 3, $R^2 = T \times T$ has an even multiplicity function. ∎

4. Examples. 1. We saw in Corollary 4 that if T is weakly mixing, and $R(x, y) = (y, Tx)$, then $R^2 = T \times T$, so R is weakly mixing. Furthermore,

$$R \circ (T \times T^{-1}) = (T^{-1} \times T) \circ R,$$

and $R^2 = T \times T$ has an even multiplicity function. It was recently shown by Ryzhikov [12] (and independently by Ageev [1]) that $R(x, y) = (y, Tx)$ has simple spectrum for the generic transformation T, so that $T \times T$ has a homogeneous spectrum of multiplicity two in this case. This answered an important open question of Rokhlin (see [4] for a discussion of these results).

2. If T has minimal self-joinings, then it was shown in [7] that $R(x, y) = (y, Tx)$ is prime with trivial centralizer (this also follows from the work of Rudolph [11]). This gives an explicit example of the situation where $ST = T^{-1}S$, with S prime and T weakly mixing (just replace S by R and T by $T \times T^{-1}$).

3. We now give examples of ergodic transformations S and T which are not weakly mixing, satisfying $ST = T^{-1}S$. Necessarily, -1 has to be the unique eigenvalue of both S and T. Simply take S_1 and T_1 to be weakly mixing automorphisms and satisfying $S_1T_1 = T_1^{-1}S_1$. Take $T_0 = S_0$ as rotations on the two-point space $Y = \{0, 1\}$. Let $T = T_1 \times T_0$ and $S = S_1 \times S_0$. Then clearly S and T have the required property.

We see that it is impossible to find ergodic T for which $ST = T^{-1}S$ where S is the 2-adic adding machine. Although S^2 is not ergodic, if such a T were to exist, the only allowable eigenvalue for both S and T would be -1. In fact, if $ST = T^{-1}S$ for T ergodic, having finite uniform rank, S has to have finite order (from [7]).

4. It is possible to give examples of ergodic transformations S for which there exists an order n transformation T (for any integer $n \geq 1$) satisfying

$ST = T^{-1}S$. We illustrate this with the case $n = 3$. Let $\sigma : \mathbb{Z}_3 \to \mathbb{Z}_3$ be the automorphism of the cyclic group $\mathbb{Z}_3 = \{0, 1, 2\}$, $\sigma(0) = 0$, $\sigma(1) = 2$, $\sigma(2) = 1$. Let $S_0 : X \to X$ be ergodic, and $\phi : X \to \mathbb{Z}_3$ a cocycle for which the automorphism extension

$$S_{\phi,\sigma} : X \times \mathbb{Z}_3 \to X \times \mathbb{Z}_3, \qquad S_{\phi,\sigma}(x, g) = (S_0 x, \phi(x) + \sigma(g)),$$

is ergodic with respect to the usual product measure $\widetilde{\mu}$. Define $T : X \times \mathbb{Z}_3 \to X \times \mathbb{Z}_3$ by $T(x, g) = (x, g + 1)$. It is easily checked that $S_{\phi,\sigma}T = T^{-1}S_{\phi,\sigma}$ and $T^3 = I$.

Theorem 3 now implies that $S^2_{\phi,\sigma}$ has an even multiplicity function on the orthogonal complement of the subspace

$$\{f \in L^2(X \times \mathbb{Z}_3, \widetilde{\mu}) : \widehat{T}^2(f) = f\} = \{f \in L^2(X \times \mathbb{Z}_3, \widetilde{\mu}) : \widehat{T}(f) = f\}.$$

5. Using the construction of Example 4 we can give an example of a rank one transformation S which is weakly mixing and such that S^2 has a non-simple spectrum. Simply take S to be $S_{\phi,\sigma}$ satisfying $S_{\phi,\sigma}T = T^{-1}S_{\phi,\sigma}$ as in Example 4, where $T^3 = I$. The construction is started by choosing S_0 to be a weakly mixing rank one map for which the cocycle ϕ is constant on each of the levels (except for the top level) in the construction of S_0. By modifying the usual construction of rank one weakly mixing $\mathbb{Z}_3$-extensions (e.g., see Oseledets [10]), we can ensure that the automorphism extension $S_{\phi,\sigma}$ is weakly mixing and has rank one.

6. If S is ergodic and $ST = T^{-1}S$ where $T^2 = I$, $T \neq I$, then $T = T^{-1}$ and $ST = TS$. It follows that S is a 2-point extension of some factor map S_0. For example, this holds for $(T \times T) \circ F = F \circ (T \times T)$, where T is weakly mixing and F is the "flip" map $F(x, y) = (y, x)$. The factor map in this case is the symmetric cartesian square $T^{2\odot}$.

Conversely, if S is an ergodic 2-point extension of S_0 and σ is the coordinate interchange map, then $S\sigma = \sigma^{-1}S$.

7. Here is an example with $ST = T^{-1}S$, S^2 ergodic (in fact, S is mixing), and T aperiodic, but not ergodic.

Let $\alpha \in [0, 1)$ be irrational, and define $T : [0, 1)^{\mathbb{Z}} \to [0, 1)^{\mathbb{Z}}$ by

$$T(\ldots, x_{-1}, \overset{*}{x}_0, x_1, \ldots) = (\ldots, x_{-1} - \alpha, x_0 \overset{*}{+} \alpha, x_1 - \alpha, \ldots)$$

(the $*$ denotes the zeroth coordinate). If S is the shift map, it is easy to verify that $ST = T^{-1}S$, and that T is aperiodic but not ergodic. Similar examples can be constructed with both S and T weakly mixing.

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