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LINEAR GROWTH OF THE DERIVATIVE FOR MEASURE-PRESERVING DIFFEOMORPHISMS

BY

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Abstract. We consider measure-preserving diffeomorphisms of the torus with zero entropy. We prove that every ergodic C^1-diffeomorphism with linear growth of the derivative is algebraically conjugate to a skew product of an irrational rotation on the circle and a circle C^1-cocycle. We also show that for no positive $\beta \neq 1$ does there exist an ergodic C^2-diffeomorphism whose derivative has polynomial growth with degree β.

1. Introduction. Let M be a compact Riemannian C^1-manifold, $\mathcal{B}$ its Borel σ-algebra and μ its probability Lebesgue measure. Assume that $f : (M, \mathcal{B}, \mu) \to (M, \mathcal{B}, \mu)$ is a measure-preserving C^1-diffeomorphism of the manifold M.

DEFINITION 1. We say that the derivative of f has *linear growth* if the sequence

$$n^{-1} D f^n : M \to \mathcal{L}(TM)$$

converges μ-a.e. to a measurable μ-nonzero function $g : M \to \mathcal{L}(TM)$, i.e. there exists a set $A \in \mathcal{B}$ such that $\mu(A) > 0$ and $g(x) \neq 0$ for all $x \in A$.

Our purpose is to study ergodic diffeomorphisms of the torus with linear growth of the derivative.

By $\mathbb{T}^2$ (resp. $\mathbb{T}$) we will mean the torus $\mathbb{R}^2/\mathbb{Z}^2$ (resp. the circle $\mathbb{R}/\mathbb{Z}$) which most often will be treated as the square $[0,1) \times [0,1)$ (resp. the interval $[0,1)$) with addition mod 1; λ will denote Lebesgue measure on $\mathbb{T}^2$. An example of an ergodic diffeomorphism with linear growth of the derivative is a skew product of an irrational rotation on the circle and a circle C^1-cocycle with nonzero topological degree. Let $\alpha \in \mathbb{T}$ be an irrational number and let $\varphi : \mathbb{T} \to \mathbb{T}$ be a C^1-cocycle. We denote by $d(\varphi)$ the topological degree of φ. Consider the skew product $T_\varphi : (\mathbb{T}^2, \lambda) \to (\mathbb{T}^2, \lambda)$ defined by

$$T_\varphi(x_1, x_2) = (x_1 + \alpha, x_2 + \varphi(x_1)).$$

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LEMMA 1. *The sequence $n^{-1}DT_\varphi^n$ converges uniformly to the matrix*

$$\begin{bmatrix} 0 & 0 \\ d(\varphi) & 0 \end{bmatrix}.$$

Proof. Observe that

$$n^{-1}DT_\varphi^n(x_1,x_2) = \begin{bmatrix} n^{-1} & 0 \\ n^{-1}\sum_{k=0}^{n-1} D\varphi(x_1+k\alpha) & n^{-1} \end{bmatrix}.$$

By the Ergodic Theorem, the sequence $n^{-1}\sum_{k=0}^{n-1} D\varphi(\cdot + k\alpha)$ converges uniformly to $\int_{\mathbb{T}} D\varphi(x)\,dx = d(\varphi)$. ■

It follows that if $d(\varphi) \neq 0$, then T_φ is an ergodic (see [3]) diffeomorphism with linear growth of the derivative.

We will say that diffeomorphisms f_1 and f_2 of $\mathbb{T}^2$ are *algebraically conjugate* if there exists a group automorphism $\psi : \mathbb{T}^2 \to \mathbb{T}^2$ such that $f_1 \circ \psi = \psi \circ f_2$. It is clear that if f_1 has linear growth of the derivative and f_1 and f_2 are algebraically conjugate, then f_2 has linear growth of the derivative. Therefore every C^1-diffeomorphism of $\mathbb{T}^2$ algebraically conjugate to a skew product T_φ with $d(\varphi) \neq 0$ has linear growth of the derivative.

The aim of this paper is to prove that every ergodic measure-preserving C^1-diffeomorphism of the torus with linear growth of the derivative is algebraically conjugate to a skew product of an irrational rotation on the circle and a circle C^1-cocycle with nonzero degree. In [3], A. Iwanik, M. Lemańczyk and D. Rudolph have proved that if φ is a C^2-cocycle with $d(\varphi) \neq 0$, then the skew product T_φ has countable Lebesgue spectrum on the orthocomplement of the space of functions depending only on the first variable. Therefore every ergodic measure-preserving C^2-diffeomorphism of the torus with linear growth of the derivative has countable Lebesgue spectrum on the orthocomplement of its eigenfunctions.

It would be interesting to modify the definition of linear growth of the derivative. For example, one could study a weaker property that there exist positive constants a, b such that

$$0 < a \le \|Df^n\|/n \le b$$

for every natural n. Of course, if a diffeomorphism is C^1-conjugate to a skew product of an irrational rotation on the circle and a circle C^1-cocycle with nonzero degree, then it satisfies this weaker condition and is ergodic. The converse might also be true.

2. Linear growth. We will identify functions on $\mathbb{T}^2$ with $\mathbb{Z}^2$-periodic functions (i.e. periodic of period 1 in each coordinate) on $\mathbb{R}^2$. Assume that $f : (\mathbb{T}^2,\lambda) \to (\mathbb{T}^2,\lambda)$ is a measure-preserving C^1-diffeomorphism. Then there exists a matrix $\{a_{ij}\}_{i,j=1,2} \in M_2(\mathbb{Z})$ and C^1-functions $\widetilde{f}_1, \widetilde{f}_2 : \mathbb{T}^2 \to \mathbb{R}$ such that

$$f(x_1, x_2) = (a_{11}x_1 + a_{12}x_2 + \widetilde{f}_1(x_1, x_2), a_{21}x_1 + a_{22}x_2 + \widetilde{f}_2(x_1, x_2)).$$

Denote by $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}$ the functions given by

$$f_1(x_1, x_2) = a_{11}x_1 + a_{12}x_2 + \widetilde{f}_1(x_1, x_2),$$
$$f_2(x_1, x_2) = a_{21}x_1 + a_{22}x_2 + \widetilde{f}_2(x_1, x_2).$$

Then

$$\left|\det \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\overline{x}) & \frac{\partial f_1}{\partial x_2}(\overline{x}) \\ \frac{\partial f_2}{\partial x_1}(\overline{x}) & \frac{\partial f_2}{\partial x_2}(\overline{x}) \end{bmatrix}\right| = 1 \quad \text{for all } \overline{x} \in \mathbb{R}^2.$$

Suppose that the diffeomorphism f is ergodic. We will need the following lemmas.

LEMMA 2. *If the sequence $n^{-1}Df^n : \mathbb{T}^2 \to M_2(\mathbb{R})$ converges λ-a.e. to a measurable function $g : \mathbb{T}^2 \to M_2(\mathbb{R})$, then*

$$g(\overline{x}) = g(f^n\overline{x})Df^n(\overline{x})$$

for λ-a.e. $\overline{x} \in \mathbb{T}^2$ and all natural n.

Proof. Let $A \subset \mathbb{T}^2$ be a full measure f-invariant set such that if $\overline{x} \in A$, then $\lim_{n\to\infty} n^{-1}Df^n(\overline{x}) = g(\overline{x})$. Assume that $\overline{x} \in A$. Then for any natural m, n we have

$$\frac{m+n}{m}\frac{1}{m+n}Df^{m+n}(\overline{x}) = \frac{1}{m}Df^m(f^n\overline{x})Df^n(\overline{x})$$

and $f^n\overline{x} \in A$. Letting $m \to \infty$, we obtain

$$g(\overline{x}) = g(f^n\overline{x})Df^n(\overline{x}) \quad \text{for } \overline{x} \in A \text{ and } n \in \mathbb{N}. \blacksquare$$

LEMMA 3.

$$\lambda \otimes \lambda(\{(\overline{x}, \overline{y}) \in \mathbb{T}^2 \times \mathbb{T}^2 : g(\overline{y})g(\overline{x}) = 0\}) = 1$$

and

$$\lambda(\{\overline{x} \in \mathbb{T}^2 : g(\overline{x})^2 = 0\}) = 1.$$

Proof. Choose a sequence $\{A_k\}_{k\in\mathbb{N}}$ of measurable subsets of A (see proof of Lemma 2) such that the function $g : A_k \to M_2(\mathbb{R})$ is continuous, all open subsets of A_k (in the induced topology) have positive measure and $\lambda(A_k) > 1 - 1/k$ for any natural k. Since the transformation $f_{A_k} : (A_k, \lambda_{A_k}) \to (A_k, \lambda_{A_k})$ induced by f on A_k is ergodic, for every natural k we can find a measurable subset $B_k \subset A_k$ such that for any $\overline{x} \in B_k$ the sequence $\{f_{A_k}^n\overline{x}\}_{n\in\mathbb{N}}$ is dense in A_k in the induced topology and $\lambda(B_k) = \lambda(A_k)$.

Let $\overline{x}, \overline{y} \in B_k$. Then there exists an increasing sequence $\{m_i\}_{i\in\mathbb{N}}$ of natural numbers such that $f_{A_k}^{m_i}\overline{x} \to \overline{y}$. Hence there exists an increasing sequence $\{n_i\}_{i\in\mathbb{N}}$ of natural numbers such that $f^{n_i}\overline{x} \to \overline{y}$ and $f^{n_i}\overline{x} \in A_k$

for all $i \in \mathbb{N}$. Since $g : A_k \to M_2(\mathbb{R})$ is continuous, we get $g(f_{n_i}\overline{x}) \to g(\overline{y})$. Since

$$\frac{1}{n_i} g(\overline{x}) = g(f^{n_i}\overline{x}) \frac{1}{n_i} Df^{n_i}(\overline{x}),$$

letting $i \to \infty$ we obtain $g(\overline{y})g(\overline{x}) = 0$. Therefore

$$B_k \times B_k \subset \{(\overline{x}, \overline{y}) \in \mathbb{T}^2 \times \mathbb{T}^2 : g(\overline{y})g(\overline{x}) = 0\}$$

and

$$B_k \subset \{\overline{x} \in \mathbb{T}^2 : g(\overline{x})^2 = 0\}$$

for any natural k. It follows that

$$\lambda \otimes \lambda(\{(\overline{x}, \overline{y}) \in \mathbb{T}^2 \times \mathbb{T}^2 : g(\overline{y})g(\overline{x}) = 0\}) > (1 - 1/k)^2$$

and

$$\lambda(\{\overline{x} \in \mathbb{T}^2 : g(\overline{x})^2 = 0\}) > 1 - 1/k$$

for any natural k, which proves the lemma. ■

Lemma 4. *Let $A, B \in M_2(\mathbb{R})$ be nonzero matrices. Suppose that*

$$A^2 = B^2 = AB = BA = 0.$$

Then there exist real numbers $a, b \neq 0$ and c such that

$$A = \begin{bmatrix} ac & -ac^2 \\ a & -ac \end{bmatrix} \quad \textit{and} \quad B = \begin{bmatrix} bc & -bc^2 \\ b & -bc \end{bmatrix}$$

or

$$A = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \quad \textit{and} \quad B = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}.$$

Proof. Since $A^2 = 0$ and $A \neq 0$, we immediately see that the Jordan form of the matrix A is $\left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$. It follows that there exist matrices $C = \{c_{ij}\}_{i,j=1,2}$, $C' = \{c'_{ij}\}_{i,j=1,2} \in M_2(\mathbb{C})$ such that $\det C = \det C' = 1$ and

$$A = C \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} C^{-1} = \begin{bmatrix} c_{12}c_{22} & -c_{12}^2 \\ c_{22}^2 & -c_{12}c_{22} \end{bmatrix},$$

$$B = C' \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (C')^{-1} = \begin{bmatrix} c'_{12}c'_{22} & -c'^2_{12} \\ c'^2_{22} & -c'_{12}c'_{22} \end{bmatrix}.$$

Since the matrices A and B commute, their eigenvectors belonging to 0, i.e. (c_{12}, c_{22}) and (c'_{12}, c'_{22}), generate the same subspace. Therefore there exist real numbers $a, b \neq 0$ and c such that

$$A = \begin{bmatrix} ac & -ac^2 \\ a & -ac \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} bc & -bc^2 \\ b & -bc \end{bmatrix}$$

or

$$A = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}. \ ■$$

LEMMA 5. *Suppose that* $f : (\mathbb{T}^2, \lambda) \to (\mathbb{T}^2, \lambda)$ *is an ergodic measure-preserving* C^1*-diffeomorphism such that the sequence* $n^{-1}Df^n$ *converges* λ*-a.e. to a nonzero measurable function* $g : \mathbb{T}^2 \to M_2(\mathbb{R})$*. Then there exist a measurable function* $h : \mathbb{T}^2 \to \mathbb{R}$ *and* $c \in \mathbb{R}$ *such that*

$$g(\overline{x}) = h(\overline{x}) \begin{bmatrix} c & -c^2 \\ 1 & -c \end{bmatrix} \quad \textit{for } \lambda\textit{-a.e. } \overline{x} \in \mathbb{T}^2$$

or

$$g(\overline{x}) = h(\overline{x}) \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \textit{for } \lambda\textit{-a.e. } \overline{x} \in \mathbb{T}^2.$$

Moreover, $h(\overline{x}) \neq 0$ *for* λ*-a.e.* $\overline{x} \in \mathbb{T}^2$.

Proof. Denote by $F \subset \mathbb{T}^2$ the set of all points $\overline{x} \in \mathbb{T}^2$ with $g(\overline{x}) \neq 0$. By Lemma 2, the set F is f-invariant. As f is ergodic and $\lambda(F) > 0$ we have $\lambda(F) = 1$. By Lemma 3, we can find $\overline{y} \in \mathbb{T}^2$ such that $g(\overline{y}) \neq 0$, $g(\overline{y})^2 = 0$ and $g(\overline{x}) \neq 0$, $g(\overline{x})^2 = g(\overline{x})g(\overline{y}) = g(\overline{y})g(\overline{x}) = 0$ for λ-a.e. $\overline{x} \in \mathbb{T}^2$. An application of Lemma 4 completes the proof. ∎

Assume that $f : (\mathbb{T}^2, \lambda) \to (\mathbb{T}^2, \lambda)$ is an ergodic measure-preserving C^1-diffeomorphism with linear growth of the derivative. Then the sequence $n^{-1}Df^n$ converges λ-a.e. to a function $g : \mathbb{T}^2 \to M_2(\mathbb{R})$. In the remainder of this section we assume that g can be represented as

$$g = h \begin{bmatrix} c & -c^2 \\ 1 & -c \end{bmatrix},$$

where $h : \mathbb{T}^2 \to \mathbb{R}$ and $c \in \mathbb{R}$. We can do it because the second case

$$g = h \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

reduces to $c = 0$ after interchanging the coordinates, which is an algebraic isomorphism.

Now by Lemma 2,

$$h(\overline{x})h(f\overline{x})^{-1} \begin{bmatrix} c & -c^2 \\ 1 & -c \end{bmatrix} = \begin{bmatrix} c & -c^2 \\ 1 & -c \end{bmatrix} Df(\overline{x}) \tag{1}$$

for λ-a.e. $\overline{x} \in \mathbb{T}^2$. It follows that

$$\begin{aligned} h(\overline{x})h(f\overline{x})^{-1}c &= c\frac{\partial f_1}{\partial x_1}(\overline{x}) - c^2\frac{\partial f_2}{\partial x_1}(\overline{x}), \\ -h(\overline{x})h(f\overline{x})^{-1}c &= \frac{\partial f_1}{\partial x_2}(\overline{x}) - c\frac{\partial f_2}{\partial x_2}(\overline{x}) \end{aligned}$$

for λ-a.e. $\overline{x} \in \mathbb{T}^2$. Therefore

$$-c\frac{\partial}{\partial x_1}(f_1(\overline{x}) - cf_2(\overline{x})) = \frac{\partial}{\partial x_2}(f_1(\overline{x}) - cf_2(\overline{x}))$$

for λ-a.e. $\overline{x} \in \mathbb{T}^2$. Since the functions $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}$ are of class C^1 the equality holds for every $\overline{x} \in \mathbb{R}^2$. Then there exists a C^1-function $u : \mathbb{R} \to \mathbb{R}$ such that

$$f_1(x_1, x_2) - cf_2(x_1, x_2) = u(x_1 - cx_2). \tag{2}$$

LEMMA 6. *If c is irrational, then $f(x_1, x_2) = (x_1 + d, x_2 + e)$, where $d, e \in \mathbb{R}$.*

Proof. Represent the diffeomorphism f as

$$f_1(x_1, x_2) = a_{11}x_1 + a_{12}x_2 + \widetilde{f}_1(x_1, x_2),$$
$$f_2(x_1, x_2) = a_{21}x_1 + a_{22}x_2 + \widetilde{f}_2(x_1, x_2),$$

where $\{a_{ij}\}_{i,j=1,2} \in M_2(\mathbb{Z})$ and $\widetilde{f}_1, \widetilde{f}_2 : \mathbb{T}^2 \to \mathbb{R}$. From (2),

$$u(x_1 - cx_2) = (a_{11} - ca_{21})x_1 + (a_{12} - ca_{22})x_2 + \widetilde{f}_1(x_1, x_2) - c\widetilde{f}_2(x_1, x_2). \tag{3}$$

Since the function $\widetilde{f}_1 - c\widetilde{f}_2 : \mathbb{R}^2 \to \mathbb{R}$ is $\mathbb{Z}^2$-periodic, there exists $(\widetilde{x}_1, \widetilde{x}_2) \in \mathbb{R}^2$ such that

$$\frac{\partial \widetilde{f}_1}{\partial x_1}(\widetilde{x}_1, \widetilde{x}_2) - c\frac{\partial \widetilde{f}_2}{\partial x_1}(\widetilde{x}_1, \widetilde{x}_2) = \frac{\partial \widetilde{f}_1}{\partial x_2}(\widetilde{x}_1, \widetilde{x}_2) - c\frac{\partial \widetilde{f}_2}{\partial x_2}(\widetilde{x}_1, \widetilde{x}_2) = 0.$$

From (3) it follows that

$$Du(\widetilde{x}_1 - c\widetilde{x}_2) = a_{11} - ca_{21}, \quad -cDu(\widetilde{x}_1 - c\widetilde{x}_2) = a_{12} - ca_{22}.$$

Hence

$$a_{12} - ca_{22} = -c(a_{11} - ca_{21}). \tag{4}$$

Then

$$u(x_1 - cx_2) = (a_{11} - ca_{21})(x_1 - cx_2) + \widetilde{f}_1(x_1, x_2) - c\widetilde{f}_2(x_1, x_2).$$

Let $v : \mathbb{R} \to \mathbb{R}$ be given by $v(x) = u(x) - (a_{11} - ca_{21})x$. As $\widetilde{f}_1 - c\widetilde{f}_2$ is $\mathbb{Z}^2$-periodic we have

$$v(x+1) = \widetilde{f}_1(x+1, 0) - c\widetilde{f}_2(x+1, 0) = \widetilde{f}_1(x, 0) - c\widetilde{f}_2(x, 0) = v(x)$$

and

$$v(x+c) = \widetilde{f}_1(x, -1) - c\widetilde{f}_2(x, -1) = \widetilde{f}_1(x, 0) - c\widetilde{f}_2(x, 0) = v(x).$$

Since v is continuous and c is irrational we conclude that the function v is constant and equal to a real number v. Therefore $\widetilde{f}_1 - c\widetilde{f}_2 = v$ and

$$f(x_1, x_2) = (a_{11}x_1 + a_{12}x_2 + c\widetilde{f}_2(x_1, x_2) + v, a_{21}x_1 + a_{22}x_2 + \widetilde{f}_2(x_1, x_2)).$$

As the diffeomorphism f preserves the measure λ we have $\det Df = \varepsilon$, where $\varepsilon \in \{-1, 1\}$. Then

$$\begin{aligned}\varepsilon &= \Big(a_{11} + c\frac{\partial \widetilde{f}_2}{\partial x_1}\Big)\Big(a_{22} + \frac{\partial \widetilde{f}_2}{\partial x_2}\Big) - \Big(a_{12} + c\frac{\partial \widetilde{f}_2}{\partial x_2}\Big)\Big(a_{21} + \frac{\partial \widetilde{f}_2}{\partial x_1}\Big) \\ &= a_{11}a_{22} - a_{12}a_{21} + (ca_{22} - a_{12})\frac{\partial \widetilde{f}_2}{\partial x_1} + (a_{11} - ca_{21})\frac{\partial \widetilde{f}_2}{\partial x_2} \\ &= a_{11}a_{22} - a_{12}a_{21} + (ca_{22} - a_{12})\Big(\frac{\partial \widetilde{f}_2}{\partial x_1} + c\frac{\partial \widetilde{f}_2}{\partial x_2}\Big),\end{aligned}$$

by (4). Since for a certain $\overline{x} \in \mathbb{R}$, $\frac{\partial \widetilde{f}_2}{\partial x_1}(\overline{x}) = \frac{\partial \widetilde{f}_2}{\partial x_2}(\overline{x}) = 0$ we see that

$$a_{11}a_{22} - a_{12}a_{21} = \varepsilon \quad \text{and} \quad \frac{\partial \widetilde{f}_2}{\partial x_1} + c\frac{\partial \widetilde{f}_2}{\partial x_2} = 0.$$

Therefore there exists a C^1-function $s : \mathbb{R} \to \mathbb{R}$ such that

$$s(cx_1 - x_2) = \widetilde{f}_2(x_1, x_2).$$

Since $\widetilde{f}_2$ is $\mathbb{Z}^2$-periodic and c is irrational, the function s is constant and equal to a real number s. It follows that

$$f(x_1, x_2) = (a_{11}x_1 + a_{12}x_2 + d, a_{21}x_1 + a_{22}x_2 + e),$$

where $d = cs + v$ and $e = s$. Then

$$Df^n = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^n$$

for any natural n. It follows that the function g is constant and finally that h is constant. From (1), we get

$$\begin{bmatrix} c & -c^2 \\ 1 & -c \end{bmatrix} = \begin{bmatrix} c & -c^2 \\ 1 & -c \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

Hence $1 = a_{11} - ca_{21}$ and $-c = a_{12} - ca_{22}$. Since c is irrational, we conclude that $a_{11} = 1$, $a_{12} = 0$, $a_{21} = 0$, $a_{22} = 1$. ∎

LEMMA 7. *If c is rational, then there exist a group automorphism $\psi : \mathbb{T}^2 \to \mathbb{T}^2$, a real number α, a C^1-function $\varphi : \mathbb{T} \to \mathbb{T}$ and $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$ such that*

$$\psi \circ f \circ \psi^{-1}(x_1, x_2) = (\varepsilon_1 x_1 + \alpha, \varepsilon_2 x_2 + \varphi(x_1)).$$

Proof. Denote by p and q the integers such that $q > 0$, $\gcd(p, q) = 1$ and $c = p/q$. Choose $a, b \in \mathbb{Z}$ with $ap + bq = 1$. Consider the group automorphism $\psi : \mathbb{T}^2 \to \mathbb{T}^2$ defined by $\psi(x_1, x_2) = (qx_1 - px_2, ax_1 + bx_2)$. Let $\widehat{f} = \psi \circ f \circ \psi^{-1}$ and let $\pi_i : \mathbb{T}^2 \to \mathbb{T}$ be the projection on the ith coordinate for $i = 1, 2$.

From (2),

$$\begin{aligned}\widehat{f}_1(x_1,x_2) &= qf_1\circ\psi^{-1}(x_1,x_2) - pf_2\circ\psi^{-1}(x_1,x_2)\\ &= qu(\pi_1\circ\psi^{-1}(x_1,x_2) - \frac{p}{q}\pi_2\circ\psi^{-1}(x_1,x_2)) = qu\left(\frac{1}{q}x_1\right).\end{aligned}$$

Therefore, $\widehat{f}_1$ depends only on the first variable. Then

$$D\widehat{f} = \begin{bmatrix} \frac{\partial \widehat{f}_1}{\partial x_1} & 0 \\ \frac{\partial \widehat{f}_2}{\partial x_1} & \frac{\partial \widehat{f}_2}{\partial x_2} \end{bmatrix}$$

and

$$\frac{\partial \widehat{f}_1}{\partial x_1}\frac{\partial \widehat{f}_2}{\partial x_2} = \det D\widehat{f} = \varepsilon \in \{-1,1\}.$$

Since $\frac{\partial \widehat{f}_2}{\partial x_2}(x_1,x_2) = \varepsilon/\frac{\partial \widehat{f}_1}{\partial x_1}(x_1,0)$, there exists a C^1-function $\varphi:\mathbb{T}\to\mathbb{T}$ such that

$$\widehat{f}_2(x_1,x_2) = \frac{\varepsilon}{\frac{\partial \widehat{f}_1}{\partial x_1}(x_1,0)}x_2 + \varphi(x_1).$$

Hence $\varepsilon/\frac{\partial \widehat{f}_1}{\partial x_1}(x_1,0)$ is an integer constant. As the map

$$\mathbb{T}\ni x\mapsto \widehat{f}_1(x,0)\in\mathbb{T}$$

is continuous, it follows that $\frac{\partial \widehat{f}_1}{\partial x_1}(x_1,0) = \varepsilon_1 \in \{-1,1\}$. Therefore

$$\widehat{f}(x_1,x_2) = (\varepsilon_1 x_1 + \alpha, \varepsilon_1\varepsilon x_2 + \varphi(x_1)). \ \blacksquare$$

THEOREM 8. *Every ergodic measure-preserving C^1-diffeomorphism of $\mathbb{T}^2$ with linear growth of the derivative is algebraically conjugate to a skew product of an irrational rotation on $\mathbb{T}$ and a circle C^1-cocycle with nonzero degree.*

Proof. Let $f:(\mathbb{T}^2,\lambda)\to(\mathbb{T}^2,\lambda)$ be an ergodic C^1-diffeomorphism with linear growth of the derivative. Then the sequence $n^{-1}Df^n$ converges λ-a.e. to a nonzero measurable function $g:\mathbb{T}^2\to M_2(\mathbb{R})$. By Lemma 5, there exist a measurable function $h:\mathbb{T}^2\to\mathbb{R}$ and $c\in\mathbb{R}$ such that

$$g(\overline{x}) = h(\overline{x})\begin{bmatrix} c & -c^2 \\ 1 & -c\end{bmatrix}$$

for λ-a.e. $\overline{x}\in\mathbb{T}^2$. First note that c is rational. Suppose, contrary to our claim, that c is irrational. By Lemma 6, $Df^n = \mathbb{I}$ for all natural n. Therefore the sequence $n^{-1}Df^n$ converges uniformly to zero, which is impossible.

By Lemma 7, there exist a group automorphism $\psi:\mathbb{T}^2\to\mathbb{T}^2$, a real number α, a C^1-function $\varphi:\mathbb{T}\to\mathbb{T}$ and $\varepsilon_1,\varepsilon_2\in\{-1,1\}$ such that

$$\psi\circ f\circ\psi^{-1}(x_1,x_2) = (\varepsilon_1 x_1 + \alpha, \varepsilon_2 x_2 + \varphi(x_1)).$$

As f is ergodic, the map

$$\mathbb{T} \ni x \mapsto \varepsilon_1 x + \alpha \in \mathbb{T}$$

is ergodic. It follows immediately that $\varepsilon_1 = 1$ and α is irrational.

Next note that $\varepsilon_2 = 1$. Suppose, contrary to our claim, that $\varepsilon_2 = -1$. Then

$$\begin{aligned}(2n)^{-1}&D(\psi \circ f^{2n} \circ \psi^{-1})(x_1, x_2)\\ &= \begin{bmatrix} (2n)^{-1} & 0 \\ (2n)^{-1}\sum_{k=0}^{n-1}(D\varphi(x_1+\alpha+2k\alpha) - D\varphi(x_1+2k\alpha)) & (2n)^{-1}\end{bmatrix}.\end{aligned}$$

By the Ergodic Theorem,

$$\frac{1}{2n}\sum_{k=0}^{n-1}(D\varphi(x_1+\alpha+2k\alpha)-D\varphi(x_1+2k\alpha)) \to \frac{1}{2}\int_{\mathbb{T}}(D\varphi(x+\alpha)-D\varphi(x))\,dx = 0$$

uniformly. Therefore the sequence $(2n)^{-1}Df^{2n}$ converges uniformly to zero, which is impossible. It follows that

$$\psi \circ f \circ \psi^{-1}(x_1, x_2) = (x_1 + \alpha, x_2 + \varphi(x_1)),$$

where α is irrational. By Lemma 1, the sequence $n^{-1}D(\psi \circ f^n \circ \psi^{-1})$ converges uniformly to the matrix $\left[\begin{smallmatrix} 0 & 0 \\ d(\varphi) & 0\end{smallmatrix}\right]$. It follows that the topological degree of φ is not zero, which completes the proof. ∎

For measure-preserving C^1-diffeomorphisms Lemma 1 and Theorem 8 give the following characterization of the property of being algebraically conjugate to a skew product of an irrational rotation and a C^1-cocycle with nonzero degree.

COROLLARY 1. *Let $f : (\mathbb{T}^2, \lambda) \to (\mathbb{T}^2, \lambda)$ be a measure-preserving C^1-diffeomorphism. Then the following are equivalent*:

(i) *f is ergodic and has linear growth of the derivative*;

(ii) *f is algebraically conjugate to a skew product of an irrational rotation on the circle and a circle C^1-cocycle with nonzero degree.*

3. Polynomial growth. Assume that $f : (M, \mathcal{B}, \mu) \to (M, \mathcal{B}, \mu)$ is a measure-preserving C^2-diffeomorphism of a compact Riemannian C^2-manifold M. Let β be a positive number. We say that the derivative of f has *polynomial growth with degree β* if the sequence $n^{-\beta}Df^n$ converges μ-a.e. to a measurable μ-nonzero function.

It is clear that replacing n by n^β in the lemmas of the previous section we obtain the following property. Every ergodic measure-preserving C^2-diffeomorphism whose derivative has polynomial growth with degree β is algebraically conjugate to a diffeomorphism $\widehat{f} : (\mathbb{T}^2, \lambda) \to (\mathbb{T}^2, \lambda)$ of the form

$$\widehat{f}(x_1,x_2)=(x_1+\alpha,\varepsilon x_2+\varphi(x_1)),$$

where α is irrational, $\varphi:\mathbb{T}\to\mathbb{T}$ is a C^2-cocycle and $\varepsilon\in\{-1,1\}$. Note that $\varepsilon=1$. Suppose, contrary to our claim, that $\varepsilon=-1$. Then

$$(2n)^{-\beta}D\widehat{f}^{\,2n}(x_1,x_2)$$
$$=\begin{bmatrix}(2n)^{-\beta} & 0\\ (2n)^{-\beta}\sum_{k=0}^{n-1}(D\varphi(x_1+\alpha+2k\alpha)-D\varphi(x_1+2k\alpha)) & (2n)^{-\beta}\end{bmatrix}.$$

Recall (see [2], p. 73) that if $\{q_n\}_{n\in\mathbb{N}}$ is the sequence of denominators of an irrational number γ and $\xi:\mathbb{T}\to\mathbb{R}$ is a function of bounded variation then

$$\Big|\sum_{k=0}^{q_n-1}\xi(x+k\gamma)-q_n\int_{\mathbb{T}}\xi(t)\,dt\Big|\le\operatorname{Var}\xi$$

for any $x\in\mathbb{T}$ and $n\in\mathbb{N}$.

Denote by $\{q_n\}_{n\in\mathbb{N}}$ the sequence of denominators of 2α. As $\int_{\mathbb{T}}(D\varphi(t+\alpha)-D\varphi(t))\,dt=0$, we obtain

$$\Big|\sum_{k=0}^{q_n-1}(D\varphi(x+\alpha+2k\alpha)-D\varphi(x+2k\alpha))\Big|\le 2\operatorname{Var}D\varphi$$

for any $x\in\mathbb{T}$. Hence the sequence $(2q_n)^{-\beta}D\widehat{f}^{\,2q_n}$ converges uniformly to zero, which is impossible. Therefore $\varepsilon=1$.

Since the derivative of $\widehat{f}$ has polynomial growth with degree β and

$$n^{-\beta}D\widehat{f}^{n}(x_1,x_2)=\begin{bmatrix}n^{-\beta} & 0\\ n^{-\beta}\sum_{k=0}^{n-1}D\varphi(x_1+k\alpha) & n^{-\beta}\end{bmatrix}$$

it follows that the sequence $n^{-\beta}\sum_{k=0}^{n-1}D\varphi(\cdot+k\alpha)$ converges a.e. to a nonzero measurable function $h:\mathbb{T}\to\mathbb{R}$. Choose $x\in\mathbb{T}$ such that

$$\lim_{n\to\infty}n^{-\beta}\sum_{k=0}^{n-1}D\varphi(x+k\alpha)=h(x)\neq 0.$$

Now denote by $\{q_n\}_{n\in\mathbb{N}}$ the sequence of denominators of α. Since

$$\Big|\sum_{k=0}^{q_n-1}D\varphi(x+k\alpha)-q_n\int_{\mathbb{T}}D\varphi(t)\,dt\Big|\le\operatorname{Var}D\varphi$$

we have

$$\lim_{n\to\infty}\Big(q_n^{-\beta}\sum_{k=0}^{q_n-1}D\varphi(x+k\alpha)-q_n^{1-\beta}\int_{\mathbb{T}}D\varphi(t)\,dt\Big)=0.$$

Hence

$$\lim_{n\to\infty}q_n^{1-\beta}\int_{\mathbb{T}}D\varphi(t)\,dt=h(x)\neq 0.$$

It follows that $\beta = 1$ and $d(\varphi) = \int_{\mathbb{T}} D\varphi(t)\,dt \neq 0$. From the above we deduce

THEOREM 9. *For no positive $\beta \neq 1$ does there exist an ergodic measure-preserving C^2-diffeomorphism whose derivative has polynomial growth with degree β.*

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