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SYMMETRIC COCYCLES AND CLASSICAL EXPONENTIAL SUMS

BY

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Abstract. This paper considers certain classical exponential sums as examples of cocycles with additional symmetries. Thus we simplify the proof of a result of Anderson and Pitt concerning the density of lacunary exponential partial sums $\sum_{k=0}^{n}\exp(2\pi i m^k x)$, $n=1,2,\ldots$, for fixed integer $m\ge 2$. Also, with the help of Hardy and Littlewood's approximate functional equation, but otherwise by elementary considerations, we improve a previous result of the author for certain examples of Weyl sum: if $\theta\in[0,1]\setminus\mathbb{Q}$ has continued fraction representation $[a_1,a_2,\ldots]$ such that $\sum_n 1/a_n<\infty$, and $|\theta-p/q|<1/q^{4+\varepsilon}$ infinitely often for some $\varepsilon>0$, then, for Lebesgue almost all $x\in[0,1]$, the partial sums $\sum_{k=0}^{n}\exp(2\pi i(k^2\theta+2kx))$, $n=1,2,\ldots$, are dense in $\mathbb{C}$.

1. Introduction. The use of cocycles to generate and study classical exponential series is well established [F], [G], [Pu], [Fo], giving results which are sometimes difficult to obtain without a dynamical approach. In this paper, exploiting an idea of symmetry that is seen most naturally from the dynamical point of view, we analyse two contrasting examples of complex-valued cocycle, each giving information about a corresponding exponential sum.

The first example (§2, Example 1) simplifies the analysis of certain lacunary series studied by Anderson and Pitt [AP2]. The dynamics which generate such series are "hyperbolic", containing many periodic points and a rich proximal structure. The second example (§2, Example 2), taking up the greater part of the paper, is the quadratic Weyl sum. A circle extension of a rotation underlies the dynamics of this series, and such a strictly ergodic system, which is only one step removed from the rigidity of a group rotation, is opposite in most senses to the hyperbolic system of Example 1. Nevertheless, we find a useful property common to both these examples: sufficient *symmetry*; and this simple idea, described in §2, allows us to deduce strong results about the divergence of the series from comparatively weak assumptions.

Before giving more detail of the results, we first describe how each of the examples relates to recent studies of two classes of dynamical cocycle.

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The skew products defined by cocycles with non-compact group values are an important source of dynamics over non-compact or infinite measure spaces. Necessary and sufficient conditions for such dynamics to be ergodic, conservative (recurrent) or dissipative are well understood [Sch1], [At1], [D].

A cocycle is also conveniently viewed as a generalized random walk in a topological group [At2], [D], and we prefer this picture here. The ergodicity of the skew product implies that the corresponding random walk is generically dense, but only in the topological setting is this implication reversed. We find in this paper, as in the pioneering paper [AP2] in a more special context, that we can show the density of certain generalized random walks almost surely without going as far as proving measurable ergodicity.

In general, it is difficult to decide whether a given cocycle is ergodic or even recurrent. Conditions and techniques have been worked out for two large classes of non-compact cocycle however: $\mathbb{R}^n$ cocycles over subshifts of finite type or other hyperbolic dynamical systems; and real cocycles over minimal rotations.

For cocycles over hyperbolic systems, we have, for example, the work of [G], [Co] exploiting a close connection between cocycles and periodic point or proximal structure developed in [Liv], [Kre], [PS] and [Sch2]. Here the information to be gained just by looking at the cocycle on periodic points or over the right-closing relation is often sufficient. Example 1 is a special example of such a cocycle.

For real-valued cocycles, as in the case of classical random walks, we benefit from the one-dimensionality of the group in which the cocycle takes values. Quite generally we have a simple criterion for recurrence of integrable real cocycles [At2], [Kry]. And, strictly for real cocycles over minimal rotations, we have Hedlund's characterization of transitivity in the topological setting [He] (generalized to the $\mathbb{R}^k$-valued case in [At3]), and the recent analysis of several authors [LM] (topological setting), [Pa], [I] (measure-preserving setting) and [LPV] (more generally). Example 2 is most closely related to this class, although, by being based on an extension of a rotation and by taking complex values, it stands apart in two significant respects.

Now we give more detail of the results of this paper leading up to Theorem 1.3, the main result.

The first example (§2, Example 1) concerns the lacunary series

$$\sum_{k=0}^{n} \exp(2\pi i m^k x), \qquad n = 1, 2, \ldots$$

By combining the more general results of their preceding paper [AP1] with a sophisticated application of Kummer theory, Anderson and Pitt showed that, for all integers $m \geq 2$ and almost all $x \in [0, 1]$, this sequence of partial sums is dense in $\mathbb{C}$. Guivarc'h [G] has shown a stronger version of the same result, namely the ergodicity of the corresponding skew product.

However, by applying the idea of symmetry given in §2, we can give a completely elementary proof of the step between [AP1] and the density almost surely for the cases $m = 6, 8, 9, 10\ldots$, thereby avoiding much of the analysis used in [AP2]. The pivotal observation is simply that a non-zero subsemigroup of $\mathbb{R}^2$, invariant under rotation by $2\pi/(m-1)$, is dense. See §2.

The second example (§2, Example 2) concerns the sequence of partial Weyl sums

$$\sum_{k=0}^{n} \exp(2\pi i(k^2\theta + kx)), \qquad n = 1, 2, \ldots$$

These are well known to be related to the distribution of the sequence $n^2\theta \bmod 1$ [KN], [W] and to other more deep number-theoretical facts [Vin], [Va], [N], [M].

However, the partial Weyl sums are less well studied simply as a sequence in the plane, that is, as a generalized random walk. In this paper and in [Fo] we ask when the partial Weyl sums form a dense set of numbers in $\mathbb{C}$ and, with restrictions on the parameter θ, obtain a positive answer for topologically many x [Fo] and, now in this paper, for full Lebesgue measure of x (Theorem 1.3). Once again the idea of symmetry is pivotal to the argument and the observation we use is that a non-zero subsemigroup of $\mathbb{R}^2$, invariant under rotation by any angle, is dense.

Definition 1.1. Given $\theta \in [0, 1]$ define

$$B(\theta) = \Big\{ x \in [0, 1] : \sum_{k=0}^{n} \exp(2\pi i(k^2\theta + kx)),\ n = 1, 2, \ldots,\ \text{is dense in } \mathbb{C} \Big\}.$$

From [Fo] we have:

Theorem 1.2. *Suppose that $0 < \theta < 1$ is an irrational number such that $\liminf_q q^{3/2}\|q\theta\| < \infty$. Then $B(\theta)$ is dense G_δ.* ∎

Here $\|\cdot\|$ refers to nearest integer. The condition on θ certainly implies that it is transcendental and restricts θ to a set of measure 0 and Hausdorff dimension 4/7.

It is important to note that some restriction on the rational approximation of θ is necessary since the result of Theorem 1.2 is known classically to be false for θ with bounded partial quotients [HL]; indeed $B(\theta) = \emptyset$ in this case. In [Fo] it is conjectured that Theorem 1 applies to all θ whose partial quotients tend to infinity.

Also, some restriction on x is needed. From other results in [HL], we have $0 \notin B(\theta)$ for any choice of θ.

The main result of this paper is the following refinement of Theorem 1.2 in many cases (θ is now found in a set of Hausdorff dimension at least $1/3$):

THEOREM 1.3. *Suppose that* $\theta \in [0,1] \setminus \mathbb{Q}$ *has continued fraction representation* $[a_1, a_2, \ldots]$ *such that* $\sum_n 1/a_n < \infty$, *and suppose* $\liminf_q q^{3+\varepsilon}\|q\theta\| = 0$ *for some* $\varepsilon > 0$. *Then* $B(\theta)$ *is of full Lebesgue measure in* $[0,1]$.

The central argument is found at the end of §2, but §§3,4 and the Appendix are all concerned with the analytic estimates needed to make the argument work. §3 takes what we need from an approximate functional equation due to Hardy and Littlewood, quoted as Theorem A.7 ahead. §4 takes this result further, adding in other results quoted in the Appendix, in order to complete the proof of Theorem 1.3. The Appendix sets aside those analytic estimates, used in §§3 and 4, but which are either direct from the literature or of intermediate relevance to the purpose of the paper.

We emphasize that the classical and modern estimates of Weyl sums [HL], [W], [Vin], [Va], [M] have nothing to say directly about the problem considered above; rather those estimates are concerned principally with upper bounds on the modulus and with further number-theoretical issues such as Waring's problem (see e.g. [Vin], [Va]). We believe that the dynamical approach adds significantly to the analytic tools available for the study of such exponential sums.

The intention of this paper, as in [Fo], is to generate "hard" estimates only strong enough to make the "softer" dynamical results work smoothly. In proving a more refined result, however, the hard analysis is necessarily more intrusive than in [Fo], but, even so, the work here, except for A.7, is "pre-Hardy–Littlewood". Not much more adeptness in estimation may be needed to improve the details of the result significantly. In particular, the important question of ergodicity of Weyl sum cocycles remains open for any value of θ.

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2. Dynamical generalities. To set the notation, we begin with some elementary dynamical constructions.

DEFINITIONS 2.1. Suppose that X is a compact metric space and that $T : X \to X$ is a continuous surjection which preserves a Borel probability measure μ on X.

Suppose that G is an abelian metric separable locally compact group.

Let $f : X \to G$ be a continuous map. From these, a *skew product* may be defined, namely, a homeomorphism

$$T_f : X \times G \to X \times G, \qquad T_f(x,g) = (Tx, f(x) + g).$$

The cocycle which is generated by this is written $a_f(x,n) \in G$, that is,

$$T_f^n(x,g) = (T^n x, a_f(x,n) + g) \qquad \forall n \in \mathbb{N},\ g \in G$$

and its connection with partial sums is made clear from the following equalities:

$$a_f(x,n) = \begin{cases} \sum_{k=0}^{n-1} f(T^k x), & n \geq 0, \\ -\sum_{k=n}^{-1} f(T^k x), & n < 0, \end{cases}$$

The cocycle is defined for negative values of n only if T^{-1} is defined.

The ergodicity and recurrence properties, conservation and dissipation, of this skew product are governed conveniently by the Essential (or Asymptotic) Values, $E(f)$ [Sch1], [FM]. However, when we are looking for the density of the cocycle values, it is enough to consider the following sets (see also [AP2] where, in the context of Example 1 ahead, G_x^+ is written $A(x,x)$):

DEFINITIONS 2.2 (see [Fo]). Let $\Omega_{(x,g)}^+$ be the forward limit points of the orbit of (x,g) under the skew action:

$$\Omega_{(x,g)}^+ = \{(y,h) : \exists n_k \to \infty : T_f^{n_k}(x,g) \to (y,h)\}.$$

Given $x \in X$, let

$$G_x^+ = \{g \in G : (x,g) \in \Omega_{(x,e)}\}.$$

LEMMA 2.3. *Suppose that (X,T) is a compact topological dynamical system and that a skew extension, $(X \times G, T_f)$, etc. are constructed as above. Then*:

(i) *For all $x \in X$, G_x^+ is either empty or is a closed subsemigroup of G.*
(ii) *For all $x \in X$, $G_{Tx}^+ = G_x^+$.*

If μ is a Borel probability measure defined on X, invariant and ergodic with respect to T, then the semigroup G_x^+ is almost everywhere the same.

P r o o f. Part (i) is [Fo] (Lemma 4(i)), but see also [AP2] for its proof for Example 1.

(ii) The formula $a_f(Tx,n) = a_f(x,n) - f(x) + f(T^n x)$ shows that if $T^{n_k}x \to x$, then $a_f(x,n_k) \to g$ if and only if $a_f(Tx,n_k) \to g$.

To show the constancy of the group, note that part (ii) and ergodicity of μ show that, for each $g \in G$ and $\varepsilon > 0$, $\mu(x : d_G(g, G_x^+) \le \varepsilon) = 0$ or 1. Thus by selecting a dense countable collection of $g \in G$ and a countable sequence of $\varepsilon \to 0$, and exploiting the fact that G_x^+ is closed, we arrive at the desired conclusion. ∎

REMARK 2.4. It is clear that $G_x^+ = G$ implies that $\sum_{k=0}^{n} f(T^k x)$, $n = 0, 1, \ldots$, is dense in G; and it is by proving the former result that the results in [AP2], [Fo] and Theorem 1.3 of this paper follow.

Note that only part (i) of the lemma above remains true for non-abelian groups G.

Recall that, by [Fo] (Lemma 4), if $G_x^+ = G$ for some $x \in X$ with dense T-orbit in X, then T_f is topologically transitive; and this implies that $G_x^+ = G$ for a dense G_δ of $x \in X$.

However, the statements (a) $G_x^+ = G$ for almost all $x \in X$, and (b) T_f is (measurably) ergodic, seem only to support an implication one way: (b)⇒(a).

Now we consider a natural notion of symmetry for topological and measurable cocycles, arising from the interaction of a symmetry on the system underlying the cocycle and on the group in which the cocycle takes its values. We introduce this abstractly since we have two diverse applications in mind and since it is likely that more applications will follow.

CONSTRUCTION 2.5. Suppose that (X,T) is a compact topological dynamical system with a continuous function $f : X \to G$ taking values in a topological group G.

Suppose that H is an abstract group represented both as a subgroup of the homeomorphisms of X, $\phi_h : h \in H$, and as a subgroup of the continuous group automorphisms of G, $\gamma_h : h \in H$. Suppose further that:

(i) H and T commute: $\phi_h T = T\phi_h$ for all $h \in H$.

(ii) H and f interact as $f(\phi_h x) = \gamma_h f(x)$ for all $x \in X$, $h \in H$.

We refer to H as a *group of symmetries* of the triple (X,T,f).

EXAMPLE 1. Suppose that X is the unit circle, represented as the unit interval mod 1. Fix an integer $m \ge 2$ and let $Tx = mx$ mod 1. Let $f(x) = \exp(2\pi i x)$, a continuous function from X to the complex numbers.

Note that the cocycles $a_f(x,n)$ are precisely the exponential sums $\sum_{j=0}^{n-1} \exp(2\pi i m^j x)$ considered in [AP2], [G].

Let H be the $2(m-1)$-element dihedral group, represented as the symmetries of the plane generated by a reflection and a rotation by $2\pi/(m-1)$. For $h \in H$ a rotation by $2\pi k/(m-1)$, set $\phi_h(x) = x + k/(m-1) \bmod 1$, and $\gamma_h(z) = z\exp(2\pi ih/(m-1))$. For h the generating reflection, set $\phi_h(x) = -x$ and $\gamma_h(z) = \overline{z}$.

The symmetry with respect to reflection/conjugation was noted in [AP2].

Example 2 [F]. Suppose that X is the 2-torus, represented as pairs of reals mod 1. Let θ be an irrational real number and let $T(x,y) = (x+\theta, y + 2x + \theta)$. Let $f(x,y) = \exp(2\pi iy)$. This gives the quadratic Weyl sums: $a_f((x,y),n) = \sum_{j=0}^{n-1} \exp(2\pi i(j^2\theta + 2jx + y))$.

Let H be the circle group represented as the unit interval mod 1. Define $\phi_h(x,y) = (x, y+h)$ and $\gamma_h(z) = z\exp(2\pi ih)$.

Definition 2.6. It is straightforward to modify this topological definition to a measurable version of symmetry. We start with a probability measure-preserving system, $(X, \mathcal{A}, \mu, T)$, and a measurable function, $f : X \to G$. A *group of symmetries*, H, on $(X, \mathcal{A}, \mu, T, f)$ has the same defining formulae, H having been represented both as a subgroup of the bimeasurable bijections of $(X, \mathcal{A})$ and as a subgroup of continuous automorphisms of G.

We say that H acts *incompressibly* on X if, when A is Borel, we have $\mu(\phi_h A) > 0$ if and only if $\mu(A) > 0$.

Examples. Both Examples 1 and 2 demonstrate symmetries in the measurable sense when we associate Lebesgue measure to the topological dynamical systems; and in each case the H action on X is measure-preserving.

The principal function of symmetry is summarized in the following easily proved proposition:

Proposition 2.7. *Suppose that H is a group of symmetries for a cocycle over a compact topological dynamical system, (X, T, f). Then for all $x \in X$ and $h \in H$, $\gamma_h(G_x^+) = G_{\phi_h x}^+$.*

Suppose further that G is abelian and μ is a Borel probability measure on X, invariant and ergodic with respect to T, and that H acts incompressibly on X. Then for almost all $x \in X$, G_x^+ is H-invariant.

Proof. The first part is straightforward by definition.

From Lemma 2.3 we know that $G_x^+ = G_0$, say, almost surely. Since H acts incompressibly, $G_{\phi_h x}^+ = G_0$ almost surely also. The equation of the first part implies $\gamma_h G_0 = G_0$ therefore, as required. ■

The main use of this is to enlarge G_x^+ from one or two points. We observe how powerful this can be by applying it to the two examples above:

Example 1 (ctd.). The analysis of Anderson and Pitt [AP1] shows that, for almost all $x \in X$, G_x^+ is a cocompact subgroup of $\mathbb{C}$.

However, if $m = 6, 8, 9, 10, \ldots$, then the $(m-1)$-fold rotational symmetry implied by Proposition 3.2 forces $G_x^+ = \mathbb{C}$ (recall that G_x^+ is closed). This implies density of the partial exponential sums almost surely, and so we reproduce the results of [AP2] in these cases.

To complete the analysis, the cases $m = 2, 3, 4, 5, 7$ can be treated ad hoc using simplifications of the Kummer-theoretic analysis of [AP2] or by the criteria given in [G].

As mentioned before, Guivarc'h [G] has shown the ergodicity of these cocycles as examples of more general dynamical techniques. However, even for this harder problem, the idea of symmetry simplifies the calculations and we give the following helpful proposition.

Proposition 2.8. *Suppose that* (X, T) *is a compact topological system*, $f : X \to G$ *a continuous map*, *and that* H *is a group of symmetries for* (X, T, f). *Then* $E_{\mathrm{top}}(f)$ (*topological essential values as in* [At1]) *is invariant under the action of* H *on* G.

When we have an ergodic measure, μ, *defined on* X, *and* H *acts incompressibly on* $(X, \mathcal{A}, \mu)$, *then a corresponding result holds for the following groups*: *the* (*measure-preserving*) *essential values* $E(f)$ ([Sch1], [FM]); Δ_f; *and* Γ_f; *the latter two groups defined in the case of a suitable periodic point or right-closing structure* (see [Kre], [PS], [Sch2]).

Proof. We give the details of the first part. Suppose that $g \in E_{\mathrm{top}}(f)$, that U is a non-empty open neighbourhood of g in G and that V is a positive open subset of X. Therefore there is an integer, n, such that $V \cap T^{-n}V \cap \{x \in X : a(x, n) \in U\} \neq \emptyset$.

Given $h \in H$, apply ϕ_h to the set above to find that $V' \cap T^{-n}V' \cap \{x \in X : a(\phi_h^{-1}x, n) \in U\} \neq \emptyset$, where $V' = \phi_h V$, an open set. However, $a(\phi_h^{-1}x, n) = \gamma_h^{-1}(a(x, n))$ and so we deduce that $V' \cap T^{-n}V' \cap \{x \in X : a(x, n) \in U'\} \neq \emptyset$ where $U' = \gamma_h U$, an open neighbourhood of $\gamma_h(g)$.

Now, since ϕ_h and γ_h are open continuous we may assume that V' and U' can be made arbitrarily small. Thus $\gamma_h(g) \in E_{\mathrm{top}}(f)$, as required.

The other parts of the theorem follow by similar considerations. ■

Example 2 (ctd.). Since H is a rotational symmetry of all angles, the argument of Example 1 works here as well. For example, Proposition 3.2 shows that, if there is a non-zero element in $E_{\mathrm{top}}(f)$, then in fact $E_{\mathrm{top}}(f) = \mathbb{C}$. This observation was exploited fully in [Fo] to deduce the topological transitivity of certain Weyl sum cocycles, giving Theorem 1.2 above.

Proof of Theorem 1.3. This symmetry can be exploited again in order to determine almost sure density of Weyl sum cocycles. In Corollary 4.8 we show, under the relevant conditions, that, for Lebesgue almost all $x \in X$, G_x^+ contains elements on the circle of radius $1/2$. By the foregoing argument, this is enough to give $G_x^+ = \mathbb{C}$ almost surely, hence, by Remark 2.4, the proof of Theorem 1.3. ■

However, to prove Corollary 4.8 requires a surprising amount of effort and this is the aim of the rest of the paper.

3. Analytic estimates of Weyl sums I: from A.7. In this section, we mix a little dynamical technique with the estimate of Hardy and Littlewood quoted in Theorem A.7. The end result is a lower bound on the size of the Weyl sums.

DEFINITION 3.1. For each $\theta \in (0,1]$, define the *Gauss map* $S\theta = \{1/\theta\}$, where we write $\{t\}$ for the fractional part of t. Also write $[t]$ for the integer part of $t \in \mathbb{R}$.

See [Kh], [Bi] for an analysis of the useful properties of S. See also Lemma A.6 ahead for a corollary of that analysis.

LEMMA 3.2. *Suppose that* $\theta \in [0,1]\backslash\mathbb{Q}$ *and* $\theta = [a_1, a_2, \ldots]$ *is its continued fraction representation. Then* $S^n\theta$ *is well defined for all* $n \geq 0$ *and* $S^{n-1}\theta \in (1/(a_n+1), 1/a_n)$ *for each* $n \geq 1$. ■

DEFINITION 3.3. Suppose that $0 < \theta, x < 1$ are given. Then define

$$\widetilde{S}(\theta, x) = (S\theta, \{-x/\theta\}) = (\{1/\theta\}, \{-x/\theta\}).$$

Define $U_\theta x = \{-x/\theta\}$, so that $\widetilde{S}(\theta, x) = (S\theta, U_\theta x)$. Generalize this to $U_\theta^{(m)}x$ for the second coordinate entry in $\widetilde{S}^m(\theta, x)$.

REMARK 3.4. The map $\widetilde{S}$ is considered by Schweiger [Schw] among many examples of fibred dynamical systems. There one can find a formula for an invariant measure on $[0,1]^2$, absolutely continuous with respect to Lebesgue measure and invariant and ergodic with respect to $\widetilde{S}$. However, by virtue of the assumptions we make about θ in this paper, the orbits we consider, $\widetilde{S}^m(\theta, x)$, are not generic: we can use no results from Schweiger's work.

However, we repair this problem in next few lemmas and show how to recover information for almost all x despite the peculiar properties of θ.

LEMMA 3.5. *For all* θ *and all* $A \subset [0,1]$ *measurable*,

$$(1-\theta)\lambda(A) \leq \lambda(U_\theta^{-1}A) \leq (1+\theta)\lambda(A).$$

Proof. By considering the graph of U_θ, we see that $U_\theta^{-1}A$ is a union of $m-1$ disjoint translates of a dilation of A (by scale θ) and subset of another such translate, where $(m-1)\theta < 1 < m\theta$. Thus we have $\lambda(U_\theta^{-1}A) \geq (m-1)\theta\lambda(A) \geq (1-\theta)\lambda(A)$, and $\lambda(U_\theta^{-1}A) \leq m\theta\lambda(A) \leq (1+\theta)\lambda(A)$, as required. ■

By applying the estimate above repeatedly, and using Lemma 3.1, we obtain the following useful result.

COROLLARY 3.6. *Suppose that* $\theta = [a_1, a_2, \ldots]$ *as above and* $\sum 1/a_n < \infty$. *Then* $C^- \lambda(A) \le \lambda(U_\theta^{(m)-1} A) \le C^+ \lambda(A)$ *for all Borel measurable* A, *where* $C^- = \prod_{j\ge 1}(1 - 1/a_j)$ *and* $C^+ = \prod_{j \ge 1}(1 + 1/a_j)$. ■

DEFINITION 3.7. Write $b(x, k) = \sum_{j=0}^{k-1} \exp(2\pi ijx)$. See Lemma A.1 for the properties of this that we use.

LEMMA 3.8. *With the assumptions and notation of Corollary* 3.6, *for all* $0 < \eta < 1/2$ *and all* $m \ge 1$,

$$\lambda\{x : \|U_\theta^{(m)} x\| < \eta\} \ge 2C^- \eta.$$

Therefore for any choice of $C_0 \ge 1$ *and* $m \ge 1$,

$$\lambda\{x : |b(U_\theta^{(m)} x, [2\pi C_0] + 1)| \ge C_0\} \ge C^-/([2\pi C_0] + 1) \ge C^-/(10C_0).$$

Proof. By Corollary 3.6, $\lambda\{x : \|U_\theta^{(m)} x\| < \eta\} \ge C^- \lambda\{x : \|x\| < \eta\}$, giving the first inequality immediately.

For the second, consider the following general estimate: Let $c \in \mathbb{Z}$ and let $\eta = 1/(2c)$. If $\|x\| < \eta$, then $\|cx\| = c\|x\|$ and we have the elementary inequality $|b(x, c)| \ge \|cx\|/(2\pi\|x\|) = c/(2\pi)$.

Now put $c = [2\pi C_0] + 1$ to find that

$$\{x : |b(U_\theta^{(m)} x, [2\pi C_0] + 1)| \ge C_0\} \supset \{x : \|U_\theta^{(m)} x\| < 1/(2[2\pi C_0] + 2)\}.$$

Then the first part of the lemma gives the result. ■

DEFINITION 3.9. Write $\sigma_m(\theta) = \sqrt{S^{m-1}\theta}\, \sigma_{m-1}(\theta)$, $m \ge 2$, inductively with $\sigma_1(\theta) = \sqrt{\theta}$. Furthermore, given $k \ge 0$, define inductively $k(m) = [k(m-1) S^{m-1}\theta]$ and $k(0) = k$.

From the basic theory of continued fractions [Kh] we have

LEMMA 3.10. *Suppose that* $\theta \notin \mathbb{Q}$ *with continued fraction representation* $\theta = [a_1, a_2, \ldots]$ *such that* $\liminf_n a_n \ge 2$. *Then there is a constant*, C_1 (*depending only on* θ), *such that* $\sigma_m(S^k\theta) \le C_1 2^{-m/2}$ *and* $k(m) \le C_1 k 2^{-m/2}$. ■

DEFINITION 3.11. Write $\psi(\theta, x, n) = |\sum_{k=0}^{n-1} \exp(\pi i(k^2\theta + 2kx))|$ and note the functional equation of Theorem A.7, which may be rewritten

$$\sigma_1(\theta)\psi(\theta, x, k) = \psi(\widetilde{S}(\theta, x), k(1)) + O(1)$$

in the notation given above.

LEMMA 3.12. *With the assumptions of Lemma* 3.10,

$$\sigma_m(\theta)\psi(\theta, x, k) = |b(U_\theta^{(m)} x, k(m))| + O(1 + k(m)^3 \|S^m\theta\|)$$

where the constant multiple in the error is absolute.

P r o o f. The estimate of Theorem A.7 gives

$$\sigma_m(\theta)\psi(\theta,x,k)=\sigma_{m-1}(S\theta)\psi(S\theta,U_\theta x,[k\theta])+O(\sigma_{m-1}(S\theta)),$$

the constant in the error term being given by the error term in Theorem A.7. Therefore, the error term is $O(2^{-m/2})$, the implicit constant being absolute.

So we sustain by induction the hypothesis that

$$\sigma_m(\theta)\psi(\theta,x,k)=\psi(S^m\theta,U_\theta^{(m)}x,k(m))+O\Big(\sum_{j=1}^{m}2^{-j/2}\Big)$$

and the error is $O(1)$ therefore.

Now we have the following general estimate:

$$\psi(\theta,x,k)=\Big|\sum_{j=0}^{k-1}\exp(\pi i(j^2\theta+2jx))\Big|=\Big|\sum_{j=0}^{k-1}(\exp(2\pi ijx)+O(j^2\|\theta\|))\Big|,$$

from which we get $\psi(\theta,x,k)=|b(x,k)|+O(k^3\|\theta\|)$ with the implicit constant in the error being absolute.

Applying this to $\psi(S^m\theta,U_\theta^{(m)}x,k(m))$ gives the lemma. ∎

Now, assuming the conditions of 3.8, we combine the arguments of 3.12 and 3.8.

Suppose that C_0 is chosen much larger than three times the constant in the error of Lemma 3.12. Now choose m_0 so that $([2\pi C_0]+1)^3\|S^m\theta\|\le 1$ for all $m\ge m_0$.

Suppose further that we have chosen x and $m\ge m_0$ so that $|b(U_\theta^{(m)}x,[2\pi C_0]+1)|\ge C_0$, and suppose that $k=k(0)$ has been chosen so that $k(m)=[2\pi C_0]+1$ (such a choice can always be made independently of x). By construction therefore, we have, by Lemma 3.12, $\sigma_m(\theta)\psi(\theta,x,k)\ge C_0/3$.

This together with Lemma 3.8 gives the following which is basic to the proof of Proposition 4.3.

PROPOSITION 3.13. *Suppose that* $\theta\in[0,1]\setminus\mathbb{Q}$ *and* $\theta=[a_1,a_2,\ldots]$ *is its continued fraction representation with* $\sum 1/a_n<\infty$. *Then there is a* $\varrho>0$ *such that for all* $C>0$, *there is a* k *such that* $\lambda\{x:|\psi(\theta,x,k)|\ge C\}\ge\varrho$.

P r o o f. By the preamble, we find that for each $m\ge m_0$, there is a k such that $\{x:\sigma_m(\theta)|\psi(\theta,x,k)|\ge C_0/3\}\supset\{x:|b(U_\theta^{(m)}x,[2\pi C_0]+1)|\ge C_0\}$. By Lemma 3.8, this latter set has measure at least $\varrho=C^-/(10C_0)>0$. Now, for given C, pick $m\ge m_0$ (by Lemma 3.10) such that $C_0/(3\sigma_m(\theta))\ge C$. ∎

4. Analytic estimates of Weyl sums II. Now we draw the calculation back to Example 2, looking for non-zero elements in G^+.

Definition 4.1. In this section, we write

$$a(x,n)=\sum_{k=0}^{n-1}\exp(2\pi i(k^2\theta+2kx))$$

and, as in 3.7,

$$b(x,m)=\sum_{k=0}^{m-1}\exp(2\pi ikx).$$

Note that $a_f((x,0),n)=a(x,n)$ where the left-hand side refers to the cocycle construction of Example 2.

Terminology. An irrational number, θ, is best approximated by the sequence of rational numbers, p_k/q_k, produced from the continued fraction approximation (see for example [Kh], [HW]). In what follows we attach the term *approximation denominator for θ* to each of the numbers q_k.

The goal of this section is to show that for almost all $x\in[0,1]$, there is a $z\in\mathbb{C}$, $|z|=1/2$, and a sequence $n_k\to\infty$ so that $T^{n_k}(x,0)\to(0,0)$ and such that $a(x,n_k)\to z$, i.e. so that $z\in G^+_{(x,0)}$. The final step from this fact to the proof of Theorem 1.3 is noted at the end of §2.

First we outline the general tactic: One of our principal problems is to find $|a(x,n_k)|\le 1$. Lemma A.4 ahead shows that we can approximate $a(x,mn)$ by $a(x,n)b(2nx,m)$ whenever n is an approximation denominator for θ. So, although $a(x,n)$ may be very large, with the extra degree of freedom allowed by adjusting m, we can hope to bring $a(x,mn)$ into the unit disc; our sequence n_k will therefore consist of multiples of approximation denominators for θ.

On the other hand we have to keep $a(x,n_k)$ away from 0 and, if we are to follow the construction above, we hope that the $a(x,n)$ will not converge to 0 as n runs over approximation denominators for θ. This fact is surprisingly difficult to establish (Proposition 4.3) and exploits the results of §3, which in turn use estimates of Hardy and Littlewood (Theorem A.7).

Nevertheless, our hope realized, we do indeed control the size of $a(x,mn)$ almost surely, but at the expense of allowing a large value for m, thereby requiring a tighter control of the rational approximation of θ.

The control on $T^{mn}(x,0)=(x+nm\theta,(nm)^2\theta+2nmx)$, meanwhile, is ensured by picking subsequences of approximation denominators which keep $2nx$ close to 0, but not so close that the earlier estimates fail. This latter consideration is uppermost in the following definition.

Definition 4.2. Suppose that $\delta>0$ is given and Q is an infinite subset of $\mathbb{N}$. Suppose that $f_n:n\in Q$ is a sequence of real-valued functions defined on $[0,1]$. Then write

$$\limsup_{n\in Q}{}^{\delta} f_n(x)$$

for the upper limit of $f_{n_j}(x)$ as (n_j) runs over all subsequences, $n_j \to \infty$, of Q such that $\limsup_j \|2n_j x\| < \delta$ and $\liminf_j \|2n_j x\| > \delta/2$.

Note that by Lemma A.2, for almost all $x \in [0,1]$ there exist subsequences available for the definition above.

Section 3 gives the key to the following result.

PROPOSITION 4.3. *Suppose that* $\theta \in [0,1] \setminus \mathbb{Q}$ *has continued fraction representation* $[a_1, a_2, \ldots]$ *such that* $\sum_n 1/a_n < \infty$. *Let* $\delta > 0$ *and let* Q *be an infinite subset of the set of approximation denominators for* θ (*so that, in particular*, $\lim_{n\in Q} n\|n\theta\| = 0$). *Then* $\lambda\{x \in [0,1] : \limsup^{\delta}_{n\in Q} |a(x,n)| = \infty\} = 1$.

Proof. Recall the constant ϱ obtained in Proposition 3.13, so that for all $C > 0$, there is a k such that $\lambda(x : |a(x,k)| \ge C) \ge \varrho$.

Therefore if $C > 0$, we can find k such that $|a(x,k)| \ge 4(2C+1)/\delta$ for all x in a set, B, of measure at least ϱ.

Corollary A.5 gives

$$2\max\{|a(x+k\theta,n)|, |a(x,n)|\} \ge \|2nx\| \cdot |a(x,k)| + O(k^2\|n\theta\| + kn\|n\theta\|).$$

By assumption, the error term is less than 1 for all $n \in Q$ large enough.

From this we deduce that

$$\lambda\{x : |a(x,n)| \ge C,\ 7\delta/12 < \|2xn\| < 11\delta/12) \ge (1/2)\lambda\{x \in B : 2\delta/3 < \|2nx\| < 5\delta/6\}$$

for all $n \in Q$ large enough. This, with Lemma A.2, gives $\lambda\{\limsup^{\delta}_{n\in Q} |a(x,n)| \ge C\} \ge \varrho$ and therefore, since C is arbitrary, we have $\lambda(A) \ge \varrho$, where $A = \{x : \limsup^{\delta}_{n\in Q} |a(x,n)| = \infty\}$.

We seek to show that $\lambda(A) = 1$ finally. Consider the map $x \mapsto x + \theta$ which is ergodic with respect to Lebesgue measure λ. It is enough therefore to show that $x \in A$ implies $x + \theta \in A$. However, $\limsup^{\delta}_{n\in Q} |a(x+\theta,n)| = \limsup^{\delta}_{n\in Q} |a(x,n)|$ as $\limsup_{n\in Q'} \|2n(x+\theta)\| = \limsup_{n\in Q'} \|2nx\|$ for any subsequence, Q', of Q. ∎

REMARK 4.4. For irrational θ such that $\liminf_q q\|q\theta\| > 0$, note the estimate $|a(x,n)| \ge c_\theta\sqrt{n}$ uniformly in $x \in [0,1]$ (see [HL]). Thus, using Lemma A.2, we see that the result of Proposition 4.3 follows for such θ as well. It seems unlikely that the θ which occupy the gap between these two conditions fail the conclusion of 4.3, but we do not have a proof of this.

Now we proceed to apply this result and others from the Appendix in the construction outlined at the start of this section.

DEFINITIONS 4.5. Let $D(x,n,M)=\{|a(x,mn)| : 0\le m<M\}$.

Suppose that $\gamma>0$ and that I is an interval in $\mathbb{R}$. We say that a subset $D\subset\mathbb{R}$ is *γ-dense in I* if, for all $x\in I$, there is a $y\in D\cap I$ such that $|x-y|<\gamma$.

PROPOSITION 4.6. *Suppose that $\theta\in[0,1]\setminus\mathbb{Q}$ has continued fraction representation $[a_1,a_2,\ldots]$ such that $\sum_n 1/a_n<\infty$. Suppose that Q_0 is a subsequence of approximation denominators for θ. Suppose furthermore that $\varepsilon>0$ and that $M_n>n^{1/2+2\varepsilon}$ and $\gamma_n>n^{1/2+\varepsilon}(M_n^{\varepsilon-1}+M_n^3 n\|n\theta\|)$ are defined for each approximation denominator, n, for θ. Then there is a subset, P, of $[0,1]$ of full measure such that if $x\in P$ and $\delta>0$, then for infinitely many $n\in Q_0$ we have*:

(i) *$D(x,n,M_n)$ is γ_n-dense in $[0,1]$.*

(ii) *If $0\le m<M_n$ and $|a(x,mn)|\in[0,1]$ (i.e. $\in D(x,n,M_n)\cap[0,1]$), then $\|2mnx\|\le 4\pi\delta(\delta/2+\gamma_n)$.*

Proof. First we describe the proof roughly, giving a little more detail to the tactic outlined at the beginning of the section.

From Lemma A.4 we know that

$$a(x,mn)=a(x,n)(b(2nx,m)+O(M^3n\|n\theta\|))\quad \text{for } 0\le m<M.$$

The elements of $D(x,n,M)$ can therefore be approximated by products $a(x,n)b(2nx,m) : 0\le m<M$, numbers which, by Lemma A.1, are γ-dense in the interval $[0,|a(x,n)|]$, where $\gamma\simeq|a(x,n)|(4\pi/(q\|2nx\|))$ and $q<M$ is an approximation denominator for x. The error in this approximation for γ is $O(|a(x,n)|M^3n\|n\theta\|)$.

On the one hand, by Lemma A.3, we can bound $|a(x,n)|$ above by $n^{1/2+\eta}$ for most x, giving control of the error and partial control of γ. On the other hand, by Proposition 4.3, we can bound $|a(x,n)|$ from below (for infinitely many n), without destroying the useful bounds on $\|2nx\|$; this completes control over γ and makes sure that $[0,|a(x,n)|]$ contains the unit interval.

Now we start the proof in earnest. Assume the conditions of the proposition and, for each approximation denominator, n, of θ, take $\eta_n\to0$ sufficiently slowly that $M_0(\eta_n)<M_n$ (from Lemma A.6) and $n^{\eta_n}\to\infty$. Pick a subset Q of Q_0 so that $\sum_{n\in Q}\eta_n<\infty$ and $\sum_{n\in Q}n^{-2\eta_n}<\infty$.

By Lemma A.1, $\{|b(2nx,m)| : 0\le m<M\}$ is $4\pi/(q\|2nx\|)$-dense in $[0,1]$ whenever $2nx$ has a continued fraction approximation p/q and $q<M$. By Lemma A.6, we construct the set, $P_1(\eta_n)$, of measure at least $1-\eta_n$. Note that $P_1'(n)=\{x\in[0,1] : 2nx\in P_1(\eta_n)\}$ also has measure at least $1-\eta_n$. By construction, for $x\in P_1'(n)$ and $M>M_0(\eta_n)$, there is a choice of approximation denominator q for $2nx$ so that $M^{1-\eta_n}<q<M$.

Consequently, $\{|b(2nx,m)| : 0 \le m < M\}$ is $4\pi/(M^{1-\eta_n}\|2nx\|)$-dense in $[0,1]$.

Also, by construction of Q, for almost all $x \in [0,1]$, $x \in P_1'(n)$ for all but finitely many $n \in Q$. So we deduce:

FACT 1. *For almost all* $x \in [0,1]$, *there is an* n_0 *such that for* $n \in Q$ *and* $n \ge n_0$, *the set* $\{|b(2nx,m)| : 0 \le m < M_n\}$ *is* $4\pi/(M_n^{1-\eta_n}\|2nx\|)$*-dense in* $[0,1]$.

Now consider the result of Proposition 4.3 that $\limsup^{\delta}_{n\in Q} |a(x,n)| = \infty$ almost surely. In particular, we have

FACT 2. *For almost all* $x \in [0,1]$, *there are infinitely many* $n \in Q$ *such that* $|a(x,n)| \ge 2/\delta$ *and* $\delta/2 < \|2nx\| < \delta$.

And finally, by Lemma A.3 and the construction of Q, we have

FACT 3. *For almost all* $x \in [0,1]$, *there is an* n_1 *such that for* $n \in Q$, $n \ge n_1$, *we have* $|a(x,n)| \le n^{1/2+\eta_n}$.

Combining these three facts together allows us to make the following sequence of deductions for almost all $x \in [0,1]$:

First (from Facts 1 and 2), there are infinitely many $n \in Q$ such that $\{|b(2nx,m)| : 0 \le m < M_n\}$ is $8\pi/(M_n^{1-\eta_n}\delta)$-dense in $[0,1]$. Consequently (using Fact 3) for such n, $\{|a(x,n)|\cdot|b(2nx,m)| : 0 \le m < M_n\}$ is $16\pi n^{1/2+\eta_n}/(M_n^{1-\eta_n}\delta^2)$-dense in $[0,|a(x,n)|] \supset [0,1]$. By Lemma A.4 therefore we find that $D(x,n,M_n)$ is $(4n^{1/2+\eta_n}/\delta)[4\pi M_n^{\eta_n-1}/\delta + M_n^3 n\|n\theta\|]$-dense in $[0,1]$. Note that, by construction, γ_n majorizes this expression whenever n is large enough.

Secondly, for the same set of $n \in Q$, a choice of m so that $|a(x,nm)| \le 1$ implies (by A.4 and Fact 2) that

$$|b(2nx,m)| \le 1/|a(x,n)| + O(M_n^3 n^{3/2+\eta_n}\|n\theta\|) < \delta/2 + \gamma_n$$

with n large enough. However, by Lemma A.1,

$$\|2mnx\| \le 2\pi|b(2nx,m)|\cdot\|2nx\|$$

and this is majorized by $2\pi(\delta/2+\gamma_n)|a(x,mn)|/|a(x,n)| \le 4\pi\delta(\delta/2+\gamma_n)$, as required.

Thus we find $P(\delta)$ of full measure for which the results of the proposition hold, δ fixed. We now let $P = \bigcap_{k\in\mathbb{N}} P(1/k)$ to get the full result. ∎

This gives the construction to be used in the next section to find elements in $G^+_{(x,0)}$ for Example 2.

COROLLARY 4.7. *Suppose that* $\theta \in [0,1]\setminus\mathbb{Q}$ *has continued fraction representation* $[a_1,a_2,\ldots]$ *and* $\sum_n 1/a_n < \infty$. *Suppose further that* $\varepsilon > 0$ *and* $\liminf_q q^{3+\varepsilon}\|q\theta\| = 0$. *Then for almost all* $x \in [0,1]$ *there is a sequence* $n_k \to \infty$ *such that*:

(i) $\|n_k\theta\| \to 0$.
(ii) $\|n_k^2\theta + 2n_k x\| \to 0$.
(iii) $|a(x, n_k)| \to 1/2$.

Proof. Choose $M_n = n^{1/2+\varepsilon/4}$. By the assumptions about θ,

$$\lim n^{1/2+\varepsilon/8} M_n^3 n \|n\theta\| = 0$$

as n runs to infinity over some subsequence, Q_0, of approximation denominators for θ. Also, more elementarily, $n^{1/2+\varepsilon/8} M_n^{\varepsilon-1} \to 0$. Therefore, we may choose $\gamma_n \to 0$ so that the conditions of Proposition 4.6 hold (for $\varepsilon/8$ instead of ε). From the conclusion of that proposition, we have a set of full measure, P, so that, for each $x \in P$, there is an infinite sequence of approximation denominators, n, with m_n such that $0 \le m_n < n^{1/2+\varepsilon/4}$, and for which $\|2m_n nx\| \to 0$ and $|a(x, m_n n)| \to 1/2$.

The sequence to be used in the conclusion of the corollary is therefore $m_n n$ as n runs through the infinite set of approximation denominators chosen for x by Proposition 4.6.

The check for parts (i) and (ii) is straightforward: $\|m_n n\theta\| \le m_n\|n\theta\| \le n\|n\theta\| \to 0$. More strictly, $\|(m_n n)^2\theta + 2m_n nx\| \le m_n^2 n\|n\theta\| + \|2m_n nx\| \le n^3\|n\theta\| + \|2m_n nx\| \to 0$ by construction. ∎

The following makes the crucial point in the proof Theorem 1.3.

Corollary 4.8. *Suppose that $\theta \in [0,1] \setminus \mathbb{Q}$ has continued fraction representation $[a_1, a_2, \ldots]$ and $\sum_n 1/a_n < \infty$. Suppose also that $\varepsilon > 0$ and that $\liminf_q q^{3+\varepsilon}\|q\theta\| = 0$. Then, in Example 2, for almost all $x \in [0,1]$ and all $y \in [0,1]$, $G^+_{(x,y)}$ contains a point of modulus $1/2$.* ∎

Appendix: General analytic facts. In this section we present the general analysis behind the estimates used in §§3 and 4.

Recall Definition 4.1, in particular $b(x,m) = \sum_{j=0}^{m-1} \exp(2\pi ijx)$ which we consider first.

Lemma A.1. *For all x and m, we have the estimate $\|mx\|/(2\pi\|x\|) \le |b(x,m)| \le 2\pi\|mx\|/\|x\|$. The points $\{b(x,m) : m \in \mathbb{Z}\}$ are distributed on a circle in the complex plane, having centre $1/(1-e^{2\pi ix})$ and passing through 0. Moreover, if q is an approximation denominator for x and if $M \ge q$, then $\{|b(x,m)| : 0 \le m < M\}$ is a set of real numbers $4\pi/(q\|x\|)$-dense in $[0, 1/(2\pi\|x\|)]$.*

Proof (see also [DM-F]). The first two parts are an easy application of the geometric series formula. The final part follows as $\{e^{2\pi imx} : 0 \le m \le M\}$ is $4\pi/q$-dense on the unit circle, so that $b(x,m) = (e^{2\pi imx}-1)/(e^{2\pi ix}-1)$ is $4\pi/(q\|x\|)$-dense on the circle described in the second part. ∎

The following is basic in the theory of uniform distribution.

LEMMA A.2 [KN]. *Suppose that* $n_k \to \infty$ *is a strictly increasing sequence of integers. Then, for Lebesgue almost all* $x \in [0,1]$, $n_k x$ *is uniformly distributed mod* 1 (*in fact we shall need only that the sequence is dense*). ∎

We use the following elementary estimate of the size of Weyl sums (recall $a(x,n) = \sum_{j=0}^{n-1} \exp(2\pi i(j^2\theta + 2jx))$).

LEMMA A.3. *Suppose that* $\eta_n \to 0$ *is a sequence of real numbers and that* Q *is a sequence of positive integers such that* $\sum_{n\in Q} n^{-2\eta_n} < \infty$. *Then, for almost all* $x \in [0,1]$, *there is an* n_1 *so that for all* $n \in Q$, $n \ge n_1$, *we have* $|a(x,n)| \le n^{1/2+\eta_n}$.

Proof. Consider the elementary integral $\int_0^1 |a(x,n)|^2\,dx = n$ and the estimate $\lambda\{x : |a(x,n)| \ge n^{1/2+\eta_n}\} \le n^{-2\eta_n}$ coming immediately from it. The conditions of the lemma allow us to apply the Borel–Cantelli Lemma to deduce the conclusion immediately. ∎

Here are two formulae which relate different Weyl sums when the θ involved has a particularly strong continued fraction approximation. In fact, the errors in the first estimate control strongly the restrictions on the value of θ in this paper.

LEMMA A.4. *Suppose that* $0 < \theta, x < 1$. *Then, for all* $n, m, k \in \mathbb{N}$, *we have the following estimates*:

$$a(x,mn) = a(x,n)(b(2nx,m) + O(m^3 n\|n\theta\|))$$

and

$$\beta a(x+k\theta,n) = a(x,n) + a(x,k)(\exp(4\pi i n x) - 1) + O(k^2\|n\theta\| + kn\|n\theta\|)$$

where $|\beta| = 1$.

Proof. These are each a sequence of equalities. For the first:

$$\begin{aligned} a(x,mn) &= \sum_{k=0}^{m-1}\sum_{j=0}^{n-1} \exp(2\pi i((kn+j)^2\theta + 2(kn+j)x)) \\ &= \sum_{k=0}^{m-1}\sum_{j=0}^{n-1} \exp(2\pi i(j^2\theta + 2jx))\exp(2\pi i(k^2n^2\theta + 2nkj\theta + 2knx)) \\ &= \sum_{k=0}^{m-1} a(x,n)(\exp(2\pi i 2knx) + O(k^2 n\|n\theta\|)) \end{aligned}$$

as required.

For the second:

$$\begin{aligned} a(x+k\theta,n) &= \sum_{j=0}^{n-1} \exp(2\pi i(j^2\theta+2jk\theta+2jx)) \\ &= \sum_{j=0}^{n-1} \exp(2\pi i((j+k)^2\theta-k^2\theta+2(j+k)x-2kx)) \\ &= \overline{\beta}\sum_{j=k}^{n+k-1} \exp(2\pi i(j^2\theta+2jx)) \end{aligned}$$

where $\beta=\exp(2\pi i(k^2\theta+2kx))$. If we drop the factor of $\overline{\beta}$ for the moment, this last expression equals

$$a(x,n)+\sum_{j=0}^{k-1}(\exp(2\pi i((j+n)^2\theta+2(j+n)x))-\exp(2\pi i(j^2\theta+2jx)))$$
$$=a(x,n)+\sum_{j=0}^{k-1}\exp(2\pi i(j^2\theta+2jx))(\exp(2\pi i(2jn\theta+n^2\theta+2nx))-1).$$

But we estimate $\exp(2\pi i(2jn\theta+n^2\theta+2nx))=\exp(4\pi inx)+O((j+n)\|n\theta\|)$ whence the estimate above. ■

The second estimate is used indirectly to show that $a(x,n)$ can be large (see Fact 2 in the proof of Proposition 4.6). Here is the precise form that is used.

COROLLARY A.5. *We have the following estimate*:

$$\big||a(x+k\theta,n)|-|a(x,n)|\big|\ge|a(x,k)|\cdot\|2nx\|+O(k^2\|n\theta\|+kn\|n\theta\|).$$

In particular,

$$2\max\{|a(x,n)|,|a(x+k\theta,n)|\}\ge|a(x,k)|\cdot\|2nx\|+O(k^2\|n\theta\|+kn\|n\theta\|).\ ■$$

Now we control the size of approximation denominators for generic $x\in[0,1]$.

LEMMA A.6. *For each $\eta>0$, there is a subset $P_1(\eta)$ of $[0,1]$, of measure at least $1-\eta$, and an $M_0(\eta)>0$ such that for each $x\in P_1(\eta)$ and for any $M>M_0$, there is a continued fraction approximation p/q for x such that $M^{1-\eta}<q<M$.*

Proof. By a famous result of Khinchin (see [Kh], [Bi]) we know that there is an $\alpha>1$ such that, for almost all $x\in[0,1]$, $(1/k)\log q_k(x)\to\alpha$ as $k\to\infty$, where $q_k(x)$ is the kth approximation denominator for x.

Find $\delta > 0$ such that $1-\eta < \log(\alpha-\delta)/\log(\alpha+\delta)$. Now find k_0 such that the set $P = \{x : (\alpha-\delta)^k < q_k(x) < (\alpha+\delta)^k \ \forall k \ge k_0\}$ has measure at least $1-\eta$ and $(1+1/k_0)(1-\eta) < \log(\alpha-\delta)/\log(\alpha+\delta)$. This ensures that $(\alpha+\delta)^{k+1} < (\alpha-\delta)^{k/(1-\eta)}$ for all $k \ge k_0$, and so consecutive intervals $I_k = [(\alpha+\delta)^k, (\alpha-\delta)^{k/(1-\eta)}]$, $k \ge k_0$, overlap.

In particular, if $M > (\alpha+\delta)^{k_0}$, then we find $k \ge k_0$ such that $M \in I_k$. However, in that case, $M^{1-\eta} < (\alpha-\delta)^k$ and $M > (\alpha+\delta)^k$ and so, by construction, we have $\{x : M^{1-\eta} < q_k(x) < M\} \supset P$ as required. ∎

Finally, we turn to a more subtle estimate on the size of Weyl sum. Recall $\psi(\theta,x,n) = |\sum_{k=0}^{n-1}\exp(\pi i(k^2\theta+2kx))|$, a convenient formula for the Weyl sum when we wish to consider it as a function of θ as well as of x and n. Hardy and Littlewood give an approximate functional equation for ψ:

THEOREM A.7 ([HL], 2.128, 2.17). *If* $0 < \theta, x < 1$ *and* $n \ge 1$, *then*

$$\sqrt{\theta}\,\psi(\theta,x,n) = \psi(\{1/\theta\},\{-x/\theta\},[n\theta]) + O(1)$$

where the constant implied in Landau's error notation is absolute. ∎

REFERENCES

[AP1] J. M. Anderson and L. D. Pitt, *On recurrence properties of certain lacunary series I*, J. Reine Angew. Math. 377 (1987), 65–82.

[AP2] —, —, *On recurrence properties of certain lacunary series II*, ibid., 83–96.

[At1] G. Atkinson, *Non-compact extensions of transformations*, Ph.D. Thesis, Univ. of Warwick, 1976.

[At2] —, *Recurrence of cocycles and random walks*, J. London Math. Soc. (2) 13 (1976), 486–488.

[At3] —, *A class of transitive cylinder transformations*, ibid. 17 (1978), 263–270.

[Bi] P. Billingsley, *Ergodic Theory and Information*, Wiley Ser. Probab. Math. Statist., Wiley, New York, 1965.

[Co] Z. Coelho, *On the asymptotic range of cocycles for shifts of finite type*, Ergodic Theory Dynam. Systems 13 (1993), 249–262.

[D] F. M. Dekking, *On transience and recurrence of generalised random walks*, Z. Wahrsch. Verw. Gebiete 61 (1982), 459–465.

[DM-F] F. M. Dekking and M. Mendès-France, *Uniform distribution modulo one: a geometrical viewpoint*, J. Reine Angew. Math. 329 (1981), 143–153.

[FM] J. Feldman and C. Moore, *Ergodic equivalence relations, cohomology and von Neumann algebras I*, Trans. Amer. Math. Soc. 234 (1977), 289–324.

[Fo] A. H. Forrest, *The limit points of Weyl sums and other continuous cocycles*, J. London Math. Soc. (2) 54 (1996), 440–452.

[F] H. Furstenberg, *Recurrence in Ergodic Theory and Combinatorial Number Theory*, Princeton Univ. Press, 1981.

[G] Y. Guivarc'h, *Propriétés ergodiques, en mesure infini, de certains systèmes dynamiques fibrés*, Ergodic Theory Dynam. Systems 9 (1989), 433–453.

[HL] G. H. Hardy and J. E. Littlewood, *The trigonometric series associated with the elliptic θ-functions*, Acta Math. 37 (1914), 193–239.

[HW] G. H. Hardy and E. M. Wright, *Introduction to the Theory of Numbers*, Clarendon Press, Oxford, 1979.

[He] G. A. Hedlund, *A class of transformations of the plane*, Proc. Cambridge Philos. Soc. 51 (1955), 554–564.

[I] A. Iwanik, *Ergodicity for piecewise smooth cocycles over toral rotations*, Fund. Math. 157 (1998), 235–244.

[Kh] A. Ya. Khintchine [A. Ya. Khinchin], *Continued Fractions*, Noordhoff, Groningen, 1963.

[Kre] W. Krieger, *On the finitary isomorphisms of Markov shifts that have finite expected coding time*, Z. Wahrsch. Verw. Gebiete 65 (1983), 323–328.

[Kry] A. B. Krygin, *An example of a cylindrical cascade with anomalous metric properties*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. 30 (1975), no. 5, 26–32 (in Russian).

[KN] L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences*, Wiley, New York, 1974.

[Liv] A. N. Livšic, *Cohomology of dynamical systems*, Math. USSR-Izv. 6 (1972), 1278–1301.

[LM] M. Lemańczyk and M. K. Mentzen, *Topological ergodicity of real cocycles over minimal rotations*, preprint, April 1999.

[LPV] M. Lemańczyk, F. Parreau and D. Volný, *Ergodic properties of real cocycles and pseudo-homogeneous Banach spaces*, Trans. Amer. Math. Soc. 348 (1996), 4919–4938.

[M] H. L. Montgomery, *Ten Lectures on the Interface between Analytic Number Theory and Harmonic Analysis*, CBMS Regional Conf. Ser. in Math. 84, Amer. Math. Soc., Providence, RI, 1994.

[N] M. B. Nathanson, *Additive Number Theory—The Classical Bases*, Grad. Texts in Math. 164, Springer, New York, 1996.

[PS] W. Parry and K. Schmidt, *Coboundaries and homomorphisms for nonsingular actions and a problem of H. Helson*, Invent. Math. 76 (1984), 15–32.

[Pa] D. A. Pask, *Skew products over the irrational rotation*, Israel J. Math. 69 (1990), 65–74.

[Pu] L. D. Pustyl'nikov, *New estimates of Weyl sums and the remainder term in the law of distribution of the fractional part of a polynomial*, Ergodic Theory Dynam. Systems 11 (1991), 515–534.

[Sch1] K. Schmidt, *Cocycles of Ergodic Transformation Groups*, Lecture Notes in Math. 1, MacMillan of India, 1977.

[Sch2] —, *Hyperbolic structure preserving isomorphisms of Markov shifts*, Israel J. Math. 55 (1986) 213–228.

[Schw] F. Schweiger, *Ergodic Theory of Fibred Systems and Metric Number Theory*, Clarendon Press, Oxford, 1995.

[Va] R. C. Vaughan, *The Hardy–Littlewood Method*, Cambridge Tract 80, Cambridge Univ. Press, 1981.

[Vin] I. M. Vinogradov, *The Method of Exponential Sums in the Theory of Numbers*, Nauka, Moscow, 1971 (in Russian).

[W] H. Weyl, *Über die Gleichverteilung von Zahlen mod. Eins*, Math. Ann. 77 (1916), 313–352.

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