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ON THE ALGEBRA OF CONSTANTS OF POLYNOMIAL DERIVATIONS IN TWO VARIABLES

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Abstract. Let d be a k-derivation of $k[x,y]$, where k is a field of characteristic zero. Denote by $\widetilde{d}$ the unique extension of d to $k(x,y)$. We prove that if $\ker d \neq k$, then $\ker \widetilde{d} = (\ker d)_0$, where $(\ker d)_0$ is the field of fractions of $\ker d$.

1. Introduction. Let k be a field of characteristic zero. Let $k[x_1,\ldots,x_n]$ be a polynomial ring in n variables over k and let d be a k-derivation of $k[x_1,\ldots,x_n]$. Denote by $k[x_1,\ldots,x_n]^d$ the ring of constants (the kernel) of d and let $\widetilde{d}$ be the unique extension of d to the quotient field $(k[x_1,\ldots,x_n])_0 = k(x_1,\ldots,x_n)$ of $k[x_1,\ldots,x_n]$. It is well known ([1] 8.1.5) that if d is locally nilpotent then $k(x_1,\ldots,x_n)^{\widetilde{d}} = (k[x_1,\ldots,x_n]^d)_0$. However if we do not assume that d is locally nilpotent, this equality is not valid even for the polynomial ring in two variables. Indeed, consider the derivation d defined by

$$d(x) = x, \quad d(y) = y.$$

Obviously, $k[x,y]^d = k$. But $k(x,y)^{\widetilde{d}} \neq k$ because $x/y \in k(x,y)^{\widetilde{d}}$. It turns out that in the polynomial ring in two variables the equality $(k[x,y]^d)_0 = k(x,y)^{\widetilde{d}}$ holds under an additional assumption.

THEOREM. *Let d be a k-derivation of $k[x,y]$. If $k[x,y]^d \neq k$, then $(k[x,y]^d)_0 = k(x,y)^{\widetilde{d}}$.*

This theorem (for $k = \mathbb{R}$) appears in the paper of S. Sato [2]. The proof given there is incorrect, because the formula for $\deg_y h$ (see the second line on page 14 in [2]) does not hold in some cases. The aim of this note is to give a complete proof of the Theorem.

2. Proof of Theorem. Let us set $d = f\partial/\partial x + g\partial/\partial y$ for polynomials $f, g \in k[x,y]$. If at least one of the elements f, g is zero, then the proof is

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straightforward, because then it is easy to compute $k[x,y]^d$ and $k(x,y)^{\widetilde{d}}$. We may assume that f and g are both nonzero polynomials.

Since $k[x,y]^d \neq k$, the transcendence degree of $k(x,y)^{\widetilde{d}}$ over k is greater or equal to 1. By the condition $d \neq 0$, this transcendence degree equals 1. Hence, by the Lüroth Theorem, $k(x,y)^{\widetilde{d}} = k(\theta)$ for some $\theta \in k(x,y) \setminus k$. Let us set $\theta = F/G$ for relatively prime elements F, G of $k[x,y]$. Since $k(\theta) = k(1/\theta)$, we may assume that $\deg_y F \geq \deg_y G$, where $\deg_y F$ denotes the degree of F with respect to y. By the condition $k[x,y]^d \neq k$, there exists an element $h \in k[x,y]^d \setminus k$. Then we have $\deg_y h > 0$ and $\deg_x h > 0$ because, if $\deg_y h = 0$, we have $h \in k[x]$. Hence we have $d(h) = f(x,y)\partial h/\partial x = 0$ and $\partial h/\partial x = 0$. Therefore $h \in k$ and we have a contradiction. In the same way, we have $\deg_x h > 0$. Let

$$F = f_n y^n + f_{n-1} y^{n-1} + \ldots + f_0,$$
$$G = g_m y^m + g_{m-1} y^{m-1} + \ldots + g_0,$$

where $n = \deg_y F$, $m = \deg_y G$ and f_i, $g_j \in k[x]$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Now, let us consider two cases.

Case 1: $n = m$ and $\deg_x f_n = \deg_x g_n = r$. Then let $f_n = c_r x^r + \ldots + c_0$ and $g_n = d_r x^r + \ldots + d_0$ where c_i, $d_i \in k$ for $i = 1, \ldots, r$. Consider the element $\theta - c_r/d_r$. It is not equal to zero, because $\theta \notin k$. Obviously $\theta - c_r/d_r = H/G$, where H is the polynomial in $k[x,y]$ equal to $F - (c_r/d_r)G$. Then H and G are relatively prime, because F and G are relatively prime. We also see that either $\deg_y H < \deg_y G$ or they are equal but coefficients of the highest power of y in H and G are polynomials in $k[x]$ of different degrees. Then we put $\theta' = 1/(\theta - c_r/d_r)$ instead of θ and we are in the following second case.

Case 2: $n > m$, or $n = m$ but $\deg_x f_n \neq \deg_x g_n$. Since $h \in k[x,y]^d \subseteq k(x,y)^{\widetilde{d}} = k(\theta)$, we can write

$$h = \frac{\sum_{i=0}^{t} a_i \theta^i}{\sum_{i=0}^{s} b_i \theta^i} = \frac{\sum_{i=0}^{t} a_i \left(\frac{F}{G}\right)^i}{\sum_{i=0}^{s} b_i \left(\frac{F}{G}\right)^i} = \frac{\sum_{i=0}^{t} a_i G^{t-i} F^i}{\sum_{i=0}^{s} b_i G^{s-i} F^i} G^{s-t}$$

for $a_i, b_i \in k$ and $a_t \neq 0$, $b_s \neq 0$. We proceed to show that in this case we have

$$\deg_y h = (t-s)(\deg_y F - \deg_y G) = (t-s)(n-m).$$

It is clear that $\deg_y G^{s-t} = -(t-s)m$ and it is sufficient to prove that $\deg_y(\sum_{i=0}^{t} a_i G^{t-i} F^i) = tn$ and $\deg_y(\sum_{i=0}^{s} b_i G^{s-i} F^i) = sn$. Assume, without loss of generality, that the degree of $\sum_{i=0}^{t} a_i G^{t-i} F^i$ is not equal to tn. If $n > m$ then each term of the form $G^{t-i} F^i$ has a different degree with respect to y. Since the highest degree (equal to nt) has $G^0 F^t$ and $a_t \neq 0$, the equality $\deg_y(\sum_{i=0}^{t} a_i G^{t-i} F^i) = tn$ holds. Hence, we may assume that

$n = m$ and $\deg_x f_n \neq \deg_x g_n$. Obviously, $\deg_y(\sum_{i=0}^{t} a_i G^{t-i} F^i) \leq tn$. If the inequality is strict, then it follows easily that the coefficient of y^{nt} equals 0. Therefore $\sum_{i=0}^{t} a_i g_n^{t-i} f_n^i = 0$. Since $\deg_x f_n \neq \deg_x g_n$, all polynomials of the form $g_n^{t-i} f_n^i$ have different degrees. Since at least one of the elements $a_1, \ldots, a_t$ is nonzero, it follows that the above sum cannot be equal to 0. This proves the formula for $\deg_y h$. Because $\deg_y h > 0$, we get $n > m$ and $t > s$.

The equality $h(x,y)G^{t-s}(\sum_{i=0}^{s} b_i G^{s-i} F^i) = \sum_{i=0}^{t} a_i G^{t-i} F^i$ implies that the polynomial

$$a_t F^t + \sum_{i=0}^{t-1} (a_i G^{t-i-1} F^i) G$$

is divisible by G and hence F^t is divisible by G. But $(G, F) = 1$, so we have $G \in k$ and $\theta \in k[x,y]$. This completes the proof. ∎

Let us end the paper with the following question. Let d be a k-derivation of the polynomial ring $k[x_1, \ldots, x_n]$. Assume that the transcendence degree of $k[x_1, \ldots, x_n]^d$ is equal to $n-1$. Is it true that $k(x_1, \ldots, x_n)^{\widetilde{d}} = (k[x_1, \ldots, x_n]^d)_0$? A positive answer to this question would be a natural generalization of the Theorem. Note (for example [1] 7.1.1) that for any nonzero k-derivation of $k[x_1, \ldots, x_n]$ the transcendence degree of $k[x_1, \ldots, x_n]^d$ is less than or equal to $n-1$.

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