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PROPERTIES OF G-ATOMS AND FULL GALOIS COVERING REDUCTION TO STABILIZERS

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Dedicated to Professor Helmut Lenzing on the occasion of his 60th birthday

Abstract. Given a group G of k-linear automorphisms of a locally bounded k-category R it is proved that the endomorphism algebra $\mathrm{End}_R(B)$ of a G-atom B is a local semiprimary ring (Theorem 2.9); consequently, the injective $\mathrm{End}_R(B)$-module $(\mathrm{End}_R(B))^*$ is indecomposable (Corollary 3.1) and the socle of the tensor product functor $-\otimes_R B^*$ is simple (Theorem 4.4). The problem when the Galois covering reduction to stabilizers with respect to a set $\mathcal{U}$ of periodic G-atoms (defined by the functors $\Phi^{\mathcal{U}}:\coprod_{B\in\mathcal{U}}\mathrm{mod}\,kG_B\to\mathrm{mod}(R/G)$ and $\Psi^{\mathcal{U}}:\mathrm{mod}(R/G)\to\prod_{B\in\mathcal{U}}\mathrm{mod}\,kG_B$) is full (resp. strictly full) is studied (see Theorems A, B and 6.3).

1. Introduction. The Galois covering technique has been originally invented for investigation of finite-dimensional algebras of finite representation type. It reduces the description of $\mathrm{mod}\,\Lambda$ to the analogous problem for the cover $\widetilde{\Lambda}$ of Λ, which is usually simpler (see [18, 12, 2, 14]). For the first generalizations of that method in representation infinite case the reader is referred to [9] and [8], and in a much more general situation to [10] (see also [17]). These results had many applications (see [23, 24, 25, 13]). The Galois coverings were also investigated for matrix problems in [19, 20, 21, 11], recently in a quite general situation [7]. In [3] a new, a little different approach of a one-step reduction to representation categories of stabilizers was proposed. It was formalized in [4], where the scheme of Galois covering reduction to stabilizers was introduced. There the important facts concerning the concept of full Galois covering reduction to stabilizers were formulated but the proofs were only briefly outlined.

In this paper we present full proofs of the main results in [4]: of [4, Theorem 3.3], which states that some natural conditions are sufficient

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for a Galois covering reduction to stabilizers to be full, and of [4, Theorem 5.2], which is the most important application of the previous one. We also study rather comprehensively a class of indecomposable locally finite-dimensional R-modules, called G-atoms. First of all we discuss those properties of G-atoms which are essential for Galois covering, mainly having in mind applications in the proofs of the cited theorems, but also in a quite general context.

Before we formulate our main results, we briefly sketch the situation we deal with. Let k be a field and R be a *locally bounded* k*-category*, i.e. all objects of R have local endomorphism rings, different objects are nonisomorphic, and both sums $\sum_{y\in R}\dim_k R(x,y)$ and $\sum_{y\in R}\dim_k R(y,x)$ are finite for each $x\in R$. By an R*-module* we mean a contravariant k-linear functor from R to the category of k-vector spaces. An R-module M is *locally finite-dimensional* (resp. *finite-dimensional*) if $\dim_k M(x)$ is finite for each $x\in R$ (resp. the *dimension* $\dim_k M=\sum_{x\in R}\dim_k M(x)$ of M is finite). We denote by $\mathrm{MOD}\,R$ the category of all R-modules, and by $\mathrm{Mod}\,R$ (resp. $\mathrm{mod}\,R$) the full subcategory formed by all locally finite-dimensional (resp. finite-dimensional) R-modules. By the *support* of an object M in $\mathrm{MOD}\,R$ we mean the full subcategory $\mathrm{supp}\,M$ of R formed by the set $\{x\in R : M(x)\neq 0\}$. We denote by $\mathcal{J}_R$ the Jacobson radical of the category $\mathrm{Mod}\,R$.

For any k-algebra A we denote analogously by $\mathrm{MOD}\,A$ (resp. $\mathrm{mod}\,A$) the category of all (resp. all finite-dimensional) right A-modules and by $J(A)$ the Jacobson radical of A.

Let G be a group of k-linear automorphisms of R acting freely on the set $\mathrm{ob}\,R$ of all objects of R. Then G acts on the category $\mathrm{MOD}\,R$ by translations ${}^g(-)$, which assign to each M in $\mathrm{MOD}\,R$ the R-module ${}^gM=M\circ g^{-1}$ and to each $f:M\to N$ in $\mathrm{MOD}\,R$ the R-homomorphism ${}^gf:{}^gM\to{}^gN$ given by the family $(f(g^{-1}(x)))_{x\in R}$ of k-linear maps. Given M in $\mathrm{MOD}\,R$ the subgroup

$$G_M=\{g\in G : {}^gM\simeq M\}$$

of G is called the *stabilizer* of M. We do not assume here that G acts freely on the set of isoclasses of indecomposable finite-dimensional R-modules (briefly $(\mathrm{ind}\,R)/{\simeq}$), i.e. that $G_M=\{\mathrm{id}_R\}$ for every indecomposable M in $\mathrm{mod}\,R$.

We can form the orbit category $\bar{R}=R/G$, which is again a locally bounded k-category (see [12]), and we want to study the module category $\mathrm{mod}\,\bar{R}$ in terms of the category $\mathrm{Mod}\,R$. The tool we have at our disposal is a pair of functors

$$\mathrm{MOD}\,R \underset{F_\bullet}{\overset{F_\lambda}{\rightleftarrows}} \mathrm{MOD}\,\bar{R},$$

where $F_\bullet:\mathrm{MOD}\,\bar{R}\to\mathrm{MOD}\,R$ is the "pull-up" functor associated with the canonical Galois covering functor $F : R \to \bar{R}$, assigning to each X in $\operatorname{MOD} \bar{R}$ the R-module $X \circ F$, and the "push-down" functor $F_\lambda : \operatorname{MOD} R \to \operatorname{MOD} \bar{R}$ is the left adjoint to $F_\bullet$.

The classical results from [12] state that if G acts freely on $(\operatorname{ind} R)/\simeq$ then F_λ induces an embedding of the set $((\operatorname{ind} R)/\simeq)/G$ of G-orbits into $(\operatorname{ind} \bar{R})/\simeq$.

Let H be a subgroup of the stabilizer G_M of a given M in $\operatorname{MOD} R$. By an *R-action* of H on M we mean a family

$$\mu = (\mu_g : M \to {}^{g^{-1}}M)_{g \in H}$$

of R-homomorphisms such that $\mu_e = \mathrm{id}_M$, where $e = \mathrm{id}_R$ is the unit of H, and ${}^{g_1^{-1}}\mu_{g_2} \cdot \mu_{g_1} = \mu_{g_2 g_1}$ for all $g_1, g_2 \in H$ (see [12]). Observe that if H is a free group then M admits an R-action of H (see [3, Lemma 4.1]). We denote by $\operatorname{Mod}^H R$ the category consisting of pairs (M, μ), where M is a locally finite-dimensional R-module and μ an R-action of H on M. For any $M = (M, \mu)$ and $N = (N, \nu)$ in $\operatorname{Mod}^H R$ the space of morphisms from M to N in $\operatorname{Mod}^H R$ consists of all $f \in \operatorname{Hom}_R(M, N)$ such that ${}^{g^{-1}}f \cdot \mu_g = \nu_g \cdot f$, for every $g \in H$, and is denoted by $\operatorname{Hom}_R^H(M, N)$. We denote by $\mathcal{J}_R^H$ the ideal $\operatorname{Hom}_R^H \cap \mathcal{J}_R$ of the category $\operatorname{Mod}^H R$.

A useful interpretation of $\operatorname{mod} \bar{R}$ is the category $\operatorname{Mod}_{\mathrm{f}}^G R$ consisting of pairs (M, μ) in $\operatorname{Mod}^G R$ such that $\operatorname{supp} M$ is contained in a union of finitely many G-orbits in R (see [3, 12]). The functor $F_\bullet$ associating with any X in $\operatorname{mod} \bar{R}$ the R-module $F_\bullet X$ endowed with the trivial R-action of G yields an equivalence

$$\operatorname{mod}(\bar{R}) \simeq \operatorname{Mod}_{\mathrm{f}}^G R.$$

We denote by $\mathcal{I}_{\bar{R}}$ the ideal $F_\bullet^{-1}(\mathcal{J}_R^G)$ which constitutes an essential class of morphisms in $\operatorname{mod} \bar{R}$. It is clear that $\mathcal{I}_{\bar{R}}$ is contained in the Jacobson radical $\mathcal{J}_{\bar{R}}$ but usually not conversely.

An important role in understanding the nature of objects from $\operatorname{Mod}_{\mathrm{f}}^G R$, or equivalently $\operatorname{mod} \bar{R}$, is played by a class of indecomposable locally finite-dimensional R-modules called *G-atoms*. Following [4], an indecomposable B in $\operatorname{Mod} R$ (with local endomorphism ring) is called a *G-atom* if $\operatorname{supp} B$ is contained in a union of finitely many G_B-orbits in R.

Denote by $\mathcal{A}$ a fixed set of representatives of isoclasses of all G-atoms, by $\mathcal{A}_{\mathrm{o}}$ a fixed set of representatives of G-orbits of the induced action of G on $\mathcal{A}$ and by $\bar{\mathcal{A}}$ the set of all $B \in \mathcal{A}$ such that $\operatorname{End}_R(B)/J(\operatorname{End}_R(B)) \simeq k$. Given a subset $\mathcal{U} \subset \mathcal{A}$ we set $\mathcal{U}_{\mathrm{o}} = {}^G\mathcal{U} \cap \mathcal{A}_{\mathrm{o}}$ (resp. $\bar{\mathcal{U}} = {}^G\mathcal{U} \cap \bar{\mathcal{A}}$), where ${}^G\mathcal{U}$ is the union of all orbits of elements from $\mathcal{U}$ in $\mathcal{A}$. For any $B \in \mathcal{A}$, denote by S_B a fixed set of representatives of left cosets of G_B in G, containing the unit e of G.

One can show that the set of isoclasses of R-modules M in $\operatorname{Mod} R$ such that $G_M = G$ and $\operatorname{supp} M/G$ is finite, is in bijective correspondence with the set $(\mathbb{N}^{\mathcal{A}_\circ})_0$ of all sequences $n = (n_B)_{B\in\mathcal{A}_\circ}$ of natural numbers such that almost all n_B are zero. This correspondence is given by $n \mapsto M_n$, where

$$M_n = \bigoplus_{B\in\mathcal{A}_\circ} \Big(\bigoplus_{g\in S_B} {}^g(B^{n_B})\Big)$$

(see Corollary 2.4). In consequence, $\operatorname{mod} \bar{R}$ is equivalent via $F_\bullet$ to the full subcategory of $\operatorname{Mod}_{\mathrm{f}}^G R$ formed by all pairs (M_n, μ), where $n \in (\mathbb{N}^{\mathcal{A}_\circ})_0$ and μ is an arbitrary R-action of G on M_n. Therefore to any X in $\operatorname{mod} \bar{R}$ one can attach the finite set $\operatorname{dss}(X)$, called the *direct summand support* of X, consisting of all $B \in \mathcal{A}$ such that n_B is nonzero, where $F_\bullet X \simeq M_n$.

This notion suggests restricting the investigation of $\operatorname{mod} \bar{R}$ to some of its parts. For any $\mathcal{U} \subset \mathcal{A}$ one can study the full subcategory $\operatorname{mod}_{\mathcal{U}} \bar{R}$ of $\operatorname{mod} \bar{R}$ consisting of all X in $\operatorname{mod} \bar{R}$ such that $\operatorname{dss}(X) \subset {}^G\mathcal{U}$.

The set $\mathcal{A}$ splits naturally into the disjoint union $\mathcal{A} = \mathcal{A}^{\mathrm{f}} \cup \mathcal{A}^\infty$ where $\mathcal{A}^{\mathrm{f}}$ (resp. $\mathcal{A}^\infty$) is the subset of all finite (dimensional) (resp. infinite (dimensional)) G-atoms. It is well known (see [8, Lemma] and [10, 2.3]) that if G acts freely on $(\operatorname{ind} R)/\simeq$ then the above splitting induces the splitting

$$\operatorname{mod} \bar{R} = \operatorname{mod}_{\mathcal{A}^{\mathrm{f}}} \bar{R} \vee \operatorname{mod}_{\mathcal{A}^\infty} \bar{R}$$

in the sense explained below.

Let $\mathcal{C}$ be a Krull–Schmidt category and $\mathcal{C}_0$, $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_i$, $i \in I$, full subcategories of $\mathcal{C}$, which are closed under direct sums, direct summands and isomorphisms. The notation $\mathcal{C} = \mathcal{C}_1 \vee \mathcal{C}_2$ (resp. $\mathcal{C} = \bigvee_{i\in I} \mathcal{C}_i$) means that the set of indecomposable objects in $\mathcal{C}$ splits into the disjoint union of indecomposables in $\mathcal{C}_1$ and in $\mathcal{C}_2$ (resp. in $\mathcal{C}_i$, $i \in I$). We denote by $[\mathcal{C}_0]$ the ideal of all morphisms in $\mathcal{C}$ which factor through an object from $\mathcal{C}_0$. For any ideal $\mathcal{I}$ in the category $\mathcal{C}$ the restriction of $\mathcal{I}$ to $\mathcal{C}_0$ is denoted by $\mathcal{I}_{\mathcal{C}_0}$.

The situation described above will play a model role in the further considerations. In this paper we shall "split off and partially describe" the category $\operatorname{mod}_{\mathcal{U}} \bar{R}$, for some special $\mathcal{U} \subset \mathcal{A}$, also contained in $\mathcal{A}^\infty$.

Following [4] a G-atom $B \in \mathcal{A}$ is called *periodic* if it admits an R-action of G_B (this is always the case if the group G_B is free). Denote by $\mathcal{P}$ the set of all periodic G-atoms.

Let B be a periodic G-atom B and ν_B an R-action of a G_B on B. Then (B, ν_B) is in $\operatorname{Mod}_{\mathrm{f}}^{G_B} R$ and $F_\lambda B$ has the structure of a kG_B-$\bar{R}$-bimodule, which is finitely generated free as a left kG_B-module, where kG_B is the group algebra of G_B over k (see [10, 3.6] for the precise definition of this structure). Consequently, it induces two functors

$$\Phi^B = - \otimes_{kG_B} F_\lambda B : \operatorname{mod} kG_B \to \operatorname{mod}_B \bar{R}$$

and

$$\Psi^B = (\overline{\mathcal{H}}_R(B, F_\bullet(-))^{-1} : \operatorname{mod} \bar{R} \to \operatorname{mod} kG_B$$

(see [3, 2.3 and 2.4]). Here $\overline{\mathcal{H}}_R$ denotes the factor bimodule $\mathcal{H}_R/\mathcal{J}_R$, where $\mathcal{H}_R = \operatorname{Hom}_R(-, \cdot\,)$ and $\mathcal{J}_R$ is the Jacobson radical of $\operatorname{Mod} R$.

Let $\mathcal{U} = (\mathcal{U}, \nu)$ be a pair where $\mathcal{U} \subset \mathcal{P}_{\mathrm{o}}$ is a subset of periodic G-atoms and $\nu = (\nu_B)_{B\in\mathcal{U}}$ a fixed selection of R-actions of G_B on B. We denote by

$$\Phi^{\mathcal{U}} : \coprod_{B\in\mathcal{U}} \operatorname{mod} kG_B \to \operatorname{mod} \bar{R}$$

the functor defined by the family $(\Phi^B)_{B\in\mathcal{U}}$ and by

$$\Psi^{\mathcal{U}} : \operatorname{mod} \bar{R} \to \prod_{B\in\mathcal{U}} \operatorname{mod} kG_B$$

the functor induced by the family $(\Psi^B)_{B\in\mathcal{U}}$, where Φ^B and Ψ^B are defined by the pairs (B, ν_B). Observe that the subcategory $\operatorname{Im}\Psi^{\mathcal{U}}$ is contained in the category $\coprod_{B\in\mathcal{U}} \operatorname{mod} kG_B$. Then the pair $(\Phi^{\mathcal{U}}, \Psi^{\mathcal{U}})$ of functors

$$\coprod_{B\in\mathcal{U}} \operatorname{mod} kG_B \underset{\Psi^{\mathcal{U}}}{\overset{\Phi^{\mathcal{U}}}{\rightleftarrows}} \operatorname{mod} \bar{R}$$

is called the *Galois covering reduction to stabilizers* (briefly, *GCS-reduction*) *with respect to* $\mathcal{U}$ (in fact with respect to $(\nu_B)_{B\in\mathcal{U}}$). It will be used to describe the category $\operatorname{mod}_{\mathcal{U}}\bar{R}$ in terms of the module categories of the stabilizer group algebras.

It is proved in [4, Theorem 2.2] that for any family $\mathcal{U}$ of periodic G-atoms contained in $\overline{\mathcal{P}}_{\mathrm{o}}$ (i.e. $\operatorname{End}_R(B)/\mathcal{J}(\operatorname{End}_R(B)) \simeq k$ for each $B \in \mathcal{U}$) the functor $\Phi^{\mathcal{U}} : \coprod_{B\in\mathcal{U}} \operatorname{mod} kG_B \to \operatorname{mod} \bar{R}$ is a right quasi-inverse for $\Psi^{\mathcal{U}} : \operatorname{mod} \bar{R} \to \prod_{B\in\mathcal{U}} \operatorname{mod} kG_B$ (therefore faithful) and is a representation embedding in the sense of [22] (i.e. yields an injection of the set of isoclasses of indecomposables in $\coprod_{B\in\mathcal{U}} \operatorname{mod} kG_B$ into the set of isoclasses of indecomposables in $\operatorname{mod}_{\mathcal{U}}\bar{R}$).

One can show (see Proposition 6.1) that the ideal $\operatorname{Ker}\Psi^{\mathcal{U}}$ contains the ideal $[\operatorname{mod}_{(\mathcal{A}_{\mathrm{o}}\setminus\mathcal{U})}\bar{R}]$, and consequently $\Psi^{\mathcal{U}}$ induces a functor

$$\overline{\Psi}^{\mathcal{U}} : \operatorname{mod} \bar{R}/[\operatorname{mod}_{(\mathcal{A}_{\mathrm{o}}\setminus\mathcal{U})}\bar{R}] \to \coprod_{B\in\mathcal{U}} \operatorname{mod} kG_B,$$

and that $\Phi^{\mathcal{U}}$ induces a faithful representation embedding functor

$$\overline{\Phi}^{\mathcal{U}} : \coprod_{B\in\mathcal{U}} \operatorname{mod} kG_B \to \operatorname{mod} \bar{R}/[\operatorname{mod}_{(\mathcal{A}_{\mathrm{o}}\setminus\mathcal{U})}\bar{R}]$$

($\overline{\Phi}^{\mathcal{U}}$ is a right quasi-inverse for $\overline{\Psi}^{\mathcal{U}}$).

Following [4], the GCS-reduction $(\Phi^{\mathcal{U}}, \Psi^{\mathcal{U}})$ with respect to $\mathcal{U}$ is said to be *full* provided $\Phi^{\mathcal{U}}$ and $\Psi^{\mathcal{U}}$ induce

(a) a splitting $\operatorname{mod}\bar{R} = \operatorname{mod}_{\mathcal{U}}\bar{R} \vee \operatorname{mod}_{(\mathcal{A}_{\mathrm{o}}\setminus\mathcal{U})}\bar{R}$,

(b) a bijection between the sets of isoclasses of indecomposables in the categories $\coprod_{B\in\mathcal{U}} \operatorname{mod} kG_B$ and $\operatorname{mod}_{\mathcal{U}}\bar{R}$.

It is shown (see Proposition 6.1) that then:

(c) $$\operatorname{Ker}\Psi^{\mathcal{U}}(X,Y) = \begin{cases} \mathcal{I}_{\bar{R}}(X,Y) & \text{if } X, Y \in \operatorname{mod}_{\mathcal{U}}\bar{R}, \\ \operatorname{Hom}_{\bar{R}}(X,Y) & \text{if } X \text{ or } Y \notin \operatorname{mod}_{\mathcal{U}}\bar{R}, \end{cases}$$

for any indecomposables X, Y in $\operatorname{mod}\bar{R}$,

(d) $\bar{\Phi}^{\mathcal{U}}$ and $\bar{\Psi}^{\mathcal{U}}$ defined above yield a bijection between the sets of isoclasses of indecomposables in $\coprod_{B\in\mathcal{U}} \operatorname{mod} kG_B$ and $\operatorname{mod}\bar{R}/[\operatorname{mod}_{(\mathcal{A}_{\mathrm{o}}\setminus\mathcal{U})}\bar{R}]$.

The GCS-reduction $(\Phi^{\mathcal{U}}, \Psi^{\mathcal{U}})$ with respect to $\mathcal{U}$ is called *strictly full* provided the pair $(\bar{\Phi}^{\mathcal{U}}, \bar{\Psi}^{\mathcal{U}})$ yields an equivalence of categories.

Note that if the GCS-reduction $(\Phi^{\mathcal{U}}, \Psi^{\mathcal{U}})$ is strictly full then it is full $(\operatorname{Im}\Phi^{\mathcal{U}} \subset \operatorname{mod}_{\mathcal{U}}\bar{R}$ and $[\operatorname{mod}_{(\mathcal{A}_{\mathrm{o}}\setminus\mathcal{U})}\bar{R}]_{\operatorname{mod}_{\mathcal{U}}\bar{R}} \subset (\mathcal{J}_{\bar{R}})_{\operatorname{mod}_{\mathcal{U}}\bar{R}})$.

Let B be a periodic G-atom together with an R-action ν_B of G_B on B, and H be a subgroup of G containing G_B. We say that $B = (B, \nu_B)$ *splits* (resp. *splits properly*) an object $M = (M, \mu)$ in $\operatorname{Mod}^H R$ provided both embeddings $\mathcal{J}_R(B, M) \subset \operatorname{Hom}_R(B, M)$ and $\mathcal{J}_R(M, B) \subset \operatorname{Hom}_R(B, M)$ are splittable (resp. splittable, proper) monomorphisms in $\operatorname{MOD}(kG_B)^{\mathrm{op}}$ (for the precise definition of the left kG_B-module structure see 5.1).

Let $\mathcal{C}$ be a full subcategory of $\operatorname{Mod}^H R$. We say B *splits* $\mathcal{C}$ provided B splits each M in $\mathcal{C}$.

One of the main results in this paper is the following.

THEOREM A [4, Theorem 3.3]. *Let R be a locally bounded k-category and $G \subset \operatorname{Aut}_k(R)$ be a group of k-linear automorphisms acting freely on* $\operatorname{ob} R$. *Suppose that $\mathcal{U} \subset \overline{\mathcal{P}}_{\mathrm{o}}$ is a family of G-atoms together with a selection $(\nu_B)_{B\in\mathcal{U}}$ of R-actions of G_B on B such that each (B, ν_B) splits* $\operatorname{Mod}_{\mathrm{f}}^G R$, *for $B \in \mathcal{U}$. Then the Galois covering reduction $(\Phi^{\mathcal{U}}, \Psi^{\mathcal{U}})$ to stabilizers with respect to $\mathcal{U}$ is full. In particular* (a)–(d) *as above hold.*

Following [4], we denote by $\mathcal{A}^1$ the set of all G-atoms $B \in \mathcal{A}$ (in fact infinite G-atoms) such that G_B is an infinite cyclic group, and by $\mathcal{A}^{1'}$ the subset of all $B \in \mathcal{A}^\infty$ such that G_B has an infinite cyclic subgroup of finite index. Observe that $\mathcal{A}^1 \subset \mathcal{P}$ and that for any $B \in \mathcal{A}^1$ the group algebra kG_B is isomorphic to the Laurent polynomial algebra $k[T, T^{-1}]$. It is shown in [6] that $\mathcal{A}^\infty$ coincides with $\mathcal{A}^1$ provided R is a representation-tame category over an algebraically closed field and the group G is torsionfree.

For any $B \in \mathcal{A}^1$ we denote by $\mathcal{A}^{1'}(B)$ the set of all $B' \in \mathcal{A}^{1'}$ satisfying the following conditions:

(a) $\operatorname{supp} B' \subset \widehat{\widehat{\operatorname{supp}}} B$,
(b) $G_{B'} \cap G_B \neq \{e\}$,
(c) $\operatorname{supp} B' \cap \operatorname{supp} B \neq \emptyset$.

Here for any subcategory L of R, $\widehat{L}$ denotes the full subcategory of R consisting of all $y \in \operatorname{ob} R$ such that $R(x, y)$ or $R(y, x)$ is nonzero for some $x \in \operatorname{ob} L$ (see [9]). Note that if (b) and (c) hold then $\operatorname{supp} B' \cap \operatorname{supp} B$ is infinite since so is $G_B \cap G_{B'}$.

Now we formulate a generalization of [4, Theorem 5.2].

Theorem B. *Let R be a locally bounded k-category, $G \subset \operatorname{Aut}_k(R)$ be a group of k-linear automorphisms acting freely on* $\operatorname{ob} R$, *and $\mathcal{U}$ be a subset of $\overline{\mathcal{A}^1}_{\circ}$ together with a selection $\{\nu_B\}_{B\in\mathcal{U}}$ of R-actions of G_B on B. Assume that for any $B \in \mathcal{A}^1$ and $B' \in \mathcal{A}^{1'}(B)$ each R-homomorphism $f : B \to B'$ (resp. $f : B' \to B$) factors through a direct sum of finite-dimensional R-modules. Then the Galois covering reduction $(\Phi^{\mathcal{U}}, \Psi^{\mathcal{U}})$ to stabilizers with respect to $\mathcal{U}$ is strictly full and the functors $\Phi^{\mathcal{U}} : \coprod_{B\in\mathcal{U}} \operatorname{mod} kG_B \to \operatorname{mod} \bar{R}$ and $\Psi^{\mathcal{U}} : \operatorname{mod} \bar{R} \to \prod_{B\in\mathcal{U}} \operatorname{mod} kG_B$ defined by the families $(\Phi^B)_{B\in\mathcal{U}}$ and $(\Psi^B)_{B\in\mathcal{U}}$ induce the following equivalence*:

$$\coprod_{B\in\mathcal{U}} \operatorname{mod} k[T, T^{-1}] \simeq \operatorname{mod} \bar{R}/[\operatorname{mod}_{(\mathcal{A}_\circ\setminus\mathcal{U})} \bar{R}] \simeq \operatorname{mod}_{\mathcal{U}} \bar{R}/[\operatorname{mod}_{\mathcal{A}^{\mathrm{f}}} \bar{R}]_{\operatorname{mod}_{\mathcal{U}} \bar{R}}.$$

In particular the functors $\Phi^{\mathcal{U}}$ and $\Psi^{\mathcal{U}}$ induce:

(i) *a splitting* $\operatorname{mod} \bar{R} = \operatorname{mod}_{\mathcal{U}} \bar{R} \vee \operatorname{mod}_{(\mathcal{A}_\circ\setminus\mathcal{U})} \bar{R}$,

(ii) *a bijection between the isoclasses of indecomposables in* $\operatorname{mod}_{\mathcal{U}} \bar{R}$ *and in* $\coprod_{B\in\mathcal{U}} \operatorname{mod} k[T, T^{-1}]$.

In case the group G acts freely on $(\operatorname{ind} R)/\simeq$ *the above equivalence has the form*

$$\coprod_{B\in\mathcal{U}} \operatorname{mod} k[T, T^{-1}] \simeq \underline{\underline{\operatorname{mod}}}_{\mathcal{U}} \bar{R}$$

where $\underline{\underline{\operatorname{mod}}}_{\mathcal{U}} \bar{R}$ *is defined below.*

Suppose the group G acts freely on $(\operatorname{ind} R)/\simeq$. We denote by $\operatorname{mod}_1 \bar{R}$ the full subcategory of $\operatorname{mod} \bar{R}$ consisting of the $\bar{R}$-modules of the first kind, i.e. those of the form $F_\lambda(M)$ for some M in $\operatorname{mod} R$ (see [10, 3, 4]). We denote by $\underline{\underline{\operatorname{mod}}}\, \bar{R}$ the factor category $\operatorname{mod} \bar{R}/[\operatorname{mod}_1 \bar{R}]$. For any subset $\mathcal{U} \subset \mathcal{A}$ we denote by $\underline{\underline{\operatorname{mod}}}_{\mathcal{U}} \bar{R}$ the image of $\operatorname{mod}_{\mathcal{U}} \bar{R}$ in the factor category $\underline{\underline{\operatorname{mod}}}\, \bar{R}$.

We will present the full proof of the above theorem, simpler than that announced in [4].

The major part of the paper is devoted to assembling information on the behaviour of the categories $\operatorname{Mod} R$ and $\operatorname{Mod}^G R$ indispensable for the proofs of the main results. An essential component is formed by the results describing the properties of various k-additive functors on both categories.

For every k-category $\mathcal{C}$ we denote by $\mathrm{MOD}\,\mathcal{C}$ the category of $\mathcal{C}$-modules consisting all contravariant k-linear functors from $\mathcal{C}$ to the category of k-vector spaces (as for locally bounded k-categories). For any M in $\mathrm{MOD}\,\mathcal{C}$ we denote by $\mathrm{Soc}\, M$ the socle of the $\mathcal{C}$-module M.

Given a full subcategory $\mathcal{C}_0$ of $\mathcal{C}$ and a $\mathcal{C}$-module M we denote by $M_{|\mathcal{C}_0}$ the $\mathcal{C}_0$-module which is the restriction of M to $\mathcal{C}_0$. If $f : M \to N$ is a $\mathcal{C}$-homomorphism we denote by $f_{|\mathcal{C}_0} : M_{|\mathcal{C}_0} \to N_{|\mathcal{C}_0}$ the $\mathcal{C}_0$-homomorphism which is the restriction of f to $\mathcal{C}_0$.

Let A be a k-algebra. For any $m, n \in \mathbb{N}$ we denote by $\mathrm{M}_{m\times n}(A)$ the set of all $m \times n$-matrices with coefficients in A, and by $\mathrm{M}_n(A)$ the algebra of all square $n \times n$-matrices with coefficients in A.

Throughout the paper we use in principle the notation and terminology established in [10, 3, 4].

The paper is organized as follows. In Section 2 the elementary properties of the endomorphism (local) algebras of indecomposable locally finite-dimensional R-modules, in particular G-atoms, are studied. Also, properties of the Jacobson radical $\mathcal{J}_R$ (of the category $\mathrm{Mod}\, R$) related to the uniqueness of decomposition into indecomposables in $\mathrm{Mod}\, R$ are discussed. The main result of this section states that the endomorphism algebra $\mathrm{End}_R(B)$ of a G-atom B is semiprimary and its quotient division algebra has finite dimension over the basic field (see Theorem 2.9). Section 3 is devoted to the elementary proof of indecomposability of the injective $\mathrm{End}_R(B)$-module $(\mathrm{End}_R(B))^*$ (see Theorem 3.1 and Corollary 3.1). In Section 4 the category $\mathrm{MOD}(\mathrm{Mod}\, R)^{\mathrm{op}}$ is studied. Certain properties of the injective objects in $\mathrm{MOD}(\mathrm{Mod}\, R)^{\mathrm{op}}$ are discussed. In particular it is proved that the dual to the projective module $\mathrm{Hom}_R(-, B)^*$ and the tensor product functor $-\otimes_R B^*$ for any G-atom both have a simple socle (see Theorem 4.4). Section 5 contains a discussion of the various functors considered in the previous section, which are associated with a G-atom B equipped with an R-action of the stabilizer G_B and now treated as functors from $\mathrm{Mod}^G R$ to $\mathrm{MOD}(kG_B)^{\mathrm{op}}$. In particular it is proved that the kG_B-modules $\mathcal{J}_R(B, M)$ and $\mathcal{J}_R(M, B)$ are pure injective for any M in $\mathrm{Mod}^G R$ (see Theorem 5.2). Section 6 is devoted to the proofs of Theorems A and B. A corollary of Theorem B (see Theorem 6.4) is also formulated.

Some of the results of this paper with the proofs in a very brief outline were announced in [4]. They were also presented to the Cocoyoc Conference ICRA VII in Mexico, August 1994, at Paderborn University, June 1994, at Bielefeld University, July 1994, and at Toruń University in several seminar talks.

2. Some remarks on the endomorphism algebras of indecomposable locally finite-dimensional modules. In this section we study the elementary properties of the k-algebra $\mathrm{End}_R(B)$, where B is an object in $\mathrm{Ind}\, R$, in particular a G-atom. We compare $\mathrm{End}_R(B)$ with the endomorphism algebras of certain indecomposable finite-dimensional modules. We also discuss certain properties of the Jacobson radical of the category $\mathrm{Mod}\, R$, which are important for the uniqueness of decomposition into a direct sum of indecomposables in $\mathrm{Mod}\, R$ (see [10, Lemma 2.1] for an algebraically closed field case). The original proof consists only of hints. Therefore we present a full proof for an arbitrary field (see Lemmas 2.1, 2.2 and 2.4).

2.1. *Any M in* $\mathrm{Mod}\, R$ *decomposes into a direct sum of indecomposable submodules.*

Proof. Consider the class $\mathcal{D}$ of all families $(M_i)_{i\in I}$ of nonzero submodules M_i of M having the property that $M = \bigoplus_{i\in I} M_i$. The class $\mathcal{D}$ is naturally ordered by the refinement relation defined as follows: $(M_i)_{i\in I} \le (M'_{i'})_{i'\in I'}$ if and only if there exists a surjection $f = f_{I,I'} : I' \to I$ such that $M_i = \bigoplus_{i'\in f^{-1}(i)} M_{i'}$ for every $i \in I$.

Note that it is enough to show that $(\mathcal{D}, \le)$ satisfies the assumptions of the Zorn Lemma since maximal elements of $\mathcal{D}$ consist of indecomposable R-modules. Clearly $\mathcal{D}$ is nonempty since $\{M\}$ is in $\mathcal{D}$. Take any linearly ordered subset $\mathcal{D}' = \{(M_{i_{(t)}})_{i_{(t)}\in I_{(t)}}\}_{t\in T}$ of $\mathcal{D}$. Denote by I the inverse limit of the system $\{I_{(t)}, f_{I_{(t')},I_{(t)}}\}$ of sets and maps. For any $i = (i_{(t)})_{t\in T} \in I$ set $M_i = \bigcap_{t\in T} M_{i_{(t)}}$ and $I_0 = \{i \in I : M_i \ne 0\}$.

We prove that $(M_i)_{i\in I_0}$ belongs to $\mathcal{D}$. For any $x \in \mathrm{ob}\, R$ the k-vector space $M(x)$ is finite-dimensional, therefore there exists $t_x \in T$ such that in each set $I_{j_{(t_x)}} = \{i \in I_0 : i_{(t_x)} = j_{(t_x)}\}$, $j_{(t_x)} \in I_{(t_x)}$, there is at most one i with $M_i(x) \ne 0$ and then obviously $M_i(x) = M_{i_{(t_x)}}$. Consequently, $\bigoplus_{i\in I_0} M_i(x) = \bigoplus_{i_{(t_x)}\in I_{(t_x)}} M_{i_{(t_x)}} = M(x)$ for every $x \in \mathrm{ob}\, R$ and $M = \bigoplus_{i\in I_0} M_i$. Then $(M_i)_{i\in I_0}$ belongs to $\mathcal{D}$ and obviously it is an upper bound of $\mathcal{D}'$. ■

For the uniqueness of the above decomposition see Lemma 2.4.

2.2. Lemma. *Let B be an indecomposable locally finite-dimensional R-module. Then the endomorphism algebra* $\mathrm{End}_R(B)$ *is local with Jacobson radical* $J(\mathrm{End}_R(B))$ *consisting of all locally nilpotent endomorphisms* $f \in \mathrm{End}_R(B)$ (*in the sense that each $f(x)$ is a nilpotent k-linear endomorphism for $x \in \mathrm{ob}\, R$*), *and the factor k-algebra* $\mathrm{End}_R(B)/J(\mathrm{End}_R(B))$ *has a finite dimension over k.*

Proof. It is enough to show that any $f \in \mathrm{End}_R(B)$ is either invertible or locally nilpotent, since locally nilpotent endomorphisms form a two-sided ideal in $\mathrm{End}_R(B)$. By indecomposability of B for any $f \in \mathrm{End}_R(B)$ there exists an irreducible polynomial $p \in k[t]$ such that each $k[t]$-module $B(x)$, $x \in \operatorname{ob} R$, with the $k[t]$-module structure given by $f(x)$, is isomorphic to a finite-dimensional direct sum of the form $\bigoplus_{n\in\mathbb{N}}(k[t]/(p^n))^{m_{n,x}}$. Now it is clear that if $p = t$ then all $f(x)$ are nilpotent, otherwise all are invertible (also for p of degree higher than 1).

To prove the second assertion note that if a local k-algebra A admits a k-algebra homomorphism to a finite-dimensional k-algebra then the dimensional $\dim_k(A/J(A))$ is finite. Since for any $x \in \operatorname{supp} B$, $\operatorname{End}_k(B(x))$ is a finite-dimensional k-algebra and the projection map $\pi_x : \operatorname{End}_R(B) \to \operatorname{End}_k(B(x))$ is a k-algebra homomorphism, the proof is complete. ∎

Remark. (1) $J(\operatorname{End}_R(B))$ consists of all $f \in \operatorname{End}_R(B)$ such that $f(x)$ is nilpotent for at least one $x \in \operatorname{supp} B$.

(2) Let U_1 (resp. U_2) be a full subcategory of R, B_1 (resp. B_2) an object of $\operatorname{Ind} U_1$ (resp. $\operatorname{Ind} U_2$), and f_1 (resp. f_2) an endomorphism in $\operatorname{End}_{U_1}(B_1)$ (resp. $\operatorname{End}_{U_2}(B_2)$). Suppose that $B_1(x) = B_2(x)$ and $f_1(x) = f_2(x)$ for some $x \in \operatorname{supp} B_1 \cap \operatorname{supp} B_2$. Then $f_1 \in J(\operatorname{End}_{U_1}(B_1))$ if and only if $f_2 \in J(\operatorname{End}_{U_2}(B_2))$.

2.3. For simplicity we denote the Jacobson radical $\mathcal{J}_R$ of the category $\operatorname{Mod} R$ by $\mathcal{J}$ (see [16] for the precise definition). As an immediate consequence of the above lemma, for any objects B, B' in $\operatorname{Ind} R$ we obtain

$$\mathcal{J}(B,B') = \begin{cases} J(\operatorname{End}_R(B)) & \text{if } B = B' \\ \operatorname{Hom}_R(B,B) & \text{if } B \not\simeq B'. \end{cases}$$

Before studying further properties of the ideal $\mathcal{J}$ we recall some definitions.

Let M, N be R-modules. Following [5] a family $(f_i)_{i\in I} \subset \operatorname{Hom}_R(M,N)$ is said to be *summable* if for each $x \in \operatorname{ob} R$ and $m \in M(x)$, $f_i(x)(m) = 0$ for almost all $i \in I$. In this case the well defined R-homomorphism $f = \sum_{i\in I} f_i : M \to N$, given by $f(x)(m) = \sum_{i\in I} f_i(x)(m)$ for any $x \in R$, $m \in M(x)$, is called the *sum* of the family $(f_i)_{i\in I}$.

A subspace W of $\operatorname{Hom}_R(M,N)$ is said to be *summably closed* if $\sum_{i\in I} f_i \in W$ for any summable family $(f_i)_{i\in I} \subset W$.

An ideal $\mathcal{I}$ of a full subcategory $\mathcal{C}$ of $\operatorname{MOD} R$ is said to be *summably closed* if the subspace $\mathcal{I}(M,N)$ of $\operatorname{Hom}_R(M,N)$ is summably closed for any M, N in $\mathcal{C}$.

A trivial example of a summably closed ideal in the category is $\operatorname{Mod} R$ is the ideal $\operatorname{Hom}_R(-,?)$. We will show that also $\mathcal{J}$ is a summably closed ideal in $\operatorname{Mod} R$. The first step is the following.

Lemma. *Let B, B' be objects in* $\operatorname{Ind} R$. *Then the subspace $\mathcal{J}(B,B')$ of $\operatorname{Hom}_R(B,B')$ is summably closed.*

Proof. By the remarks above it is enough to show that the subspace $J(\mathrm{End}_R(B))$ of $\mathrm{End}_R(B)$ is summably closed for each B in $\mathrm{Ind}\, R$. Take any summable family $(f_i)_{i\in I} \subset J(\mathrm{End}_R(B))$. Then for any $x \in \mathrm{ob}\, R$, $(\sum_{i\in I} f_i)(x) = (\sum_{i\in I_x} f_i)(x)$, where $I_x = \{i \in I : f_i(x) \neq 0\}$ is finite. Therefore by Lemma 2.2 the endomorphism $\sum_{i\in I} f_i$ is locally nilpotent, since $\sum_{i\in I_x} f_i \in J(\mathrm{End}_R(B))$, and it belongs to $J(\mathrm{End}_R(B))$. ∎

2.4. For any algebra E we denote by $\mathrm{u}(E)$ the group of its units and by $\bar{E}$ the factor algebra E/J, where $J = J(E)$. For any $m, n \in \mathbb{N}$ and $f \in \mathrm{M}_{m\times n}(E)$ we denote by $\bar{f}$ the image of f under the canonical projection

$$\pi : \mathrm{M}_{m\times n}(E) \to \mathrm{M}_{m\times n}(\bar{E}) \simeq \mathrm{M}_{m\times n}(E)/\mathrm{M}_{m\times n}(J).$$

Let B_i, $i \in I$, be a family of pairwise nonisomorphic objects in $\mathrm{Ind}\, R$. For any $i \in I$ we set $E_i = \mathrm{End}_R(B_i)$ and $\bar{E}_i = E_i/J_i$, where $J_i = J(\mathrm{End}_R(B_i))$.

LEMMA. *Let $(m_i)_{i\in I}$ and $(n_i)_{i\in I}$ be sequences of natural numbers such that the R-modules $M = \bigoplus_{i\in I} B_i^{m_i}$ and $N = \bigoplus_{i\in I} B_i^{n_i}$ are locally finite-dimensional. Suppose we are given an R-homomorphism $f : M \to N$ with components $f_{j,i} : B_i^{m_i} \to B_j^{n_j}$, $i, j \in I$. Then f is an isomorphism if and only if $m_i = n_i$ and $\bar{f}_{i,i} \in \mathrm{M}_{m_i}(\bar{E}_i)$ (equivalently $f_{i,i} \in \mathrm{M}_{m_i}(E_i)$) is invertible for every $i \in I$.*

SUBLEMMA. *Let E be a local ring with Jacobson radical $J = J(E)$ and n be a positive integer. Then*

$$\mathrm{u}(\mathrm{M}_n(E)) + \mathrm{M}_n(J) \subset \mathrm{u}(\mathrm{M}_n(E)).$$

Hence $f \in \mathrm{u}(\mathrm{M}_n(E))$ if and only if $\bar{f} \in \mathrm{u}(\mathrm{M}_n(\bar{E}))$, and $J(\mathrm{M}_n(E)) = M_n(J)$.

Proof. It is enough to show that if $a = (a_{i,j})_{i,j\in I} \in \mathrm{M}_n(E)$ is such that $a_{i,i} \in \mathrm{u}(E)$ for every $i \in \{1, \ldots, n\}$ and $a_{i,j} \in J$ for all $i, j \in \{1, \ldots, n\}$, $i \neq j$, then $a \in \mathrm{u}(\mathrm{M}_n(E))$. (Note that $c + b = c(1 + c^{-1}b)$ for any $c \in \mathrm{u}(\mathrm{M}_n(E))$ and $b \in J(\mathrm{M}_n(E))$.) Take any matrix a as above. Applying the Gaussian-row elimination, a can be transformed to an upper triangular matrix $a' = (a'_{i,j})_{i,j\in I} \in \mathrm{M}_n(E)$ such that $a'_{i,i} \in \mathrm{u}(\mathrm{M}_n(E))$ for every $i \in \{1, \ldots, n\}$. Then a is invertible since a' is. ∎

Proof of Lemma. Assume first that $f : M \to N$ is an isomorphism. Let an R-homomorphism $g : N \to M$ with components $g_{i,j} : B_j^{n_j} \to B_i^{m_i}$, $i, j \in I$, be the inverse of f. Then for any $i \in I$, $(g_{i,j} f_{j,i})_{j\in I}$ is a summable family of R-homomorphisms and therefore we have the equality

$$\mathrm{id}_{B^{m_i}} = \sum_{j\in I} g_{i,j} f_{j,i}$$

of the (i, i)th components of the endomorphisms id_M and gf in $\mathrm{End}_R(M)$. Since $g_{i,j} f_{j,i} \in \mathrm{M}_n(J_i)$ for any $j \in I \setminus \{i\}$ ($B_j \not\simeq B_i$), each endomorphism $g_{i,i}f_{i,i} = \mathrm{id}_{B^{m_i}} - \sum_{i\neq j\in I} g_{i,j}f_{j,i}$ is invertible by the Sublemma and Lemma 2.3, for $i \in I$. Analogously one shows that $f_{i,i}g_{i,i} \in \mathrm{u}(\mathrm{M}_{n_i}(E_i))$ and therefore each $f_{i,i}$ is invertible. Consequently, the matrix $\bar{f}_{i,i} \in \mathrm{M}_{n_i\times m_i}(\bar{E}_i)$ is invertible and $m_i = n_i$ for every $i \in I$.

Suppose now that $m_i = n_i$ for every $i \in I$ (then $M = N$) and that we are given an R-endomorphism $f : M \to N$ with components $f_{j,i} : B_i^{m_i} \to B_j^{n_j}$, $i, j \in I$, such that all $\bar{f}_{i,i}$'s are invertible.

Assume first that I is finite. Then applying Gaussian elimination, first with respect to rows and then with respect to columns, and using the Sublemma we can transform the matrix $(f_{j,i})_{i,j\in I}$ to $(f'_{j,i})_{i,j\in I}$ such that $f'_{j,i} = 0$ for all $i, j \in I$, $i \neq j$, and $f'_{i,i} \in \mathrm{u}(\mathrm{M}_{m_i}(E_i))$ for every $i \in I$. The endomorphism $f' \in \mathrm{End}_R(M)$ defined by $(f'_{j,i})_{i,j\in I}$ is invertible and therefore f itself is invertible.

To prove the general case consider for any $x \in \mathrm{ob}\, R$ the endomorphism $f_x : \bigoplus_{i\in I_x} B_i^{m_i} \to \bigoplus_{i\in I_x} B_i^{m_i}$ defined by the family of R-homomorphisms $(f_{j,i} : B_i^{m_i} \to B_j^{n_j})_{i,j\in I_x}$, where $I_x = \{i \in I : B_i^{m_i}(x) \neq 0\}$. By the first part of the proof each f_x is an isomorphism since I_x is finite. Consequently, f is an isomorphism ($f(x) = f_x(x)$ for every $x \in \mathrm{ob}\, R$). ■

COROLLARY. *Let M be in* Mod R. *Then* supp M/G *is finite and $G_M \simeq G$ if and only if M is isomorphic to M_n for some sequence $n = (n_B)_{B\in\mathcal{A}_\circ} \in (\mathbb{N}^{\mathcal{A}_\circ})_0$* (see Introduction).

2.5. PROPOSITION. *The Jacobson radical $\mathcal{J}$ is a summably closed ideal in* Mod R.

By [5, Proposition 3.1] it is enough to show that for any M and N in Mod R and fixed decompositions $M = \bigoplus_{s\in S} M_s$ and $N = \bigoplus_{t\in T} N_t$ into direct sums of indecomposable submodules (they always exist by Lemma 2.1),

$$\mathcal{J}(M,N) = \prod_{s\in S}\prod_{t\in T} \mathcal{J}(M_s, N_t).$$

Take any $f \in \mathrm{Hom}_R(M,N)$ with components $f_{t,s} \in \mathcal{J}(M_s, N_t)$, $s \in S$, $t \in T$. We have to show that for any $g \in \mathrm{Hom}_R(N,M)$ the endomorphism $\mathrm{id}_M - gf$ is invertible (see [16]). Let g have components $g_{s,t} \in \mathrm{Hom}_R(N_t, M_s)$, $s \in S$, $t \in T$. Then the (s', s)th component $(gf)_{s',s} \in \mathrm{Hom}_R(M_s, M_{s'})$ of gf is the sum of the summable family $(g_{s',t}f_{t,s})_{t\in T}$ and by Lemma 2.3 it belongs to $\mathcal{J}(M_s, M_{s'})$ for all $s, s' \in S$. Now $\mathrm{id}_M - gf$ is invertible by Lemma 2.4. Consequently, $f \in \mathcal{J}(M,N)$. ■

COROLLARY. *Let M_1, M_2 and N_i, $i \in I$, be indecomposable R-modules in* Mod R. *Suppose $f : M_1 \to N$ and $g : N \to M_2$, where $N = \bigoplus_{i\in I} N_i$* (*which is not necessarily in* Mod R), *are R-homomorphisms with components $f_i \in \mathrm{Hom}_R(M_1, N_i)$ and $g_i \in \mathrm{Hom}_R(N_i, M_2)$, $i \in I$. Assume that for* *any $i \in I$ either f_i or g_i belongs to the Jacobson radical $\mathcal{J}$. Then so does the composition gf.*

Proof. Follows directly from [5, Lemma 1.1(ii)] and the above Proposition. ■

Later we will discuss the analogous question for products in some special situation (see Lemma 2.8).

2.6. The essential role in further considerations will be played by the following notion.

DEFINITION. Let B be an object in $\operatorname{Ind} R$ and U a finite nontrivial full subcategory of $\operatorname{supp} R$. A V-module $B^{(U)} = B_V^{(U)}$ in $\operatorname{Ind} V$, where V is a full subcategory of R containing U, is called a *V-approximation* of B on U provided the following two conditions are satisfied:

(1) $B_{|U} \simeq B^{(U)}{}_{|U}$,

(2) for any $f \in \operatorname{End}_R(B)$ there exists $f^{(U)} = f_V^{(U)} \in \operatorname{End}_V(B^{(U)})$ such that $f_{|U} = f^{(U)}{}_{|U}$.

The approximation $B^{(U)}$ is called *finite* if $\dim_k B^{(U)}$ is finite. If $V = R$ then the R-module $B^{(U)}$ is simply called an *approximation* of B on U.

PROPOSITION. *Let B be in* $\operatorname{Ind} R$. *Then for any finite full subcategory U of* $\operatorname{supp} R$ *there exists a finite full subcategory U' of R containing U which admits a finite U'-approximation $B_{U'}^{(U)}$ of B on U. In particular there exists a finite approximation $B^{(U)}$ of B on U.*

Proof. Take any U as above. By [10, Lemma 4.3 and Corollary 4.4] (they are also valid if k is not algebraically closed, one has only to check some details in the proof of [10, Lemma 4.4]) there exist a finite full subcategory U' of R and an indecomposable U'-module B' such that $B_{|U} = B'_{|U}$ and $B_{|U'} = B' \oplus B''$ for some B'' in $\operatorname{mod} U'$. It is clear that for any $f \in \operatorname{End}_R(B)$ the component $f' : B' \to B'$ of the U'-homomorphism $f_{|U'} : B_{|U'} \to B_{|U'}$ satisfies $f_{|U} = f'_{|U}$. The last assertion follows directly from the first by the existence of the full and faithful functor $e_\lambda^{U'} : \operatorname{mod} U' \to \operatorname{mod} R$, which is right quasi-inverse (and left adjoint) to the restriction functor $e_\bullet^{U'} : \operatorname{mod} R \to \operatorname{mod} U'$. ■

REMARK. (1) For any f and $f^{(U)}$ satisfying the condition (2) of Definition 2.6, $f \in J(\operatorname{End}_R(B))$ if and only if $f^{(U)}$ belongs to $J(\operatorname{End}_V(B^{(U)}))$ (see Remark 2.2).

(2) The mapping $f \mapsto f^{(U)}{}_{|U}$ (see Definition 2.6) defines an algebra homomorphism

$$\operatorname{End}_R(B) \to \operatorname{End}_U(B^{(U)}{}_{|U}).$$

(3) The mapping $f \mapsto f'$ (see the proof of Proposition 2.6) induces an algebra homomorphism

$$\mathrm{End}_R(B) \to \mathrm{End}_{U'}(B')/J(\mathrm{End}_{U'}(B'))$$

(B'' has no direct summand isomorphic to B'), and consequently by (1) an algebra embedding

$$\mathrm{End}_R(B)/J(\mathrm{End}_R(B)) \hookrightarrow \mathrm{End}_{U'}(B')/J(\mathrm{End}_{U'}(B')).$$

2.7. Lemma. *Let B be in* $\mathrm{Ind}\, R$ *and U be a finite nontrivial full subcategory of* $\mathrm{supp}\, B$. *Assume that for an approximation $B' = B^{(U)}$ of B on U the factor algebra* $\mathrm{End}_R(B')/J(\mathrm{End}_R(B'))$ *is isomorphic to k. Then so is* $\mathrm{End}_R(B)/J(\mathrm{End}_R(B))$.

Proof. Take any $f \in \mathrm{End}_R(B)$. By assumption there exist $f' \in \mathrm{End}_R(B')$ and $a \in k$ such that $f_{|U} = f'_{|U}$ and $f' - a\cdot \mathrm{id}_{B'} \in J(\mathrm{End}_R(B'))$.

Then by Remark 2.2, $f - a\cdot\mathrm{id}_B \in J(\mathrm{End}_R(B))$. This directly implies the required isomorphism. ∎

Corollary. *Let $\{C_n\}_{n\in\mathbb{N}}$ be an ascending sequence of finite, full, connected subcategories of R such that $R = \bigcup_{n\in\mathbb{N}} C_n$, and $\{B_n\}_{n\in\mathbb{N}}$ a fundamental sequence w.r.t. $\{C_n\}_{n\in\mathbb{N}}$ produced by an R-module B in* $\mathrm{Ind}\, R$ (see [10, Definition 4.1]). *Assume that for infinitely many $n \in \mathbb{N}$ the factor algebra* $\mathrm{End}_{C_n}(B_n)/J(\mathrm{End}_{C_n}(B_n))$ *is isomorphic to k. Then so is* $\mathrm{End}_R(B)/J(\mathrm{End}_R(B))$. *In particular this is always the case if k is algebraically closed.*

Proof. Denote by U the full subcategory formed by $\{x\}$, where x is a fixed object in $\mathrm{supp}\, B$. By [10, Lemma 4.3 and Corollary 4.4], B_n is a finite C_n-approximation of B on U for almost all $n \in \mathbb{N}$. Then by assumption there exists $n \in \mathbb{N}$ such that the finite approximation of B on U of the form $e_\lambda^{C_n}(B_n)$ (see the proof of Proposition 2.6) satisfies the assumption of Lemma 2.7. ∎

2.8. Proposition 2.6 allows us to answer partially the question mentioned at the end of 2.5.

Lemma. *Let B be in* $\mathrm{Ind}\, R$ *and $f : B \to \prod_I B$ an R-homomorphism defined by a family of endomorphisms $f_i \in J(\mathrm{End}_R(B))$, $i \in I$. Then $gf \in J(\mathrm{End}_R(B))$ for any homomorphism $g : \prod_I B \to B$.*

Proof. Assume first that B is finite-dimensional. Then the ideal $J = J(\mathrm{End}_R(B))$ is nilpotent so there exists a positive integer $m \in \mathbb{N}$ such that $J^{m-1} \neq 0$ and $J^m = 0$. The endomorphism gf is annihilated on the right by J^{m-1} since $f_i J^{m-1} = 0$ for every $i \in I$. Consequently, $gf \in J$, since otherwise gf is invertible and $gfJ^{m-1} \neq 0$.

Now we consider the general case. Fix a nontrivial finite full subcategory U of $\operatorname{supp} B$. Then there exist U', B' and B'' as in the proof of Proposition 2.6. It is easy to see that $(gf)_{|U} = (g'f')_{|U}$, where $f' : B' \to \prod_I B'$ (resp. $g' : \prod_I B' \to B'$) denotes the appropriate component of the U'-homomorphism $f_{|U'} : B_{|U'} \to \prod_I B_{|U'}$ (resp. $g_{|U'} : \prod_I B_{|U'} \to B_{|U'}$) under the standard identification

$$\prod_I B_{|U'} = \prod_I B' \oplus \prod_I B''$$

(cf. Remark 2.6(3)). Moreover, by assumption all components f'_i, $i \in I$, of f' belong to $J(\operatorname{End}_{U'}(B'))$ (see Remark 2.2) and by the first part of the proof, $g'f' \in J(\operatorname{End}_{U'}(B'))$. Now the assertion follows immediately from Remark 2.2. ∎

2.9. The following fact is useful in the proof of the main result of this section.

LEMMA. *Let B be in* $\operatorname{Ind} R$. *Assume that $\{U_i\}_{i\in I}$ is a family of full, finite subcategories of* $\operatorname{supp} B$ *such that* $\operatorname{supp} B = \bigcup_{i\in I} U_i$, *and $\{B^i\}_{i\in I}$ a family of indecomposable R-modules such that each $B^i = B^{(U_i)}$ is a finite approximation of B on U_i. If the sequence $\{\dim_k(\operatorname{End}_R(B^i))\}_{i\in I}$ is bounded, then* $\operatorname{End}_R(B)$ *is a semiprimary k-algebra.*

Proof. Let n be an upper bound of $\{\dim_k(\operatorname{End}_R(B^i))\}_{i\in I}$. We show that $J(\operatorname{End}_R(B))^n = 0$. Take any $f_1, \ldots, f_n \in J(\operatorname{End}_R(B))$ and x in $\operatorname{supp} B$. By the assumptions there exist $i \in I$ and $f_1^{(i)}, \ldots, f_n^{(i)} \in J(\operatorname{End}_R(B^i))$ such that x is in U_i and $f_l(x) = f_l^{(i)}(x)$ for every $l = 1, \ldots, n$. It is clear that $J(\operatorname{End}_R(B^i))^n = 0$ and therefore $f_n(x) \cdot \ldots \cdot f_1(x) = 0$. Consequently, $f_n \cdot \ldots \cdot f_1 = 0$. ∎

THEOREM. *Let R be a locally bounded k-category and G a group of k-linear automorphisms acting freely on* $\operatorname{ob} R$. *Then the endomorphism algebra* $\operatorname{End}_R(B)$ *of any G-atom B is a local, semiprimary k-algebra such that* $\dim_k \operatorname{End}_R(B)/J(\operatorname{End}_R(B))$ *is finite.*

Proof. For any x in $\operatorname{supp} B$ denote by U_x the full subcategory of R formed by $\{x\}$. By Proposition 2.6 there exists a finite approximation $B^x = B^{(U_x)}$ of B on U_x. Without loss of generality we can assume that ${}^gB^x \simeq B^{gx}$ for any $x \in \operatorname{supp} B$ and $g \in G_B$. Then the sequence $\{\dim_k \operatorname{End}_R(B^x)\}_{x\in\operatorname{supp} B}$ is bounded, since $\operatorname{supp} B$ is a union of a finite number of G_B-orbits in R. Now the assertion follows directly from Lemmas 2.9 and 2.2. ∎

3. On indecomposability of $(\operatorname{End}_R(B))^*$

3.1. The main aim of this section is to give an elementary short proof of the following fact.

Theorem. *Let A be a local k-algebra with $J = J(A)$. If A is semiprimary and $\dim_k A/J$ is finite then the injective right* (*resp. left*) *A-module* $({}_A A)^*$ (*resp.* $(A_A)^*$) *has a local endomorphism ring. In particular* $({}_A A)^*$ (*resp.* $(A_A)^*$) *is indecomposable and it is an injective hull of the unique* (*up to isomorphism*) *simple right* (*resp. left*) *A-module.*

By Theorem 2.9 we obtain as an immediate consequence the following.

Corollary. *Let B be a locally finite-dimensional R-module. If B is a G-atom then the endomorphism algebra of the left* (*resp. right*) $\mathrm{End}_R(B)$-*module* $(\mathrm{End}_R(B))^*$ *is local, and consequently* ${}_{\mathrm{End}_R B}(\mathrm{End}_R(B))^*$ (*resp.* $(\mathrm{End}_R(B))^*_{\mathrm{End}_R B}$) *is indecomposable.*

3.2. For the proof of the above result we study some multiplicative structure on the k-linear space A^{**}, where A is an arbitrary k-algebra. Let

$$\bullet : A^{**} \times A^{**} \to A^{**}$$

be the k-bilinear map given by

$$(\varphi \bullet \psi)(\eta) = \varphi(\psi_\eta)$$

for $\varphi, \psi \in A^{**}$ and $\eta \in A^*$, where ψ_η denotes the k-linear form $\psi(\eta \cdot -) \in A^*$.

For any vector space V we denote by $e_V : V \to V^{**}$ the canonical embedding. For any k-vector spaces V, W we have mutually inverse k-linear maps

$$\text{(a)} \qquad \mathrm{Hom}_k(V, W^*) \underset{(-)^* \circ e_V}{\overset{(-)^* \circ e_W}{\rightleftarrows}} \mathrm{Hom}_k(W, V^*),$$

which gives the selfadjointness of the contravariant functor

$$(-)^* : \mathrm{MOD}\, k \to \mathrm{MOD}\, k,$$

and if A is a k-algebra the adjointness of the pair of functors

$$\text{(b)} \qquad \mathrm{MOD}\, A \underset{(-)^*}{\overset{(-)^*}{\rightleftarrows}} \mathrm{MOD}\, A^{\mathrm{op}}.$$

Lemma. (i) *$A^{**} = (A^{**}, \bullet)$ is a k-algebra.*

(ii) *A^{**} is naturally isomorphic to* $\mathrm{End}_A(({}_A A)^*)$.

(iii) *The canonical map $e_A : A \to A^{**}$ is an embedding of k-algebras.*

Proof. Applying the A-algebra version of (a) for the A-modules A^{**} and A, and the natural isomorphism of left A-modules $\mathrm{Hom}_A({}_A A, M) \simeq {}_A M$, we obtain the composite k-linear isomorphism

$$\mathrm{End}_A(({}_A A)^*) \xrightarrow{(-)^* \circ e_A} \mathrm{Hom}_A({}_A A, ({}_A A)^{**}) \xrightarrow{\sim} A^{**},$$

which we denote by u. The map u assigns to any $s \in \mathrm{End}_A(({}_A A)^*)$ the k-linear form $(s(-))(1)$ on A^*. We show that u yields the isomorphism of

$(\mathrm{End}_A(({}_A A)^*), \circ)$ and $(A^{**}, \bullet)$ as k-vector spaces with bilinear forms. The inverse v of u is given by $(v(\varphi))(\eta) = \varphi_\eta$ for $\eta \in A^*$. Indeed,

$$\begin{aligned} u(v(\varphi) \circ v(\psi))(\eta) &= ((v(\psi) \circ v(\varphi))(\eta))(1) = (v(\psi)(\varphi_\eta))(1) \\ &= \psi(\varphi_\eta \cdot 1) = (\psi \circ \varphi)(\eta) \end{aligned}$$

for any $\varphi, \psi \in A^{**}$ and $\eta \in A^*$. Now (i) and (ii) follow easily. The proof of (iii) is an easy check on the definitions. ■

REMARK. (i) The endomorphism algebra $\mathrm{End}_A((A_A)^*)$ of the left A-module $(A_A)^*$ is isomorphic to the k-algebra $(A^{\mathrm{op}})^{**}$.

(ii) The identity map yields an isomorphism of the A-A-bimodules A^{**} and $(A^{\mathrm{op}})^{**}$, where the bimodule structure is given by Lemma 3.2(iii).

3.3. For any subspace V of a k-vector space U we identify the double dual space V^{**} with its image $i^{**}(V^{**}) = \{\varphi \in U^{**} : \varphi(V^\perp) = 0\}$ in U^{**} via the map $i^{**} : V^{**} \to V^{**}$, where $i : V \to U$ is the canonical embedding and $V^\perp = \{\eta \in U^* : \eta(V) = 0\}$.

For any two subspaces V and W of a k-algebra A we denote by $V \cdot W$ the vector k-subspace of A spanned by all products $v \cdot w$, where $v \in V$ and $w \in W$.

LEMMA. *$V^{**} \bullet W^{**} \subset (V \cdot W)^{**}$ for any k-subspaces V and W of a k-algebra A.*

Proof. Take any $\varphi \in V^{**}$ and $\psi \in W^{**}$. To show that $\varphi \bullet \psi \in (V \cdot W)^{**}$, equivalently that $\varphi(\psi_\eta) = 0$ for all $\eta \in (V \cdot W)^\perp$, it is enough to check that $\psi_\eta = \psi(\eta \cdot -) \in A^*$ vanishes on V for every $\eta \in (V \cdot W)^\perp$. Indeed, $\psi(\eta \cdot v) = 0$ for all $v \in V$ and $\eta \in (V \cdot W)^\perp$, since $\eta \cdot v = \eta(v \cdot -)$ vanishes on W. ■

COROLLARY. *For any two-sided ideal I of A the k-subspace $I^{**} \subset A^{**}$ is a two-sided ideal of the k-algebra A^{**}, and I^{**} is nilpotent if I is. Moreover, if $\dim_k A/I$ is finite then the canonical embedding $e_A : A \to A^{**}$ induces an isomorphism of k-algebras $A/I \simeq A^{**}/I^{**}$.*

Proof. The first statement is clear by Lemma 3.3. For the second assertion observe first that $(-)^{**}$ is an exact functor and therefore we have the natural k-algebra isomorphism $(A/I)^{**} \simeq A^{**}/I^{**}$. Since $\dim_k A/I$ is finite the canonical embedding $e_{A/I} : A/I \to (A/I)^{**}$ is an isomorphism of k-algebras and consequently e_A induces the required isomorphism. ■

3.4. PROPOSITION. *Let A be a local k-algebra. If A is semiprimary and $\dim_k A/J(A)$ is finite then A^{**} is a local, semiprimary k-algebra (with $J(A^{**}) = J(A)^{**}$), and $A^{**}/J(A^{**})$ is a finite-dimensional division k-algebra isomorphic to $A/J(A)$.*

Proof. By Corollary 3.3 the k-subspace $J(A)^{**}$ is a two-sided, nilpotent ideal of the k-algebra A^{**}, ($J(A)$ is nilpotent) and $A^{**}/J(A)^{**}$ is a finite-dimensional division k-algebra isomorphic to $A/J(A)$. The nilpotency of $J(A)^{**}$ implies $J(A)^{**} \subset J(A^{**})$, since $J(A^{**})$ is the intersection of all maximal (left) ideals of A^{**}. On the other hand the ideal $J(A)^{**}$ is maximal and therefore $J(A)^{**} = J(A^{**})$. ∎

3.5. *Proof of Theorem 3.1.* By Lemma 3.2(ii) and Proposition 3.4 the k-algebra $\mathrm{End}_A(({}_AA)^*)$ is local and therefore the right injective module $({}_AA)^*$ is indecomposable. Denote by $\pi : {}_AA \to {}_AS$ the canonical projective cover of the simple left A-module ${}_AS = {}_AA/{}_AJ(A)$. Since $\dim_k A/J(A)$ is finite, the right A-module $({}_AS)^*$ is also simple. By indecomposability of $({}_AA)^*$ the morphism $\pi^* : ({}_AS)^* \to ({}_AA)^*$ yields an injective hull of $({}_AS)^*$ and the proof is finished. ∎

4. Socle of the functors $\mathcal{I}_B$ and $\mathcal{T}_{B^*}$.

We briefly discuss the functorial analog of the situation studied in the previous section.

4.1. Let $\mathcal{C}$ be an additive k-category. We introduce a k-category structure $\mathcal{C}^{**}$ defined as follows. The class of objects $\mathrm{ob}\,\mathcal{C}^{**}$ is by definition $\mathrm{ob}\,\mathcal{C}$. For any two $c_1, c_2 \in \mathrm{ob}\,\mathcal{C}^{**}$ we set $\mathcal{C}^{**}(c_1, c_2) = \mathcal{C}(c_1, c_2)^{**}$. Moreover, for an object c of $\mathcal{C}$ we distinguish the element $e_{\mathcal{C}(c,c)}(\mathrm{id}_c) \in \mathcal{C}^{**}(c, c)$. For any c_1, c_2, c_3 in $\mathrm{ob}\,\mathcal{C}^{**}$ the composition

$$\circ : \mathcal{C}^{**}(c_2, c_3) \times \mathcal{C}^{**}(c_1, c_2) \to \mathcal{C}^{**}(c_1, c_3)$$

in $\mathcal{C}^{**}$ is given by the formula

$$(\varphi \circ \psi)(\eta) = \varphi(\psi_\eta)$$

where $\varphi \in \mathcal{C}^{**}(c_2, c_3)$, $\psi \in \mathcal{C}^{**}(c_1, c_2)$, $\eta \in \mathcal{C}(c_1, c_3)^*$ and ψ_η is the k-linear form in $\mathcal{C}(c_1, c_3)^{**}$ with $\psi_\eta(f) = \psi(\eta(f \cdot -))$ for $f \in \mathcal{C}(c_2, c_3)$.

We are going to use $\mathcal{C}^{**}$ to describe some injective $\mathcal{C}$-modules. Recall that as in the case of modules over an algebra we have at our disposal the pair of contravariant functors

$$\mathrm{MOD}\,\mathcal{C} \underset{(-)^*}{\overset{(-)^*}{\rightleftarrows}} \mathrm{MOD}\,\mathcal{C}^{\mathrm{op}},$$

which are adjoint to each other. The natural isomorphism

$$\text{(a)} \qquad \mathrm{Hom}_{\mathcal{C}}(\mathcal{M}, \mathcal{N}^*) \simeq \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(\mathcal{N}, \mathcal{M}^*)$$

establishing the adjointness is induced, for given $\mathcal{M}$ in $\mathrm{MOD}\,\mathcal{C}$ and $\mathcal{N}$ in $\mathrm{MOD}\,\mathcal{C}^{\mathrm{op}}$, by the pair of k-linear isomorphisms described by 3.2(a).

For an arbitrary object c in $\mathcal{C}$ we denote by $\mathcal{I}^c$ the $\mathcal{C}$-module $\mathcal{C}(c, -)^*$ and by $\mathcal{I}_c$ the $\mathcal{C}^{\mathrm{op}}$-module $\mathcal{C}(-, c)^*$. By the Yoneda Lemma and the isomorphism (a) both modules $\mathcal{I}^c$ and $\mathcal{I}_c$ are injective.

LEMMA. (i) *$\mathcal{C}^{**}$ is an additive k-category.*

(ii) *$\mathcal{C}^{**}$ is canonically isomorphic to the full subcategory of* MOD $\mathcal{C}$ *formed by all injective modules $\mathcal{I}^c$.*

(iii) *The canonical embeddings $e_{\mathcal{C}(c_1,c_2)} : \mathcal{C}(c_1, c_2) \to \mathcal{C}^{**}(c_1, c_2)$ induce a faithful embedding functor $\mathbf{e} : \mathcal{C} \to \mathcal{C}^{**}$ of k-categories.*

Proof. For any c_1, c_2 in ob $\mathcal{C}$, by the Yoneda Lemma and (a) we obtain the composite isomorphism

$$\operatorname{Hom}_{\mathcal{C}}(\mathcal{I}^{c_1}, \mathcal{I}^{c_2}) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{C}^{\mathrm{op}}}(\mathcal{C}(c_2, -), \mathcal{C}(c_1, -)^{**}) \xrightarrow{\sim} \mathcal{C}(c_1, c_2)^{**},$$

which we denote by u_{c_1,c_2}. As in the proof of Lemma 3.2, one shows that $u_{c_1,c_3}(\Psi \circ \Phi) = u_{c_2,c_3}(\Psi) \circ u_{c_1,c_2}(\Phi)$ for any $\mathcal{C}$-homomorphisms $\Phi : \mathcal{I}^{c_1} \to \mathcal{I}^{c_2}$ and $\Psi : \mathcal{I}^{c_2} \to \mathcal{I}^{c_3}$ in MOD $\mathcal{C}$. Now the assertion follows easily. ∎

REMARK. (i) For any object c in $\mathcal{C}$ the algebra $\mathcal{C}^{**}(c, c)$ and the algebra $(\mathcal{C}(c, c))^{**}$ defined in 3.2 coincide.

(ii) The full subcategory of MOD $\mathcal{C}^{\mathrm{op}}$ formed by all injective modules of the form $\mathcal{I}_c$ is canonically isomorphic to $(\mathcal{C}^{\mathrm{op}})^{**}$.

4.2. The following is an analog of Theorem 3.1.

THEOREM. *Let $\mathcal{C}$ be an additive k-category and c be an object of $\mathcal{C}$. If $A = \mathcal{C}(c, c)$ is a semiprimary, local k-algebra such that $\dim_k A/J(A)$ is finite then $\operatorname{End}_{\mathcal{C}}(\mathcal{I}^c)$ and $\operatorname{End}_{\mathcal{C}^{\mathrm{op}}}(\mathcal{I}_c)$ have the same properties. In particular the injective modules $\mathcal{I}^c$ and $\mathcal{I}_c$ are indecomposable.*

Proof. Follows directly from Lemma 4.1, Remark 4.1 and Proposition 3.4. ∎

4.3. From now on we assume that $\mathcal{C} = \operatorname{Mod} R$. We study the properties of the injective modules $\mathcal{I}^M$ and $\mathcal{I}_M$ for an indecomposable locally finite-dimensional R-module M. For any M in Mod R we denote by $\overline{\mathcal{H}}^M$ (resp. $\overline{\mathcal{H}}_M$) the (Mod R)-module $\mathcal{H}^M/\mathcal{J}^M$ (resp. $(\operatorname{Mod} R)^{\mathrm{op}}$-module $\mathcal{H}_M/\mathcal{J}_M$), where $\mathcal{H}^M = \operatorname{Hom}_R(-, M)$ and $\mathcal{J}^M = \mathcal{J}(-, M)$ (resp. $\mathcal{H}_M = \operatorname{Hom}_R(M, -)$ and $\mathcal{J}_M = \mathcal{J}(M, -)$).

LEMMA. *Let B be in* Ind R. *Then*

(i) *both modules $\overline{\mathcal{H}}_B$ and $\overline{\mathcal{H}}^B$ are simple,*

(ii) *we have the isomorphisms $\overline{\mathcal{H}}_B^* \simeq \overline{\mathcal{H}}^B$ and $(\overline{\mathcal{H}}^B)^* \simeq \overline{\mathcal{H}}_B$.*

Proof. (i) Let $\operatorname{Mod}_{(B)} R$ denote the full subcategory of Mod R formed by all R-modules M' which have no direct summand isomorphic to B. The definition is correct by the uniqueness of decomposition into indecomposables (see Lemma 2.4). Consider the functor MOD(Mod R) $\to$ MOD $\operatorname{End}_R(B)$ which assigns to each $\mathcal{M}$ in MOD(Mod R) the right $\operatorname{End}_R(B)$-module $\mathcal{M}(B)$. The restriction of this functor to the full subcategory of MOD(Mod R)

formed by all $\mathcal{M}$ vanishing on $\mathrm{Mod}_{(B)}R$ is full and faithful. We show first that $\overline{\mathcal{H}}^B$ is zero on $\mathrm{Mod}_{(B)}R$. Note the standard formula

$$\overline{\mathcal{H}}_B(B') \simeq \begin{cases} \mathrm{End}_R(B)/J(\mathrm{End}_R(B)) & \text{if } B' \simeq B, \\ 0 & \text{if } B' \not\simeq B, \end{cases}$$

for any B' in $\mathrm{Ind}\,R$ (see 2.3). Moreover, since both Hom_R and $\mathcal{J}$ are summably closed ideals we have the formula

$$\overline{\mathcal{H}}^B(M) \simeq \prod_{i\in I} \overline{\mathcal{H}}^B(M_i)$$

for any decomposition $M = \bigoplus_{i\in I} M_i$ of a locally finite-dimensional R-module M (see [5, 1.2]). Now it is clear that $\overline{\mathcal{H}}^B(M') = 0$ for any M' in $\mathrm{Mod}_{(B)}R$. Consequently, by the above mentioned equivalence $\overline{\mathcal{H}}^B$ is a simple $(\mathrm{Mod}\,R)$-module since $\mathrm{End}_R(B)$ is a local k-algebra (see Lemma 2.2). Using analogous arguments one proves that $\overline{\mathcal{H}}_B$ is a simple $(\mathrm{Mod}\,R)^{\mathrm{op}}$-module.

(ii) Note that $\overline{\mathcal{H}}_B^*$ vanishes on $\mathrm{Mod}_{(B)}R$. The $\mathrm{End}_R(B)$-module $\overline{\mathcal{H}}_B^*(B)$ is simple since $\dim_k \mathrm{End}_R(B)/J(\mathrm{End}_R(B))$ is finite by Lemma 2.2. Therefore it is isomorphic to the $\mathrm{End}_R(B)$-module $\overline{\mathcal{H}}^B(B)$ and by the previous remark we have the isomorphism $\overline{\mathcal{H}}^B \simeq \overline{\mathcal{H}}_B^*$. ∎

4.4. For any N in $\mathrm{Mod}\,R^{\mathrm{op}}$ we denote by $\mathcal{T}_N$ the $(\mathrm{Mod}\,R)^{\mathrm{op}}$-module $- \otimes_R N$, where $\otimes_R$ is the tensor product for R-modules (see [16, 3]).

THEOREM. *Let R be a locally bounded k-category and G a group of k-linear automorphisms acting freely on* $\mathrm{ob}\,R$. *Then for any G-atom B the following hold true*:

(i) *The injective* $(\mathrm{Mod}\,R)^{\mathrm{op}}$-*module* $\mathcal{I}_B = \mathrm{Hom}_R(-, B)^*$ *is indecomposable* (*with a local endomorphism ring*).

(ii) $\mathcal{I}_B$ *is an injective hull of the simple module* $\overline{\mathcal{H}}_B$ $(\simeq (\overline{\mathcal{H}}^B)^*)$.

(iii) *The socle of* $\mathcal{I}_B$ *is simple and isomorphic to* $\overline{\mathcal{H}}_B$.

(iv) *The socle of* $\mathcal{T}_{B^*} = - \otimes_R B^*$ *is simple and isomorphic to* $\overline{\mathcal{H}}_B$.

Proof. (i) follows from Theorems 4.2 and 2.9. Denote by $\pi^B : \mathcal{H}^B \to \overline{\mathcal{H}}^B = \mathcal{H}^B/\mathcal{J}^B$ the canonical projection of $(\mathrm{Mod}\,R)$-modules. Then the dual morphism $(\pi^B)^* : (\overline{\mathcal{H}}^B)^* \to \mathcal{I}_B$ is an embedding and the assertions (ii) and (iii) follow from Lemma 4.3 and (i) by general properties of injective objects in a module category (see [1]).

To show (iv), consider the canonical embedding $e : \mathcal{T}_{B^*} \hookrightarrow \mathcal{I}_B$ of $(\mathrm{Mod}\,R)^{\mathrm{op}}$-modules defined for any M in $\mathrm{Mod}\,R$ by the compositions

$$M \otimes_R B^* \hookrightarrow (M \otimes_R B^*)^{**} \simeq \mathrm{Hom}_R(M, B^{**})^* \simeq \mathrm{Hom}_R(M, B)^*$$

(see [3, Corollary 2.4]). Clearly $\mathrm{Soc}\,\mathcal{T}_{B^*}$ embeds via e into $\mathrm{Soc}\,\mathcal{I}_B$. Since $\mathcal{I}^B$ is an injective hull of the simple module $(\overline{\mathcal{H}}^B)^*$ $(\simeq \overline{\mathcal{H}}_B)$ it follows that $e(\mathcal{T}_{B^*})$ contains $\pi^*((\overline{\mathcal{H}}^B)^*)$, and now (iv) follows from (iii). ∎

REMARK. Analogous results hold true for $(\operatorname{Mod} R)^{\mathrm{op}}$-modules.

For any G-atom B we set $\mathcal{C}_{B^*} = \mathcal{T}_{B^*}/\mathrm{Soc}\mathcal{T}_{B^*}$ $(\simeq e(\mathcal{T}_{B^*})/(\pi^B)^*((\overline{\mathcal{H}}^B)^*)$; see proof of Theorem 4.4).

5. Pure-injectivity of $\mathcal{J}(M,B)$ and $\mathcal{J}(B,M)$

5.1. From now on we assume that G is a group of k-linear automorphisms acting freely on $\operatorname{ob} R$.

Let M and N be R-modules and H be a subgroup of $G_M \cap G_N$. Recall that if μ is an R-action of H on M and ν is an R-action of H on N, then we can define the induced group action

(a) $$\operatorname{Hom}_R(\mu,\nu) : H \times \operatorname{Hom}_R(M,N) \to \operatorname{Hom}_R(M,N)$$

by $(h,f) \mapsto {}^h\nu_h \cdot {}^hf \cdot \mu_{h^{-1}}$ (see also [3, 2.4]). This defines a left kH-module structure on $\operatorname{Hom}_R(M,N)$, where kH is the group algebra of H over k. Observe that the subspace $\mathcal{J}(M,N)$ of $\operatorname{Hom}_R(M,N)$ is a kH-submodule.

Let M be an R-module, N be an R^{op}-module and H be a subgroup of $G_M \cap G_N$. If μ is an R-action of H on M and ν is an R^{op}-action of H on N, then we can define the induced group action

(b) $$\mu \otimes_R \nu : H \times M \otimes_R N \to M \otimes_R N$$

by $(h, m_x \otimes n_x) \mapsto \mu_h(m_x) \otimes \nu_h(n_x)$, where $h \in H$, $x \in \operatorname{ob} R$, $m_x \in M(x)$ and $n_x \in N(x)$ (see also [3, 2.4]). This defines a left kH-module structure on $M \otimes_R N$.

Let M be an R-module (resp. R^{op}-module) and H be a subgroup of G_M. If μ is an R-action (resp. R^{op}-action) of H on M then the family of isomorphisms

$$\{M^* = {}^{h^{-1}}({}^hM)^*f \xrightarrow{{}^{h^{-1}}(\mu_{h^{-1}})^*} {}^{h^{-1}}M^*\}_{h\in H}$$

defines an R^{op}-action (resp. R-action) of H on the R^{op}-module (resp. R-module) M^*. In the sequel the module M^* equipped with this action will be denoted by M^* for simplicity (in contrast to the notation $M^{\circledast}$ from [3, 2.1]).

We will use the same simplification for the standard left kH-module structure on the dual space of a left kH-module V, which will be simply denoted by V^*.

Let B be a periodic G-atom (i.e. admitting an R-action of G_B). If we fix an R-action ν_B on B then $\mathcal{H}_B$, $\mathcal{J}_B$, $\overline{\mathcal{H}}_B$, $\mathcal{T}_{B^*}$ and $\mathcal{C}_{B^*}$ (resp. $\mathcal{H}^B$, $\mathcal{J}^B$, $\overline{\mathcal{H}}^B$) can be regarded as functors from $\operatorname{Mod}^G R$ to $\operatorname{MOD}(kG_B)^{\mathrm{op}}$ (resp. from $(\operatorname{Mod}^G R)^{\mathrm{op}}$ to $\operatorname{MOD}(kG_B)^{\mathrm{op}}$). We keep for them the same notation with the understanding that now B is not just a single module but a pair (B, ν_B). The analogous convention will be applied for their duals. Now the canonical exact sequences

$$0 \to \mathcal{J}_B \to \mathcal{H}_B \to \overline{\mathcal{H}}_B \to 0, \quad 0 \to \mathcal{J}^B \to \mathcal{H}^B \to \overline{\mathcal{H}}^B \to 0$$

and

$$0 \to (\overline{\mathcal{H}}_B)^* \to (\mathcal{H}_B)^* \to (\mathcal{J}_B)^* \to 0, \quad 0 \to (\overline{\mathcal{H}}^B)^* \to (\mathcal{H}^B)^* \to (\mathcal{J}^B)^* \to 0$$

in $\mathrm{MOD}((\mathrm{Mod}\, R)^{\mathrm{op}})$ (resp. in $\mathrm{MOD}(\mathrm{Mod}\, R)$) become exact sequences of k-linear functors from $\mathrm{Mod}^G R$ to $\mathrm{MOD}(kG_B)^{\mathrm{op}}$ (resp. from $(\mathrm{Mod}^G R)^{\mathrm{op}}$ to $\mathrm{MOD}(kG_B)^{\mathrm{op}}$). We also have at our disposal an exact sequence

$$0 \to (\overline{\mathcal{H}}^B)^* \to \mathcal{T}_{B^*} \to \mathcal{C}_{B^*} \to 0$$

of functors from $\mathrm{Mod}^G R$ to $\mathrm{MOD}(kG_B)^{\mathrm{op}}$ (see definition of $\mathcal{C}_{B^*}$ and for more details proof of Theorem 5.2).

LEMMA. *Let $B = (B, \nu_B)$ be a periodic G-atom together with a fixed R-action of G_B on B. Suppose that* $\mathrm{End}_R(B)/J(\mathrm{End}_R(B)) \simeq k$. *Then the functors*

$$(\overline{\mathcal{H}}^B)^*, \overline{\mathcal{H}}_B : \mathrm{Mod}^G R \to \mathrm{MOD}(kG_B)^{\mathrm{op}}$$

and

$$(\overline{\mathcal{H}}_B)^*, \overline{\mathcal{H}}^B : (\mathrm{Mod}^G R)^{\mathrm{op}} \to \mathrm{MOD}(kG_B)^{\mathrm{op}}$$

are isomorphic (cf. Lemma 4.3).

Proof. For simplicity denote the stabilizer G_B by H. Take any $M = (M, \mu)$ in $\mathrm{Mod}^G R$ and consider the bilinear composition map

$$\circ : \mathrm{Hom}_R(M, B) \times \mathrm{Hom}_R(B, M) \to \mathrm{End}_R(B).$$

It is not hard to check that $\circ$ is H-equivariant (with respect to the H-module structures defined in (a)) in the sense that $h\psi \circ h\phi = h(\psi\phi)$ for all $h \in H$, $\phi \in \mathrm{Hom}_R(B, M)$ and $\psi \in \mathrm{Hom}_R(M, B)$. The map $\circ$ induces a k-bilinear form

$$\bar{\circ} : \overline{\mathcal{H}}^B(M) \times \overline{\mathcal{H}}_B(M) \to \mathrm{End}_R(B)/J(\mathrm{End}_R(B)) \simeq k.$$

It is easy to show that the form $\bar{\circ}$ is nondegenerate. Moreover, since $\mathrm{End}_R(B) = k \cdot \mathrm{id}_B \oplus J(\mathrm{End}_R(B))$ and $J(\mathrm{End}_R(B))$ is an H-submodule of $\mathrm{End}_R(B)$, the H-module $\mathrm{End}_R(B)/J(\mathrm{End}_R(B))$ is canonically isomorphic to the trivial character. Consequently, the form $\bar{\circ}$ is H-invariant in the sense that $h\bar{\psi}\,\bar{\circ}\, h\bar{\phi} = \bar{\psi}\,\bar{\circ}\,\bar{\phi}$ for all $h \in H$, $\bar{\phi} \in \overline{\mathcal{H}}_B(M)$ and $\bar{\psi} \in \overline{\mathcal{H}}^B(M)$. Therefore the associated linear isomorphisms $\overline{\mathcal{H}}^B(M) \to \overline{\mathcal{H}}_B(M)^*$ (resp. $\overline{\mathcal{H}}_B(M) \to \overline{\mathcal{H}}^B(M)^*$) given by $\bar{\psi} \mapsto (\bar{\psi}\,\bar{\circ}\, -)$ (resp. $\bar{\phi} \mapsto (-\,\bar{\circ}\,\bar{\phi})$) are H-equivariant and natural with respect to M, and so they yield the required isomorphism of functors. ∎

5.2. The main result of this section is the following.

THEOREM. *Let R be a locally bounded k-category and G a group of k-linear automorphisms acting freely on* $\mathrm{ob}\, R$. *Suppose that* $B = (B, \nu_B)$

is a periodic G-atom together with a fixed R-action of G_B on B. Then for any $M = (M, \mu)$ in $\mathrm{Mod}^G R$ *the kG_B-submodules $\mathcal{J}^B(M)$ of $\mathcal{H}^B(M)$ and $\mathcal{J}_B(M)$ of $\mathcal{H}_B(M)$ are pure-injective* (= *algebraically compact*).

Before the proof we need some preparation.

5.3. Let V be a k-linear vector space and V^* its dual. The bilinear form $V^* \times V \to k$ given by $(\eta, v) \mapsto \eta(v)$ for $v \in V$ and $\eta \in V^*$ induces two operations

$$\{\text{subspaces } W \text{ of } V\} \underset{(-)_\perp}{\overset{(-)^\perp}{\rightleftarrows}} \{\text{subspaces } F \text{ of } V^*\}$$

defined as follows:

$$W^\perp = \{\eta \in V^* : \eta(W) = 0\}, \qquad F_\perp = \bigcap_{\eta \in F} \operatorname{Ker} \eta.$$

Suppose that A is a k-algebra and V is a left A-module. Then for any A-submodule W of V the subspace $W^\perp$ is an A-submodule of the right A-module V^*_A, and for any A-submodule F of V^*_A the subspace $F_\perp$ is an A-submodule of ${}_AV$.

It is well known that for any subspaces W of V and F of V^* we have

(a) $$(W^\perp)_\perp = W, \qquad (F_\perp)^\perp \subset F;$$

moreover, if $\dim_k V$ is finite then

(b) $$(F_\perp)^\perp = F.$$

We will formulate another condition on F implying (b), that is, implying that $F = W^\perp$ ($\simeq (V/W)^*$) for some subspace W of V.

We start by observing that for the canonical embedding $e_V : V \to V^{**}$ and for any subspaces W of V and F of V^* we have the equalities

(c) $$e_V(W)_\perp = W^\perp$$

(of subspaces of V^*) and

(d) $$e_V(F_\perp) = e_V(V) \cap F^\perp$$

(of subspaces of V^{**}). Note that in the definition of $F^\perp$ one should refer to the operation $(-)^\perp$ induced by the bilinear form $V^{**} \times V^* \to k$.

Lemma. *Let F be a subspace of V^*. If $F^\perp$ is contained in $e_V(V)$ then $F = (F_\perp)^\perp$ $(= (e_V^{-1}(F^\perp))^\perp)$, and consequently $F \simeq (V/F_\perp)^*$.*

Proof. Note that by (c) the inclusion $F^\perp \subset e_V(V)$ is equivalent to the equality $F^\perp = e_V(F_\perp)$. Therefore the assumption together with (a) and (c) yields

$$F = (F^\perp)_\perp = (e_V(F_\perp))_\perp = (F_\perp)^\perp = (e_V^{-1}(F^\perp))^\perp. \ \blacksquare$$

REMARK. The inclusion $F^\perp \subset e(V)$ is not always satisfied even for subspaces of the form $F = W^\perp$, where W is a subspace of V. It is not hard to see that in this situation $F^\perp$ is contained in $e_V(V)$ if and only if $\dim_k V$ is finite.

5.4. *Proof of Theorem 5.2.* We only prove that $\mathcal{J}^B(M)$ is pure-injective, the proof for $\mathcal{J}_B(M)$ is analogous. It is enough to show that the left kG_B-module $\mathcal{J}^B(M)$ is of the form X^* for some right kG_B-module X (see [15]). Using the canonical identification

$$(\mathcal{T}_{B^*})^* \simeq \mathcal{H}^B$$

of k-linear functors from $(\mathrm{Mod}^G R)^{\mathrm{op}}$ to $\mathrm{MOD}(kG_B)^{\mathrm{op}}$ (see [3, Lemma 2.4]) we can interpret $\mathcal{J}^B(M)$ as a kG_B-submodule of $(\mathcal{T}_{B^*})^*(M) = (M \otimes_R B^*)^*$. We know from the proof of Theorem 4.4(iv) that $e(\mathcal{T}_{B^*})$ contains $\pi^*((\overline{\mathcal{H}}^B)^*)$, where e is the embedding given by the composition

$$\mathcal{T}_{B^*} \hookrightarrow (\mathcal{T}_{B^*})^{**} \simeq (\mathcal{H}^B)^* = \mathcal{I}_B$$

and $(\pi^B)^* : (\mathcal{H}^B/\mathcal{J}^B)^* \to (\mathcal{H}^B)^*$ is the dual of the canonical projection $\pi^B : \mathcal{H}^B \to \overline{\mathcal{H}}^B = \mathcal{H}^B/\mathcal{J}^B$ (both e and $(\pi^B)^*$ are morphisms of k-linear functors from $\mathrm{Mod}^G R$ to $\mathrm{MOD}(kG_B)^{\mathrm{op}}$). Evaluating this inclusion at M we find that $\mathcal{J}(M,B)^\perp$ is contained (via the identification $(\mathcal{H}^B)^* \simeq (\mathcal{T}_{B^*})^{**}$) in $e_{M\otimes_R B^*}(M \otimes_R B^*)$. Now the existence of the required kG_B-module X $(= \mathcal{C}_{B^*}(M))$ follows immediately from Lemma 5.3. ∎

REMARK. (1) $\mathcal{J}(M,B)$ and $\mathcal{J}(B,M)$ are also pure-injective as $\mathrm{End}_R(B)$-modules.

(2) $\mathcal{H}^B(M)$ and $\mathcal{H}_B(M)$ are obviously pure-injective modules over kG_B and $\mathrm{End}_R(B)$.

(3) It is not clear when the embeddings of kG_B-modules $\mathcal{J}^B(M) \hookrightarrow \mathcal{H}^B(M)$ (resp. $\mathcal{J}_B(M) \hookrightarrow \mathcal{H}_B(M)$) are pure.

COROLLARY. *The functors*

$$\mathcal{J}_B, (\mathcal{C}_{B^*})^* : \mathrm{Mod}^G R \to \mathrm{MOD}(kG_B)^{\mathrm{op}}$$

are isomorphic. In particular, if G_B is an infinite cyclic group, then the kG_B-module $\mathcal{J}_B(M)$ is injective if and only if the finitely generated kG_B-module $\mathcal{C}_{B^*}(M)$ is free.

Proof. The first assertion follows immediately from the proof of Theorem 5.2. To prove the second one, fix M in $\mathrm{Mod}^G R$. Clearly if the kG_B-module $\mathcal{C}_{B^*}(M)$ is projective then the kG_B-module $\mathcal{J}_B(M)$ is injective $(\mathcal{J}_B(M) \simeq \mathcal{C}_{B^*}(M)^*)$. Note also that the kG_B-module $\mathcal{C}_{B^*}(M)$ is always finitely generated since so is $\bigoplus_{x\in R} M(x) \otimes B^*(x)$ ($\mathrm{supp}\, B/G_B$ is finite).

Suppose now that G_B is an infinite cyclic group and the kG_B-module $\mathcal{J}_B(M)$ is injective. Then the group algebra kG_B is a principal ideal domain

($\simeq K[T,T^{-1}]$). Hence there exist finitely generated kG_B-modules F and T such that F is free, $\dim_k T$ is finite and $\mathcal{C}_{B^*}(M) = T \oplus F$. Thus we obtain isomorphisms $\mathcal{J}_B(M) \simeq \mathcal{C}_{B^*}(M)^* \simeq F^* \oplus T^*$, and so the finite-dimensional kG_B-module T^* is injective. Consequently, $T = 0$ and $\mathcal{C}_{B^*}(M) = F$. ■

6. Proofs of the main results

6.1. Let $B=(B,\nu_B)$ be a periodic G-atom together with a fixed R-action of G_B on B. We denote by $\widetilde{\Phi}^B$ the functor

$$F_\bullet \circ ((\ )^{-1} \otimes_{kG_B} F_\lambda B) : \mathrm{mod}(kG_B)^{\mathrm{op}} \to \mathrm{Mod}_{\mathrm{f}}^G R$$

and by $\widetilde{\Psi}^B$ the functor

$$\overline{\mathcal{H}}_B : \mathrm{Mod}_{\mathrm{f}}^G R \to \mathrm{mod}(kG_B)^{\mathrm{op}}.$$

It is clear that $(\ )^{-1} \circ \widetilde{\Psi}^B \circ F_\bullet = \Psi^B$ and $F_\bullet \circ \Phi^B \circ (\ )^{-1} = \widetilde{\Phi}^B$, where $(\ )^{-1} : \mathrm{MOD}(kG)^{\mathrm{op}} \to \mathrm{MOD}\, kG$ is the standard equivalence.

In the proofs we also refer to the alternative description of $\widetilde{\Phi}^B$ as the composition of the tensor product functor

$$- \otimes_k B : \mathrm{mod}(kG_B)^{\mathrm{op}} \to \mathrm{Mod}_{\mathrm{f}}^{G_B} R$$

and the induction functor

$$\theta = \theta_{G_B} : \mathrm{Mod}_{\mathrm{f}}^{G_B} R \to \mathrm{Mod}_{\mathrm{f}}^G R$$

(see [3, Proposition 2.3(i)]).

Recall that for any subgroup H of G and every object (B,ν_B) in $\mathrm{Mod}_{\mathrm{f}}^H R$ the functor $-\otimes_k B$ assigns to any G-representation V in $\mathrm{mod}(kH)^{\mathrm{op}}$ the R-module $V \otimes_k B$ equipped with the "twisted" R-action of H. The induction functor θ_H assigns to an object (N,ν) in $\mathrm{Mod}_{\mathrm{f}}^H R$ the induced structure $\theta_H(N)$: the R-module $\bigoplus_{g\in S_H} {}^g N$, where S_H is a fixed set of representatives of the set of left cosets G/H, equipped with the standard R-action of G induced by ν (for details see [10, 3]).

Lemma. *Let $B = (B,\nu_B)$ be a G-atom from $\overline{\mathcal{A}}_{\mathrm{o}}$ equipped with an R-action of G_B, and X, Y, Z be objects in $\mathrm{mod}\,\overline{R}$. Then the kernel of the functor Ψ^B has the following properties*:

(i) $\Psi^B(Z) = 0$ *for any Z in* $\mathrm{mod}_{(\mathcal{A}_{\mathrm{o}}\setminus\{B\})}\overline{R}$, *and consequently* $\mathrm{Ker}\,\Psi^B$ *contains the ideal* $[\mathrm{mod}_{(\mathcal{A}_{\mathrm{o}}\setminus\{B\})}\overline{R}]$,

(ii) $$\mathrm{Ker}\,\Psi^B(X,Y) = \begin{cases} \mathcal{I}_{\overline{R}}(X,Y) & \text{if } X,Y \in \mathrm{mod}_{\{B\}}\overline{R}, \\ \mathrm{Hom}_{\overline{R}}(X,Y) & \text{if } X \text{ or } Y \in \mathrm{mod}_{(\mathcal{A}_{\mathrm{o}}\setminus\{B\})}\overline{R}. \end{cases}$$

Proof. The assertion (i) and the inclusion $\mathcal{I}_{\overline{R}}(X,Y) \subset \mathrm{Ker}\,\Psi^B(X,Y)$ of (ii) follow immediately from $\Psi^B = \overline{\mathcal{H}}_B \circ F_\bullet$. The equality $\mathrm{Ker}\,\Psi^B(X,Y) = \mathrm{Hom}_{\overline{R}}(X,Y)$ follows trivially from (i).

It remains to show $\operatorname{Ker}\Psi^B(X,Y) \subset \mathcal{I}_{\overline{R}}(X,Y)$. For this purpose it is enough to prove that for any $M = (\bigoplus_{g\in S_B} {}^gB^m, \mu)$, $N = (\bigoplus_{g'\in S_B} {}^{g'}B^n, \nu)$ in $\mathrm{Mod}_{\mathrm{f}}^G R$ and a morphism $f \in \operatorname{Ker}\widetilde{\Psi}^B(M,N)$ the R-homomorphism $f = (f_{g',g})_{g',g\in S_B} : \bigoplus_{g\in S_B} {}^gB^m \to \bigoplus_{g'\in S_B} {}^{g'}B^n$ belongs to the Jacobson radical $\mathcal{J} = \mathcal{J}_R$.

Clearly the components $f_{g',g}$ such that $g \neq g'$ belong to $\mathcal{J}({}^gB^m, {}^{g'}B^n)$ for $g, g' \in S_B$. Observe that $f_{e,e} \in \mathcal{J}({}^gB, {}^{g'}B)$ since $\overline{\mathcal{H}}_B(f) = 0$ by assumption. We show that each $f_{g,g}$ belongs to $\mathcal{J}({}^gB^m, {}^gB^n)$ for $g \in S_B$. Since f is a morphism in $\mathrm{Mod}_{\mathrm{f}}^G R$ we have ${}^{h^{-1}}f \cdot \mu_h = \nu_h \cdot f$ for every $h \in G$. Then for any $g, g_1' \in S_B$, looking at the (g_1', g)-components of the above equality we obtain the following equalities in $\operatorname{Hom}_R({}^gB^m, {}^{h^{-1}g_1'}B^n)$:

$$(*)_{(h,g_1',g)} \qquad \sum_{g_1\in S_B} {}^{h^{-1}}f_{g_1',g_1} \cdot \mu_h^{(g_1,g)} = \sum_{g'\in S_B} \nu_h^{(g_1',g')} \cdot f_{g',g}$$

where $\mu_h^{(g_1,g)} : {}^gB^m \to {}^{h^{-1}g_1}B^m$ (resp. $\nu_h^{(g_1',g')} : {}^{g'}B^n \to {}^{h^{-1}g_1'}B^n$) is the (g_1, g)-component (resp. (g_1', g')-component) of the R-homomorphism

$$\mu_h : \bigoplus_{g\in S_B} {}^gB^m \to {}^{h^{-1}}\Big(\bigoplus_{g_1\in S_B} {}^{g_1}B^m\Big),$$

respectively

$$\nu_h : \bigoplus_{g'\in S_B} {}^{g'}B^n \to {}^{h^{-1}}\Big(\bigoplus_{g_1'\in S_B} {}^{g_1'}B^n\Big),$$

defining the R-action μ (resp. ν) of G_B. Assume now that $g_1' = e$ and $h = g^{-1}$. Note that $\mu_h^{(g_1,g)}, \nu_h^{(e,g')} \in \mathcal{J}$ for $g_1 \neq e$ and $g' \neq g$; also ${}^gf_{e,e} \in \mathcal{J}$. Then $(*)_{(g^{-1},e,g)}$ implies by Lemma 2.3 that $\nu_h^{(e,g)} \cdot f_{g,g} \in \mathcal{J}$. But by Lemma 2.4, $\nu_h^{(e,g)}$ is an R-isomorphism and therefore $f_{g,g} \in \mathcal{J}$ for every $g \in S_B$. Consequently, by Proposition 2.5, $f \in \mathcal{J}$. ∎

Proposition. *Let $\mathcal{U} \subset \overline{\mathcal{P}}_{\mathrm{o}}$ be a family of periodic G-atoms together with a selection $(\nu_B)_{B\in\mathcal{U}}$ of R-actions of G_B on B. Then*:

(i) $\operatorname{Ker}\Psi^{\mathcal{U}}$ *contains the ideal* $[\mathrm{mod}_{(\mathcal{A}_{\mathrm{o}}\setminus\mathcal{U})}\bar{R}]$,

(ii) $\Phi^{\mathcal{U}}$ *induces a faithful representation embedding functor*

$$\bar{\Phi}^{\mathcal{U}} : \coprod_{B\in\mathcal{U}} \mathrm{mod}\, kG_B \to \mathrm{mod}\, \bar{R}/[\mathrm{mod}_{(\mathcal{A}_{\mathrm{o}}\setminus\mathcal{U})}\bar{R}].$$

Suppose that the GCS-reduction $(\Phi^{\mathcal{U}}, \Psi^{\mathcal{U}})$ (w.r.t. $\mathcal{U}$) is full. Then:

(iii) $$\operatorname{Ker}\Psi^{\mathcal{U}}(X,Y) = \begin{cases} \mathcal{I}_{\overline{R}}(X,Y) & \text{if } X, Y \in \mathrm{mod}_{\mathcal{U}}\bar{R}, \\ \operatorname{Hom}_{\overline{R}}(X,Y) & \text{if } X \text{ or } Y \notin \mathrm{mod}_{\mathcal{U}}\bar{R}, \end{cases}$$

for any indecomposables X, Y in $\mathrm{mod}\,\bar{R}$,

(iv) *the functor* $\bar{\Phi}^{\mathcal{U}} : \coprod_{B\in\mathcal{U}} \operatorname{mod} kG_B \to \operatorname{mod} \bar{R}/[\operatorname{mod}_{(\mathcal{A}_\circ\setminus\mathcal{U})}\bar{R}]$ *together with the functor* $\bar{\Psi}^{\mathcal{U}} : \operatorname{mod} \bar{R}/[\operatorname{mod}_{(\mathcal{A}_\circ\setminus\mathcal{U})}\bar{R}] \to \coprod_{B\in\mathcal{U}} \operatorname{mod} kG_B$ *induced by* $\Psi^{\mathcal{U}}$ *yields a bijection between the sets of isoclasses of indecomposables in the categories* $\coprod_{B\in\mathcal{U}} \operatorname{mod} kG_B$ *and* $\operatorname{mod} \bar{R}/[\operatorname{mod}_{(\mathcal{A}_\circ\setminus\mathcal{U})}\bar{R}]$.

Proof. (i) Follows immediately from Lemma 6.1(i).

(ii) We know that $\Psi^{\mathcal{U}}\Phi^{\mathcal{U}} \simeq \mathrm{id}_{\coprod_{B\in\mathcal{U}} \operatorname{mod} kG_B}$ and that $\Phi^{\mathcal{U}}$ is a representation embedding functor (see [4, Proposition 2.3 and Theorem 2.2]). Since $\operatorname{Ker}\Psi^{\mathcal{U}}$ contains $[\operatorname{mod}_{(\mathcal{A}_\circ\setminus\mathcal{U})}\bar{R}]$, the functor $\Psi^{\mathcal{U}}$ factorizes as

$$\Psi^{\mathcal{U}} : \operatorname{mod} \bar{R} \xrightarrow{\Pi} \operatorname{mod} \bar{R}/[\operatorname{mod}_{(\mathcal{A}_\circ\setminus\mathcal{U})}\bar{R}] \xrightarrow{\bar{\Psi}^{\mathcal{U}}} \coprod_{B\in\mathcal{U}} \operatorname{mod} kG_B$$

where Π denotes the canonical projection. Consequently,

$$\bar{\Phi}^{\mathcal{U}} = \Pi\Phi^{\mathcal{U}} : \coprod_{B\in\mathcal{U}} \operatorname{mod} kG_B \to \operatorname{mod} \bar{R}/[\operatorname{mod}_{(\mathcal{A}_\circ\setminus\mathcal{U})}\bar{R}]$$

is a representation embedding since $\Phi^{\mathcal{U}}$ is (note that $\operatorname{Im}\Phi^{\mathcal{U}} \subset \operatorname{mod}_{\mathcal{U}}\bar{R}$ and $(\operatorname{Ker}\Pi)_{\operatorname{mod}_{\mathcal{U}}\bar{R}} \subset (\mathcal{J}_{\bar{R}})_{\operatorname{mod}_{\mathcal{U}}\bar{R}}$).

(iii) Note that

$$(*) \qquad \operatorname{mod} \bar{R} = \operatorname{mod}_{(\mathcal{A}_\circ\setminus\mathcal{U})}\bar{R} \vee \bigvee_{B\in\mathcal{U}} \operatorname{mod}_{\{B\}}\bar{R}.$$

Fix a pair X, Y of $\bar{R}$-modules. If X and Y do not belong simultaneously to $\operatorname{mod}_{\{B_0\}}\bar{R}$ for any $B_0 \in \mathcal{U}$ then for each $B \in \mathcal{U}$ either $\Psi^B(X)$ or $\Psi^B(Y)$ is zero (see Lemma 6.1). Consequently, $\operatorname{Ker}\Psi^{\mathcal{U}}(X,Y) = \operatorname{Hom}_{\bar{R}}(X,Y)$. Note that $\mathcal{I}_{\bar{R}}(X,Y) = \operatorname{Hom}_{\bar{R}}(X,Y)$ provided X is in $\operatorname{mod}_{\{B\}}\bar{R}$ and Y is in $\operatorname{mod}_{\{B'\}}\bar{R}$ for distinct $B, B' \in \mathcal{U}$. If both X and Y belong to $\operatorname{mod}_{\{B\}}\bar{R}$ for some $B \in \mathcal{U}$ then $\operatorname{Ker}\Psi^{\mathcal{U}}(X,Y) = \operatorname{Ker}\Psi^B(X,Y) = \mathcal{I}_{\bar{R}}(X,Y)$ (see Lemma 6.1).

(iv) follows directly from (ii) and from conditions (a), (b) of the definition of full reduction. ∎

6.2. *Proof of Theorem A.* The following fact together with [4, Theorem 2.2] proves that the GCS-reduction $(\Phi^{\mathcal{U}}, \Psi^{\mathcal{U}})$ satisfying the assumptions of Theorem A is full. (It implies (a) and (b) in the definition of full GCS-reduction, (c) and (d) follow by Proposition 6.1).

PROPOSITION. *Let* $B = (B, \nu_B)$ *be a periodic* G*-atom in* $\bar{\mathcal{A}}$ *with an* R*-action* ν_B *of* G_B *on* B. *Assume that* $M = (M, \mu)$ *is an object in* $\operatorname{Mod}_{\mathrm{f}}^G R$ *such that* $B = (B, \nu_B)$ *splits properly* M. *Then* $\widetilde{\Phi}^B\widetilde{\Psi}^B(M)$ *is a direct summand of* M *in* $\operatorname{Mod}_{\mathrm{f}}^G R$. *In particular*, *if* M *is an indecomposable object in* $\operatorname{Mod}_{\mathrm{f}}^G$ *then* $\widetilde{\Phi}^B\widetilde{\Psi}^B(M) \simeq M$.

Proof. To prove that $\widetilde{\Phi}^B\widetilde{\Psi}^B(M)$ ($\simeq \theta(\overline{\mathcal{H}}_B(M)\otimes_k B)$) is a direct summand of M we construct a splittable monomorphism $\boldsymbol{i} : \overline{\mathcal{H}}_B(M)\otimes_k B \to M$ in $\mathrm{Mod}^{G_B} R$.

The canonical kG_B-module embedding $\varepsilon_B(M) : \mathcal{J}_B(M) \to \mathcal{H}_B(M)$ splits, hence there exists a kG_B-homomorphism $\overline{\mathcal{H}}_B(M) \stackrel{i^M}{\to} \mathcal{H}_B(M)$ such that $\pi_B(M)\cdot i^M = \mathrm{id}_{\overline{\mathcal{H}}_B(M)}$, where

$$0 \to \mathcal{J}_B \stackrel{\varepsilon_B}{\to} \mathcal{H}_B \stackrel{\pi_B}{\to} \overline{\mathcal{H}}_B \to 0$$

is the canonical exact sequence in $\mathrm{MOD}((\mathrm{Mod}\, R)^{\mathrm{op}})$. Since $\varepsilon_B(M)$ is proper, $\overline{\mathcal{H}}_B(M)$ is nonzero and we can simply assume that M is a direct sum $B^n \oplus M'$, where n is a positive integer and M' has no direct summand isomorphic to B. Then i^M and the canonical $\mathrm{End}_R(B)$-isomorphisms

$$\mathcal{H}_B(M) \simeq \mathcal{H}_B(B^n)\oplus \mathcal{H}_B(M') \quad \text{and} \quad \overline{\mathcal{H}}_B(M) \simeq \overline{\mathcal{H}}_B(B^n)$$

($\overline{\mathcal{H}}_B(M') = 0$) induce a kG_B-homomorphism

$$i = \begin{pmatrix} i_1 \\ i_2 \end{pmatrix} : \overline{\mathcal{H}}_B(B^n) \to \mathcal{H}_B(B^n)\oplus \mathcal{H}_B(M')$$

such that $\pi^B(B^n)\cdot i_1 = \mathrm{id}_{\overline{\mathcal{H}}_B(B^n)}$. Here the kG_B-module structure on the domain and codomain of i are induced by the above isomorphisms. Let

$$\widetilde{i} : \overline{\mathcal{H}}_B(B^n)\otimes_k B \to M = B^n \oplus M'$$

be the morphism in $\mathrm{Mod}^{G_B} R$ adjoint to i (see [3, Lemma 2.4]). We prove that $\widetilde{i}$ is a splittable monomorphism in $\mathrm{Mod}^{G_B} R$. For this purpose it is enough to construct a morphism $\widetilde{p}$ in $\mathrm{Mod}^{G_B} R$ with domain M such that $\widetilde{p}\cdot\widetilde{i}$ is an R-isomorphism. However, before we construct $\widetilde{p}$, we give a more detailed description of $\widetilde{i}$.

Denote by $w_i \in \mathrm{Hom}_R(B, B^n)$ the ith standard embedding of B into B^n, by $\overline{w}_i$ the coset of w_i in $\overline{\mathcal{H}}_B(B^n)$ and by φ'_i the image $i_2(\overline{w}_i)$ in $\mathrm{Hom}_R(B, M') = \mathcal{J}(B, M')$, where $i = 1,\ldots,n$. Observe that the equality $\pi_B(B^n)\cdot i_1 = \mathrm{id}_{\overline{\mathcal{H}}_B(B^n)}$ implies the existence of R-homomorphisms $\varphi_i \in \mathcal{J}_R(B, B^n)$ with components $\{\varphi_i^j\}_{j=1,\ldots,n}$ in $\mathcal{J}_R(B,B)$, $i = 1,\ldots,n$, such that $i_1(\overline{w}_i) = w_i + \varphi_i$.

Note that since $\overline{\mathcal{H}}_B(B) \simeq k$ as k-vector spaces, the canonical isomorphism $\overline{\mathcal{H}}_B(B)^n \simeq \overline{\mathcal{H}}_B(B^n)$ induces an R-isomorphism

(a) $$B^n \simeq \overline{\mathcal{H}}_B(B^n)\otimes_k B.$$

Denote by

$$i' = \begin{pmatrix} i'_1 \\ i'_2 \end{pmatrix} : B^n \to B^n \oplus M'$$

the composition of the isomorphism (a) and $\widetilde{i}$. It is not hard to show that

$$i'_1 = \mathrm{id}_{B^n} + \varphi \quad \text{and} \quad i'_2 = \varphi',$$

where φ is defined by the components $\{\varphi_i^j\}_{i,j=1,\dots,n}$, and φ' by the components $\{\varphi'_i\}_{i=1,\dots,n}$.

The construction of $\widetilde{i}$ was determined by the fact that $\widetilde{\Psi}^B$ by definition is equal to the top $\overline{\mathcal{H}}_B$ of the functor $\mathcal{H}_B$. The construction of $\widetilde{p}$ will reflect the alternative description of $\widetilde{\Psi}^B$ as the socle $(\overline{\mathcal{H}}_B)^*$ of the tensor product functor $\mathcal{T}_{B^*}$ (see 4.4).

As before, the canonical kG_B-embedding $\varepsilon^B(M) : \mathcal{J}^B(M) \to \mathcal{H}^B(M)$ splits and hence there exists a kG_B-homomorphism $j_M : \overline{\mathcal{H}}^B(M) \to \mathcal{H}^B(M)$ such that $\pi^B(M) \cdot j_M = \mathrm{id}_{\overline{\mathcal{H}}^B(M)}$, where

$$0 \to \mathcal{J}^B \xrightarrow{\varepsilon^B} \mathcal{H}^B \xrightarrow{\pi^B} \overline{\mathcal{H}}^B \to 0$$

is a canonical exact sequence in $\mathrm{MOD}(\mathrm{Mod}\, R)$. Consequently, the dual kG_B-homomorphism $j_M^* : \mathcal{H}^B(M)^* \to \overline{\mathcal{H}}^B(M)^*$ is a splittable epimorphism. Since $\overline{\mathcal{H}}^B(M') = 0$, we can assume that the codomain of j_M^* is equal to the kG_B-module $\overline{\mathcal{H}}^B(B^n)^*$, with the structure given by the dual of the standard isomorphism $\overline{\mathcal{H}}^B(M) \simeq \overline{\mathcal{H}}^B(B^n)$ induced by the canonical projection $M \to B^n$. Denote by

$$p : \mathcal{T}_{B^*}(M) \to (\overline{\mathcal{H}}^B)^*(B^n)$$

the composition of j_M^* and the canonical embedding of $\mathcal{T}_{B^*}$ into $(\mathcal{H}^B)^* = \mathcal{I}_B$ (see 4.4 and 5.4). Note that p is a kG_B-epimorphism since the socles of both functors $\mathcal{T}_{B^*}$ and $\mathcal{I}_B$ coincide and are equal to $(\overline{\mathcal{H}}^B)^*$ (see Theorem 4.4). Let

$$\widetilde{p} : M \to \mathrm{Hom}_k(B^*, (\overline{\mathcal{H}}^B)^*(B^n))$$

be the morphism in $\mathrm{Mod}^{G_B} R$ adjoint to p (see [3, Lemma 2.4]). We now analyze $\widetilde{p}$ in more detail, in order to prove that $\widetilde{p} \cdot \widetilde{i}$ is an R-isomorphism. Denote by p' the composition

$$p' : M \xrightarrow{\widetilde{p}} \mathrm{Hom}_k(B^*, (\overline{\mathcal{H}}^B)^*(B^n)) \xrightarrow{\sim} \mathrm{Hom}_k(\overline{\mathcal{H}}^B(B^n), B)$$

in $\mathrm{Mod}^{G_B} R$, where the second isomorphism is given by the appropriate version of the isomorphism 3.2(a) and the fact that B is locally finite-dimensional. Applying the definitions and a variant of [3, Lemma 2.4] one shows that p' has the following factorization:

$$\begin{aligned} B^n \oplus M' \xrightarrow{u_{B^n} \oplus u_{M'}} \mathrm{Hom}_k(\mathcal{H}^B(B^n), B) &\oplus \mathrm{Hom}(\mathcal{H}^B(M'), B) \\ &\xrightarrow{v} \mathrm{Hom}_k(\overline{\mathcal{H}}^B(B^n), B). \end{aligned}$$

Here $u_N : N \to \mathrm{Hom}_k(\mathrm{Hom}_R(N, B), B)$, for any N in $\mathrm{Mod}\, R$, denotes the R-homomorphism adjoint to the canonical map $\mathrm{Hom}_R(N, B) \otimes_k N \to B$, and v the R-homomorphism given by the pair $(\mathrm{Hom}_k(j_1,B),\mathrm{Hom}_k(j_2,B))$, where the k-linear map

$$j=\begin{pmatrix} j_1\\ j_2\end{pmatrix} : \overline{\mathcal{H}}^B(B^n)\to \mathcal{H}^B(B^n)\oplus\mathcal{H}^B(M')$$

is the composition of j_M and the canonical $\mathrm{End}_R(B)$-isomorphisms

$$\mathcal{H}^B(M)\simeq \mathcal{H}^B(B^n)\oplus\mathcal{H}^B(M') \quad\text{and}\quad \overline{\mathcal{H}}^B(M)\simeq\overline{\mathcal{H}}^B(B^n).$$

Denote by $r_i\in\mathrm{Hom}_R(B^n,B)$ the ith standard projection, by $\bar r_i$ the coset of r_i in $\overline{\mathcal{H}}^B(B^n)$ and by ψ_i' the image $j_2(\bar r_i)$ in $\mathrm{Hom}_R(B,M')=\mathcal{J}(B,M')$, where $i=1,\ldots,n$. Observe that the equality $\pi^B(B^n)\cdot j_1=\mathrm{id}_{\overline{\mathcal{H}}^B(M)}$ implies the existence of R-homomorphisms $\psi_i\in\mathcal{J}(B^n,B)$ with components $\{\psi_i^j\}_{j=1,\ldots,n}$ in $\mathcal{J}(B,B)$ such that $j_1(\bar r_i)=\psi_i$.

Note that since $\overline{\mathcal{H}}^B(B)\simeq k$ as k-vector spaces, the canonical isomorphism $\overline{\mathcal{H}}^B(B)^n\simeq\overline{\mathcal{H}}^B(B^n)$ induces an R-isomorphism

$$\text{(b)}\qquad \mathrm{Hom}_k(\overline{\mathcal{H}}^B(B^n),B)\simeq B^n.$$

Denote by

$$p''=(p_1'',p_2''):B^n\oplus M'\to B^n$$

the composition of p' and the isomorphism (b). It is not hard to show that

$$p_1''=\mathrm{id}_{B^n}+\psi \quad\text{and}\quad p_2''=\psi',$$

where ψ is defined by the components $\{\psi_i^j\}_{i,j=1,\ldots,n}$, and ψ' by the components $\{\psi_i'\}_{i=1,\ldots,n}$.

Now by Lemma 2.4 the composition $p''i'$ is an R-isomorphism, and therefore $\widetilde{p}\cdot\widetilde{i}$ is an isomorphism in $\mathrm{Mod}_{\mathrm f}^{G_B}R$. In this way the construction of a splittable monomorphism $\boldsymbol{i}:\overline{\mathcal{H}}_B(M)\otimes_k B\to M$ in $\mathrm{Mod}^{G_B}R$ is finished. Now applying the Lemma below and [3, Proposition 2.3(i)] we conclude that $\widetilde{\Phi}^B\widetilde{\Psi}^B(M)$ is a direct summand of M in $\mathrm{Mod}_{\mathrm f}^G R$. ■

LEMMA. *Let H be a subgroup of G, $N=(N,\nu)$ an object in* $\mathrm{Mod}_{\mathrm f}^H R$ *and $M=(M,\mu)$ an object in* $\mathrm{Mod}^G R$. *Suppose that N is a direct summand of the restriction $(M,\mu|H)$ in* $\mathrm{Mod}^H R$ *and that N satisfies the following condition*:

($*$) *for any H-atoms B,B' which are direct summands of N each element $g\in G$ such that ${}^gB\simeq B'$ belongs to H.*

Then $\theta_H(N)$, an R-module with an R-action of G induced by N, is a direct summand of M in $\mathrm{Mod}^G R$.

Proof. Let $\boldsymbol{i}:N\to M$ and $\boldsymbol{p}:M\to N$ be morphisms in $\mathrm{Mod}^H R$ such that $\boldsymbol{pi}=\mathrm{id}_N$. Denote by $\boldsymbol{I}:\theta_H(N)\to M$ and $\boldsymbol{P}:M\to\theta_H(N)$ the morphisms in $\mathrm{Mod}^G R$ adjoint to $\boldsymbol{i}$ and $\boldsymbol{p}$ (see [3, Lemma 2.3]). It is enough to show that $\boldsymbol{PI}$ is an automorphism of the R-module $\bigoplus_{g\in S_H}{}^gN$. By the very definition each (g,g)-component of $\boldsymbol{PI}$ is the identity map $\mathrm{id}_{{}^gN} = {}^g\,\mathrm{id}_N$, for $g \in S_H$. For any different $g_1, g_2 \in S_H$, the (g_1,g_2)-component of $\boldsymbol{PI}$ belongs to $\mathcal{J}({}^{g_1}N, {}^{g_2}N)$ by the condition $(*)$. Now the claim follows immediately from Lemma 2.4. ∎

6.3. Following [4, 5] for any M and N in $\operatorname{Mod} R$ we define the subspace

$$\mathcal{P}u(M,N) \subseteq \operatorname{Hom}_R(M,N)$$

to consist of all R-homomorphisms $f : M \to N$ having a factorization through a direct sum of finite-dimensional modules.

Let H be a subgroup of $G_M \cap G_N$. If μ is an R-action of H on M and ν is an R-action of H on N, then $\mathcal{P}u(M,N)$ is a kH-submodule of $\operatorname{Hom}_R(M,N)$ equipped with the standard structure (see 5.1(a)).

It is easy to see that the subspaces $\mathcal{P}u(M,N)$ define a two-sided ideal

$$\mathcal{P}u(\,\cdot\,,-) \subseteq \operatorname{Hom}_R(\,\cdot\,,-)$$

called the *pure-projective ideal* of $\operatorname{Mod} R$.

Remark. For any M, N in $\operatorname{Ind} R$ which are not simultaneously finite-dimensional we have $\mathcal{P}u(M,N) \subset \mathcal{J}_R(M,N)$. If additionally $\operatorname{supp} M \cap \operatorname{supp} N$ is a disjoint union of pairwise orthogonal finite subcategories then $\mathcal{P}u(M,N) = \mathcal{J}_R(M,N)$.

Proof of Theorem B. We show first that under the assumptions of Theorem B the GCS-reduction w.r.t. $\mathcal{U}$ is full. By Theorem A it is enough to prove the following.

Proposition. *Let $B = (B, \nu_B)$ be a G-atom in $\mathcal{A}^1$, with an arbitrary R-action ν_B of G_B on B, and $M = (M,\mu)$ an object in $\operatorname{Mod}_{\mathrm{f}}^G R$. Assume that for any $B' \in \mathcal{A}^{1'}(B)$ (see Introduction) each R-homomorphism $f : B \to B'$ (resp. $f : B' \to B$) factors through a direct sum of finite-dimensional R-modules. Then (B,ν_B) splits properly (M,μ) provided B is a direct summand of the R-module M.*

Proof. Set $H = G_B$ and $L = \operatorname{supp} B$. It is enough to prove that both kH-modules $\mathcal{J}_B(M)$ and $\mathcal{J}^B(M)$ are injective. We show the injectivity of $\mathcal{J}_B(M)$, the proof of the other case is similar.

Denote by $e_\varrho : \operatorname{Mod}^H \widehat{L} \to \operatorname{Mod}^H R$ the right adjoint functor to the restriction functor $e_\bullet : \operatorname{Mod}^H R \to \operatorname{Mod}^H \widehat{L}$. Then by [5, Lemma 2.1] the canonical morphism $M \to N$ in $\operatorname{Mod}^H R$ induces a kH-isomorphism

$$\text{(a)} \qquad \operatorname{Hom}_R(B,M) \simeq \operatorname{Hom}_R(B,N),$$

where $N = e_\varrho e_\bullet(M)$. We show that it also induces a kH-isomorphism

$$\text{(b)} \qquad \mathcal{J}(B,M) \simeq \mathcal{J}(B,N).$$

Note that by [9, Lemma 2] an R-module B' in $\operatorname{Ind} R$ is isomorphic to B if and only if B is a direct summand of $e_\varrho e_\bullet(B')$. Now fix any decomposition $M = \bigoplus_{i\in I} B_i'$ into a direct sum of objects in $\operatorname{Ind} R$ (see Lemma 2.1). Then we obtain a decomposition $N = \bigoplus_{i\in I} e_\varrho e_\bullet(B_i')$ into a direct sum of R-modules (M is a locally finite-dimensional module over the locally bounded k-category R). The isomorphism (b) now follows by the previous observation from (a) and the fact that Hom_R and $\mathcal{J}$ are summably closed ideals (see Proposition 2.5).

Now by [5, Theorem A] it is enough to prove that $\mathcal{J}(B,N) = \mathcal{P}u(B,N)$. Observe that N is an object of $\operatorname{Mod}_{\mathrm{f}}^H R$. The support of N is contained in $\widehat{\widehat{L}}$ and $\widehat{\widehat{L}}/H$ is finite (R is a locally bounded k-category and L/H is finite), and consequently $\operatorname{supp} N$ is contained in a union of finitely many H-orbits in R. Therefore N has a decomposition $N = \bigoplus_{j\in J} B_j$ into a direct sum of H-atoms. By [5, Theorem A] we obtain

$$\mathcal{P}u\Big(B, \bigoplus_{j\in J} B_j\Big) = \prod_{j\in J} \mathcal{P}u(B, B_j).$$

Since

$$\mathcal{J}_R\Big(B, \bigoplus_{j\in J} B_j\Big) = \prod_{j\in J} \mathcal{J}_R(B, B_j)$$

($\mathcal{J}$ is summably closed) we only have to show that

$$(*)_j \qquad \mathcal{P}u(B, B_j) = \mathcal{J}(B, B_j)$$

for all $j \in J$. Denote by J' the set of all $j \in J$ such that $\operatorname{supp} B_j \cap L$ is finite, and by J'' the complement of J' in J. It is clear that by Remark 6.2 we have to consider $(*)_j$ only for $j \in J''$. Take any H-atom B_j, where $j \in J''$. We know that $\operatorname{supp} B_j/H_{B_j}$ is finite and hence $\operatorname{supp} B_j/G_{B_j}$ is finite. Consequently, $H_{B_j} = G_B \cap G_{B_j}$ is infinite since $\operatorname{supp} B_j$ is infinite. This implies that B_j belongs to $\mathcal{A}^{1'}(B)$. Thus $(*)_j$ holds by Remark 6.2 and the assumptions of the proposition. In this way $\mathcal{J}(B,N) = \mathcal{P}u(B,N)$ and consequently the kH-module $\mathcal{J}(B,M)$ is injective. ∎

Now we prove the main assertion of Theorem B, the equivalence

$$\coprod_{B\in\mathcal{U}} \operatorname{mod} k[T,T^{-1}] \simeq \operatorname{mod} \bar{R}/[\operatorname{mod}_{(\mathcal{A}_{\mathrm{o}}\setminus\mathcal{U})} \bar{R}] \simeq \operatorname{mod}_{\mathcal{U}} \bar{R}/[\operatorname{mod}_{\mathcal{A}^{\mathrm{f}}} \bar{R}]_{\operatorname{mod}_{\mathcal{U}} \bar{R}}.$$

Since $\Psi^{\mathcal{U}}\Phi^{\mathcal{U}} \simeq \mathrm{id}_{\coprod_{B\in\mathcal{U}} \operatorname{mod} kG_B}$ (see [4, Theorem 2.2]) the functors $\Phi^{\mathcal{U}}$ and $\Psi^{\mathcal{U}}$ induce an equivalence

$$\text{(c)} \qquad \coprod_{B\in\mathcal{U}} \operatorname{mod} k[T,T^{-1}] \simeq \operatorname{mod} \bar{R}/\operatorname{Ker} \Psi^{\mathcal{U}} \simeq \operatorname{mod}_{\mathcal{U}} \bar{R}/(\operatorname{Ker} \Psi^{\mathcal{U}})_{\operatorname{mod}_{\mathcal{U}} \bar{R}}$$

(note that $\operatorname{Im}\Phi^{\mathcal{U}} \subset \operatorname{mod}_{\mathcal{U}}\bar{R}$). Note the following obvious inclusions of ideals:

(d) $$[\operatorname{mod}_{\mathcal{A}^{\mathrm{f}}}\bar{R}] \subset [\operatorname{mod}_{(\mathcal{A}_{\mathrm{o}}\setminus\mathcal{U})}\bar{R}] \subset \operatorname{Ker}\Psi^{\mathcal{U}}.$$

By Proposition 6.1 and the splitting (a) in the definition of full GCS-reduction the ideals $\operatorname{Ker}\Psi^{\mathcal{U}}$ and $[\operatorname{mod}_{(\mathcal{A}_{\mathrm{o}}\setminus\mathcal{U})}\bar{R}]$ can differ only on pairs of indecomposables from $\operatorname{ob}\operatorname{mod}_{\mathcal{U}}\bar{R} \times \operatorname{ob}\operatorname{mod}_{\mathcal{U}}\bar{R}$. Therefore it is enough to show the inclusion

(e) $$\mathcal{I}_{\operatorname{mod}_{\mathcal{U}}\bar{R}} \subset [\operatorname{mod}_{\mathcal{A}^{\mathrm{f}}}\bar{R}]_{\operatorname{mod}_{\mathcal{U}}\bar{R}}$$

where $\mathcal{I} = \mathcal{I}_{\bar{R}}$. Consequently, Proposition 6.1 and (d) yield $(\operatorname{Ker}\Psi^{\mathcal{U}})_{\operatorname{mod}_{\mathcal{U}}\bar{R}} = [\operatorname{mod}_{\mathcal{A}^{\mathrm{f}}}\bar{R}]_{\operatorname{mod}_{\mathcal{U}}\bar{R}}$ and $\operatorname{Ker}\Psi^{\mathcal{U}} = [\operatorname{mod}_{(\mathcal{A}_{\mathrm{o}}\setminus\mathcal{U})}\bar{R}]$, which combined with (c) gives the required equivalence.

To prove (e) take any $f \in \mathcal{I}(X', X)$ where X and X' are indecomposable $\bar{R}$-modules in $\operatorname{mod}_{\mathcal{U}}\bar{R}$. Then by the bijection (b) in the definition of full GCS-reduction there exist isomorphisms $u : X \to \Phi^B\Psi^B X$ and $u' : X' \to \Phi^{B'}\Psi^{B'}X'$ for some $B, B' \in \mathcal{U}$. It is enough to show that $f_1 = uf \in [\operatorname{mod}_{\mathcal{A}^{\mathrm{f}}}\bar{R}]$.

To do this we make use of the isomorphism

$$\Phi^B \simeq \operatorname{Hom}_{kG_B}(F_\lambda B^{kG_B}, -)$$

where $F_\lambda B^{kG_B} = \operatorname{Hom}_{kG_B}(F_\lambda B, kG_B)$ (see [3, Corollary 2.1]). Then by [3, Lemma 2.1 and Proposition 2.5(iii)] we obtain the natural isomorphisms

(f) $$\begin{aligned}\operatorname{Hom}_{\bar{R}}(X', \Phi^B\Psi^B X) &\simeq \operatorname{Hom}_{kG_B}(X' \otimes_{\bar{R}} F_\lambda B^{kG_B}, \Psi^B X)\\ &\simeq \operatorname{Hom}_{kG_B}(F_\bullet X' \otimes_R B^*, (\Psi^B X)^{-1}) = \operatorname{Hom}_{kG_B}(\mathcal{T}_{B^*}(F_\bullet X'), (\Psi^B X)^{-1})\end{aligned}$$

(we keep the notation from 5.1). Denote by $\widetilde{f_1} : \mathcal{T}_{B^*}(F_\bullet X') \to (\Psi^B X)^{-1}$ the image of f_1 under (f). We now prove that $\widetilde{f_1}$ factors through $\mathcal{C}_{B^*}(F_\bullet X')$. The case $B \neq B'$ is clear since then $\overline{\mathcal{H}}^B(F_\bullet X') = 0$ ($F_\bullet X' \in \operatorname{mod}_{\{B'\}}\bar{R}$) and $\mathcal{T}_{B^*}(F_\bullet X') = \mathcal{C}_{B^*}(F_\bullet X')$. Suppose now that $B = B'$. Since the isomorphism (f) is natural w.r.t. the first component, we have a factorization $\widetilde{f_1} = \widetilde{u} \cdot \mathcal{T}_{B^*}(F_\bullet f)$, where $\widetilde{u} : \mathcal{T}_{B^*}(F_\bullet X) \to (\Psi^B X)^{-1}$ is the image of u via (f). Observe that the kG_B-homomorphism $\mathcal{T}_{B^*}(F_\bullet f) : \mathcal{T}_{B^*}(F_\bullet X') \to \mathcal{T}_{B^*}(F_\bullet X)$ vanishes on the kG_B-submodule $(\overline{\mathcal{H}}^B)^*(F_\bullet X')$ of $\mathcal{T}_{B^*}(F_\bullet X')$ since $(\overline{\mathcal{H}}^B)^*(F_\bullet f) = 0$ ($f \in \mathcal{I}(X', X)$). Hence $\mathcal{T}_{B^*}(F_\bullet f)$ and also $\widetilde{f_1}$ factor through $\mathcal{C}_{B^*}(F_\bullet X')$. Using now the fact that (f) is natural w.r.t. the second component we conclude that f_1 factors through $\operatorname{Hom}_{kG_B}(F_\lambda B^{kG_B}, \mathcal{C}_{B^*}(F_\bullet X')^{-1})$, which is isomorphic to $F_\lambda B^m$ for some $m \in \mathbb{N}$ by Corollary 5.4 ($\mathcal{J}_B(F_\bullet X')$ is injective). Now the slight modification of [3, Lemma 4.4] (replace $\operatorname{mod}_1\bar{R}$ by $\operatorname{mod}_{\mathcal{A}^{\mathrm{f}}}\bar{R}$ when the assumption that G acts freely on $(\operatorname{ind}R)/\simeq$ is not present; the proof remains valid) yields $f_1 \in [\operatorname{mod}_{\mathcal{A}^{\mathrm{f}}}\bar{R}]$. In this way the proof of (e) and in consequence of the whole Theorem B is finished. ■

6.4. For any subset $\mathcal{U} \subset \mathcal{P}_{\mathrm{o}}$ and $n \in \mathbb{N}$ we denote by $\mathcal{U}(n)$ the set of all $B \in \mathcal{U}$ such that the rank of the free kG_B-module $F_\lambda B$ is just n.

The following fact follows directly from Theorem B by [10, Lemma 2.2] and [9, Lemma 3 and Proposition 2].

THEOREM [4, Theorem 5.2]. *Let R be a locally bounded k-category and $G \subset \mathrm{Aut}_k(R)$ be a group of k-linear automorphisms of R acting freely on $(\mathrm{ind}\, R)/\simeq$ such that $\bar{R}$ is a finite category. Assume R satisfies the following conditions*:

(i) $\mathcal{A}^\infty = \overline{\mathcal{A}^1}$,

(ii) $\mathcal{J}(B_1, B_2) = \mathcal{P}u(B_1, B_2)$ *for any* $B_1, B_2 \in \mathcal{A}^\infty$ *with* $G_{B_1} \cap G_{B_2}$ *nontrivial.*

Then the functors $\Phi^{\mathcal{A}^\infty_{\mathrm{o}}}$ and $\Psi^{\mathcal{A}^\infty_{\mathrm{o}}}$ induce an equivalence

$$\coprod_{B \in \mathcal{A}^\infty_{\mathrm{o}}} \mathrm{mod}\, k[T, T^{-1}] \simeq \underline{\underline{\mathrm{mod}}}\, \bar{R},$$

and $\bar{R}$ is tame if and only if R is tame and all sets $\mathcal{A}^\infty_{\mathrm{o}}(n)$, $n \in \mathbb{N}$, are finite.

REFERENCES

[1] I. Bucur and A. Deleanu, *Introduction to the Theory of Categories and Functors*, Wiley, 1968.

[2] K. Bongartz and P. Gabriel, *Covering spaces in representation theory*, Invent. Math. 65 (1982), 331–378.

[3] P. Dowbor, *On modules of the second kind for Galois coverings*, Fund. Math. 149 (1996), 31–54.

[4] —, *Galois covering reduction to stabilizers*, Bull. Polish Acad. Sci. Math. 44 (1996), 341–352.

[5] —, *The pure projective ideal of a module category*, Colloq. Math. 71 (1996), 203–214.

[6] —, *On stabilizers of G-atoms of representation-tame categories*, Bull. Polish Acad. Sci. Math. 46 (1998), 304–315.

[7] P. Dowbor and S. Kasjan, *Galois covering technique and non-simply connected posets of polynomial growth*, J. Pure Appl. Algebra, to appear.

[8] P. Dowbor, H. Lenzing and A. Skowroński, *Galois coverings of algebras by locally support-finite categories*, in: Lecture Notes in Math. 1177, Springer, 1986, 91–93.

[9] P. Dowbor and A. Skowroński, *On Galois coverings of tame algebras*, Arch. Math. (Basel) 44 (1985), 522–529.

[10] —, —, *Galois coverings of representation-infinite algebras*, Comment. Math. Helv. 62 (1987), 311–337.

[11] Yu. A. Drozd, S. A. Ovsienko and B. Yu. Furchin, *Categorical construction in representation theory*, in: Algebraic Structures and their Applications, University of Kiev, Kiev UMK VO, 1988, 43–73 (in Russian).

[12] P. Gabriel, *The universal cover of a representation-finite algebra*, in: Lecture Notes in Math. 903, Springer, 1982, 68–105.
[13] C. Geiss and J. A. de la Peña, *An interesting family of algebras*, Arch. Math. (Basel) 60 (1993), 25–35.
[14] E. L. Green, *Group-graded algebras and the zero relation problem*, in: Lecture Notes in Math. 903, Springer, 1982, 106–115.
[15] C. U. Jensen and H. Lenzing, *Model Theoretic Algebra*, Gordon and Breach, 1989.
[16] B. Mitchell, *Rings with several objects*, Adv. Math. 8 (1972), 1–162.
[17] Z. Pogorzały, *Regularly biserial algebras*, in: Topics in Algebra, Banach Center Publ. 26, Part 2, PWN, Warszawa, 1990, 371–384.
[18] C. Riedtmann, *Algebren, Darstellungsköcher, Überlagerungen und zurück*, Comment. Math. Helv. 55 (1980), 199–224.
[19] D. Simson, *Socle reduction and socle projective modules*, J. Algebra 108 (1986), 18–68.
[20] —, *Representations of bounded stratified posets, coverings and socle projective modules*, in: Topics in Algebra, Banach Center Publ. 26, Part 2, PWN, Warszawa, 1990, 499–533.
[21] —, *Right peak algebras of two-separate stratified posets, their Galois coverings and socle projective modules*, Comm. Algebra 20 (1992), 3541–3591.
[22] —, *On representation types of module categories and orders*, Bull. Polish Acad. Sci. Math. 41 (1993), 77–93.
[23] A. Skowroński, *Selfinjective algebras of polynomial growth*, Math. Ann. 285 (1989) 177–193.
[24] —, *Criteria for polynomial growth of algebras*, Bull. Polish Acad. Sci. Math. 42 (1994), 173–183.
[25] —, *Tame algebras with strongly simply connected Galois coverings*, Colloq. Math. 72 (1997), 335–351.

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