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A GEOMETRIC ESTIMATE FOR A PERIODIC SCHRÖDINGER OPERATOR

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Abstract. We estimate from below by geometric data the eigenvalues of the periodic Sturm–Liouville operator $-4d^2/ds^2 + \kappa^2(s)$ with potential given by the curvature of a closed curve.

1. Introduction. Let $X^3(c)$ be a 3-dimensional space form of constant curvature $c = 0$ or 1 and admitting a real Killing spinor with respect to some spin structure. Consider a compact, oriented and immersed surface $M^2 \subset X^3(c)$ with mean curvature H. The spin structure of $X^3(c)$ induces a spin structure on M^2. Denote by D the corresponding Dirac operator acting on spinor fields defined over the surface M^2. The first eigenvalue $\lambda_1^2(D)$ of the operator D^2 and the first eigenvalue μ_1 of the Schrödinger operator $\Delta + H^2 + c$ are related by the inequality

$$\lambda_1^2(D) \le \mu_1(\Delta + H^2 + c).$$

Equality holds if and only if the mean curvature H is constant (see [1], [5]). Moreover, the Killing spinor defines a map $f \mapsto \Phi(f)$ of the space $L^2(M^2)$ of functions into the space $L^2(M^2; S)$ of spinors such that

$$\|D(\Phi(f))\|^2_{L^2} = \langle \Delta f + H^2 f + cf, f\rangle_{L^2}.$$

In particular, the above inequality holds for all eigenvalues, i.e.,

$$\lambda_k^2(D) \le \mu_k(\Delta + H^2 + c).$$

This inequality was used in order to estimate the first eigenvalue of the Dirac operator defined on special surfaces of Euclidean space (see [1]). On the other hand, in case we know $\lambda_1^2(D)$, the inequality yields a lower bound for the spectrum of the Schrödinger operator $\Delta + H^2 + c$. For example, for any Riemannian metric g on the 2-dimensional sphere S^2 we have the

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inequality

$$\lambda_1^2(D) \geq \frac{4\pi}{\mathrm{vol}(S^2, g)}$$

(see [2], [6]). Consequently, we obtain

$$\frac{4\pi}{\mathrm{vol}(M^2, g)} \leq \mu_1(\Delta + H^2)$$

for any surface $M^2 \hookrightarrow \mathbb{R}^3$ of genus zero in Euclidean space $\mathbb{R}^3$. In this note we present the idea described above and, in particular, we estimate the spectrum of special periodic Schrödinger operators where the potential is given by the curvature κ of a spherical curve.

2. The 1-dimensional case. First of all, let us consider the 1-dimensional case, i.e., a curve γ of length L in a 2-dimensional space form $X^2(c)$. Let Φ be a Killing spinor of length one on $X^2(c)$:

$$\nabla_T \Phi = \frac{1}{2} c \cdot T \cdot \Phi.$$

The restriction $\varphi = \Phi_{|\gamma}$ defines a pair of spinors and the covariant derivative of φ along the curve γ is given by the formula

$$\nabla_T^\gamma(\varphi) = \frac{1}{2} cT \cdot \varphi + \frac{1}{2} \kappa_{\mathrm{g}} T \cdot N \cdot \varphi,$$

where T and N are the tangent and the normal vectors of the curve γ and κ_{g} denotes the curvature of the curve γ in $X^2(c)$ (see [5]). We compute the 1-dimensional Dirac operator

$$D(\varphi) = T \cdot \nabla_T^\gamma(\varphi) = -\frac{1}{2} c\varphi - \frac{1}{2} \kappa_{\mathrm{g}} N \cdot \varphi.$$

Let us represent the Clifford multiplication by the normal vector N:

$$N = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Then we obtain

$$|D(\varphi)|^2 = \frac{1}{4}(c^2 + \kappa^2)|\varphi|^2 = \frac{1}{4}(c^2 + \kappa_{\mathrm{g}}^2).$$

A similar computation for the spinor field $\psi = f \cdot \varphi$ yields the equation

$$|D\psi|^2 = |\dot{f}|^2 + f^2 \left(\frac{c}{4} + \frac{1}{4} \kappa_{\mathrm{g}}^2 \right).$$

Therefore, we obtain

$$\lambda_k^2(D) \leq \mu_k \left(-\frac{d^2}{ds^2} + \frac{c}{4} + \frac{1}{4} \kappa_{\mathrm{g}}^2 \right).$$

Suppose now that the spin structure on γ induced by the spin structure of $X^2(c)$ is non-trivial. Then we have $\lambda^2_{k+1}(D) = (4\pi^2/L^2)(k+1/2)^2$ (see [4]) and, in particular, we obtain

$$\frac{4\pi^2}{L^2}\left(k+\frac{1}{2}\right)^2 \le \mu_{k+1}\left(-\frac{d^2}{ds^2}+\frac{c}{4}+\frac{1}{4}\kappa_{\mathrm{g}}^2\right).$$

Theorem 1. *Let $\gamma\subset\mathbb{R}^3$ be a plane or spherical curve and denote by $\kappa^2 = c+\kappa_{\mathrm{g}}^2$ the square of its curvature. Suppose that the induced spin structure on γ is non-trivial, i.e., the tangent vector field has an odd rotation number. Then*

$$\frac{4\pi^2}{L^2} \le \mu_1\left(-4\frac{d^2}{ds^2}+\kappa^2\right),$$

where μ_1 is the first eigenvalue of the periodic Sturm–Liouville operator on the interval $[0,L]$. Moreover, equality occurs if and only if the curvature is constant.

Remark. The purely analytic Maz'ya method yields the inequality

$$\frac{\pi^2}{L^2} \le \mu\left(-4\frac{d^2}{ds^2}+\kappa^2\right)$$

(private communication of M. Shubin). A better geometric lower bound for the Sturm–Liouville operator $-4d^2/ds^2+\kappa^2$ with potential defined by the square of the curvature $\kappa(s)$ of a closed curve γ in Euclidean space seems to be unknown. We conjecture that the estimate given in Theorem 1 holds for any closed curve in $\mathbb{R}^3$. Let us compare this inequality with the well known Fenchel–Milnor inequality

$$2\pi \le \oint_\gamma \kappa.$$

Thus, by the Cauchy–Schwarz inequality we obtain

$$\frac{4\pi^2}{L^2} \le \frac{1}{L}\oint_\gamma \kappa^2.$$

Moreover, using the test function $f\equiv 1$, we have

$$\mu_1\left(-4\frac{d^2}{ds^2}+\kappa^2\right) \le \frac{1}{L}\oint_\gamma \kappa^2.$$

Suppose that γ is a simple curve in $\mathbb{R}^3$ and denote by ϱ the minimal number of generators of the fundamental group $\pi_1(\mathbb{R}^3\setminus\gamma)$. Then we have

$$2\pi\varrho \le \oint_\gamma \kappa.$$

In the spirit of this remark one should be able to prove the stronger inequality

$$\frac{4\pi^2}{L^2}\varrho^2 \leq \mu_1\left(-4\frac{d^2}{ds^2}+\kappa^2\right)$$

in case of a simple curve in $\mathbb{R}^3$.

EXAMPLES. We calculated the eigenvalue μ_1 for some classical curves in $\mathbb{R}^3$:

(a) *The lemniscate* $x = \sin(t)$, $y = \cos(t)\sin(t)$:

$$4\pi^2/L^2 = 1.06193, \qquad \mu_1 = 3.7315, \qquad \frac{1}{L}\oint_\gamma \kappa^2 = 4.36004.$$

(b) *The trefoil* $x = \sin(3t)\cos(t)$, $y = \sin(3t)\sin(t)$:

$$4\pi^2/L^2 = 0.221, \qquad \mu_1 = 5.21, \qquad \frac{1}{L}\oint_\gamma \kappa^2 = 8.16.$$

(c) *Viviani's curve* $x = 1+\cos(t)$, $y = \sin(2t)$, $z = 2\sin(t)$:

$$4\pi^2/L^2 = 0.169071, \qquad \mu_1 = 0.5335, \qquad \frac{1}{L}\oint_\gamma \kappa^2 = 0.567803.$$

(d) *The torus knot* $x = (8+3\cos(5t))\cos(2t)$, $y = (8+3\cos(5t))\sin(2t)$, $z = 5\sin(5t)$:

$$4\pi^2/L^2 = 0.00146034, \qquad \mu_1 = 0.03232, \qquad \frac{1}{L}\oint_\gamma \kappa^2 = 0.0333803.$$

(e) *The spherical spiral* $x = \cos(t)\cos(4t)$, $y = \cos(t)\sin(4t)$, $z = \sin(t)$:

$$4\pi^2/L^2 = 0.127036, \qquad \mu_1 = 1.744, \qquad \frac{1}{L}\oint_\gamma \kappa^2 = 4.93147.$$

3. The 2-dimensional Schrödinger operator. For a short curve we prove a similar inequality for the 2-dimensional periodic Schrödinger operator

$$P_{A,L} = -\left(1+\frac{A^2}{L^2}\right)\frac{\partial}{\partial t^2} - 4\frac{\partial^2}{\partial s^2} - \frac{4A}{L}\frac{\partial}{\partial t}\frac{\partial}{\partial s} + \kappa^2(s)$$

defined on $[0,2\pi]\times[0,L]$. In case $t = \text{const}$ one obtains again the inequality for the Sturm–Liouville operator.

THEOREM 2. *Let $\gamma \subset S^2 \subset \mathbb{R}^3$ be a closed, simple curve of length L bounding a region of area A, and denote by κ its curvature. Then the spectrum of the 2-dimensional periodic Schrödinger operator $P_{A,L}$ is bounded by*

$$\frac{4\pi^2}{L^2} \leq \mu_1(P_{A,L}).$$

Equality holds if and only if the curvature of γ is constant.

In general, let us consider a Riemannian manifold (Y^n, g) of dimension n as well as an S^1-principal fibre bundle $\pi : P \to Y^n$ over Y^n. Denote by $\vec{V}$ the vertical vector field on P induced by the action of the group S^1 on the total space P, i.e.,

$$\vec{V}(p) = \frac{d}{dt}(p \cdot e^{it})_{t=0}, \qquad p \in P.$$

A connection Z in the bundle P defines a decomposition of the tangent bundle $T(P) = T^{\mathrm{v}}(P) \oplus T^{\mathrm{h}}(P)$ into its vertical and horizontal subspace. We introduce a Riemannian metric g^* on the total space P, requiring that

(a) $g^*(\vec{V}, \vec{V}) = 1$,
(b) $g^*(T^{\mathrm{v}}, T^{\mathrm{h}}) = 0$,
(c) the differential $d\pi$ maps $T^{\mathrm{h}}(P)$ isometrically onto $T(Y^n)$.

A closed curve $\gamma : [0, L] \to Y^n$ of length L defines a torus $H(\gamma) := \pi^{-1}(\gamma) \subset P$ and we want to study the isometry class of this flat torus in P. Let $\alpha = e^{i\Theta} \in S^1$ be the holonomy of the connection Z along the closed curve γ. Consider a horizontal lift $\widehat{\gamma} : [0, L] \to P$ of the curve γ. Then

$$\widehat{\gamma}(L) = \widehat{\gamma}(0)e^{i\Theta}.$$

Consequently, the formula

$$\Phi(t, s) = \widehat{\gamma}(s)e^{-i\Theta s/L}e^{it}$$

defines a parametrization $\Phi : [0, 2\pi] \times [0, L] \to H(\gamma)$ of the torus $H(\gamma)$. Since

$$\frac{\partial \Phi}{\partial t} = \vec{V}, \qquad \frac{\partial \Phi}{\partial s} = dR_{e^{it}e^{-i\Theta s/L}}(\dot{\widehat{\gamma}}(s)) - \frac{\Theta}{L}\vec{V},$$

we obtain

$$g^*\left(\frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial t}\right) = 1, \quad g^*\left(\frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial s}\right) = -\frac{\Theta}{L}, \quad g^*\left(\frac{\partial \phi}{\partial s}, \frac{\partial \phi}{\partial s}\right) = 1 + \frac{\Theta^2}{L^2},$$

i.e., the torus $H(\gamma)$ is isometric to the flat torus $(\mathbb{R}^2/\Gamma_0, g^*)$, where Γ_0 is the orthogonal lattice $\Gamma_0 = 2\pi \cdot \mathbb{Z} \oplus L \cdot \mathbb{Z}$ and the metric g^* has the non-diagonal form

$$g^* = \begin{pmatrix} 1 & -\Theta/L \\ -\Theta/L & 1 + \Theta^2/L^2 \end{pmatrix}.$$

Using the transformation

$$x = -\frac{\Theta}{L}s + t, \qquad y = s,$$

we see that $H(\gamma)$ is isometric to the flat torus $(\mathbb{R}^2/\Gamma, dx^2 + dy^2)$, where the lattice Γ is generated by the two vectors

$$v_1 = \begin{pmatrix} 2\pi \\ 0 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} \Theta \\ L \end{pmatrix}.$$

In case the closed curve $\gamma : [0, L] \to Y^n$ is the oriented boundary of an oriented compact surface $M^2 \subset Y^n$, we can calculate the holonomy $\alpha = e^{i\Theta}$ along the curve γ. Indeed, let Ω^Z be the curvature form of the connection Z. It is a 2-form defined on the manifold Y^n with values in the Lie algebra of the group S^1, i.e., with values in $i \cdot \mathbb{R}^1$. The parameter Θ is given by the integral

$$\Theta = i \int_{M^2} \Omega^Z.$$

Let us consider the Hopf fibration $\pi : S^3 \to S^2$, where

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$$

is the 3-dimensional sphere of radius 1. The connection Z is given by the formula

$$Z = \tfrac{1}{2}\{\overline{z}_1 dz_1 - z_1 d\overline{z}_1 + \overline{z}_2 dz_2 - z_2 d\overline{z}_2\}$$

and its curvature form $(\omega = z_1/z_2)$

$$\Omega^Z = -\frac{d\omega \wedge d\overline{\omega}}{(1+|\omega|^2)^2} = -\frac{i}{2} dS^2$$

essentially coincides with half the volume form of the unit sphere S^2 of radius 1. However, the differential $d\pi : T^{\mathrm{h}}(S^3) \to T(S^2)$ multiplies the length of a vector by two, i.e., the Hopf fibration is a Riemannian submersion in the sense described before if we fix the metric of the sphere $S^2(1/2) = \{x \in \mathbb{R}^3 : |x| = 1/2\}$ on S^2. Consequently, for a closed simple curve $\gamma \subset S^2$ bounding a region of area A, the Hopf torus $H(\gamma) \subset S^3$ is isometric to the flat torus $\mathbb{R}^2/\Gamma$ and the lattice Γ is generated by the two vectors

$$v_1 = \begin{pmatrix} 2\pi \\ 0 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} A/2 \\ L/2 \end{pmatrix}.$$

The mean curvature H of the torus $H(\gamma) \subset S^3$ coincides with the geodesic curvature κ_{g} of the curve $\gamma \subset S^2 \subset \mathbb{R}^3$ (see [7], [8]). We now apply the inequality

$$\lambda_1^2(D) \le \mu_1(\Delta + H^2 + 1)$$

to the Hopf torus $H(\gamma) \subset S^3$. Then we obtain the estimate

$$\lambda_1^2(D) \le \mu_1(P_{A,L}),$$

where D is the Dirac operator on the flat torus $\mathbb{R}^2/\Gamma$ with respect to the induced spin structure. All spin structures of a 2-dimensional torus are classified by pairs $(\varepsilon_1, \varepsilon_2)$ of numbers $\varepsilon_i = 0, 1$. If γ is a simple curve in S^2, the induced spin structure on the Hopf torus $H(\gamma)$ is non-trivial and given by the pair $(\varepsilon_1, \varepsilon_2) = (0, 1)$. The spectrum of the Dirac operator for all flat tori is well known (see [4]): The dual lattice Γ^* is generated by

$$v_1^* = \begin{pmatrix} \frac{1}{2\pi} \\ -\frac{A}{2\pi L} \end{pmatrix}, \quad v_2^* = \begin{pmatrix} 0 \\ \frac{2}{L} \end{pmatrix}$$

and the eigenvalues of D^2 are given by

$$\begin{aligned} \lambda^2(k,l) &= 4\pi^2 \left\| k v_1^* + \left(l + \frac{1}{2}\right) v_2^* \right\|^2 \\ &= k^2 + \frac{4\pi^2}{L^2}\left((2l+1) - k\frac{A}{2\pi}\right)^2. \end{aligned}$$

We minimize $\lambda^2(k,l)$ on the integral lattice $\mathbb{Z}^2$. The isoperimetric inequality $4\pi A - A^2 \le L^2$ and $A \le \mathrm{vol}(S^2) = 4\pi$ show that $\lambda^2(k,l)$ attends its minimum at $(k,l) = (0,1)$, i.e.,

$$\frac{4\pi^2}{L^2} \le \lambda^2(k,l).$$

REMARK. Suppose now that equality holds for some curve $\gamma \subset S^2$. We consider the corresponding Hopf torus $H(\gamma) \subset S^3$ and then we obtain

$$\lambda_1^2(D) = \mu_1(\Delta + H^2 + 1).$$

Therefore, the mean curvature $H = \kappa$ is constant (see [1], [5]), i.e., γ is a curve on S^2 of constant curvature κ. Consequently, γ is a circle in a 2-dimensional plane. Denote by r its radius. Then

$$\kappa^2 = 1/r^2, \quad L = 2\pi r, \quad A = 2\pi(1 - \sqrt{1-r^2}),$$

and the inequality

$$4\pi^2/L^2 \le \kappa^2$$

is an equality for all $r \ne 0$.

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