A GEOMETRIC ESTIMATE FOR A PERIODIC SCHRÖDINGER OPERATOR

BY

THOMAS FRIEDRICH (BERLIN)

Abstract. We estimate from below by geometric data the eigenvalues of the periodic Sturm–Liouville operator $-4d^2/ds^2 + \kappa^2(s)$ with potential given by the curvature of a closed curve.

1. Introduction. Let $X^3(c)$ be a 3-dimensional space form of constant curvature $c = 0$ or $1$ and admitting a real Killing spinor with respect to some spin structure. Consider a compact, oriented and immersed surface $M^2 \subset X^3(c)$ with mean curvature $H$. The spin structure of $X^3(c)$ induces a spin structure on $M^2$. Denote by $D$ the corresponding Dirac operator acting on spinor fields defined over the surface $M^2$. The first eigenvalue $\lambda_1^2(D)$ of the operator $D^2$ and the first eigenvalue $\mu_1$ of the Schrödinger operator $\Delta + H^2 + c$ are related by the inequality

$$\lambda_1^2(D) \leq \mu_1(\Delta + H^2 + c).$$

Equality holds if and only if the mean curvature $H$ is constant (see [1], [5]). Moreover, the Killing spinor defines a map $f \mapsto \Phi(f)$ of the space $L^2(M^2)$ of functions into the space $L^2(M^2; S)$ of spinors such that

$$\|D(\Phi(f))\|_{L^2}^2 = \langle \Delta f + H^2 f + cf, f \rangle_{L^2}.$$

In particular, the above inequality holds for all eigenvalues, i.e.,

$$\lambda_k^2(D) \leq \mu_k(\Delta + H^2 + c).$$

This inequality was used in order to estimate the first eigenvalue of the Dirac operator defined on special surfaces of Euclidean space (see [1]). On the other hand, in case we know $\lambda_1^2(D)$, the inequality yields a lower bound for the spectrum of the Schrödinger operator $\Delta + H^2 + c$. For example, for any Riemannian metric $g$ on the 2-dimensional sphere $S^2$ we have the

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inequality
\[ \lambda_1^2(D) \geq \frac{4\pi}{\text{vol}(S^2, g)} \]
(see [2], [6]). Consequently, we obtain
\[ \frac{4\pi}{\text{vol}(M^2, g)} \leq \mu_1(\Delta + H^2) \]
for any surface \( M^2 \hookrightarrow \mathbb{R}^3 \) of genus zero in Euclidean space \( \mathbb{R}^3 \). In this note we present the idea described above and, in particular, we estimate the spectrum of special periodic Schrödinger operators where the potential is given by the curvature \( \kappa \) of a spherical curve.

2. The 1-dimensional case. First of all, let us consider the 1-dimensional case, i.e., a curve \( \gamma \) of length \( L \) in a 2-dimensional space form \( X^2(c) \). Let \( \Phi \) be a Killing spinor of length one on \( X^2(c) \):
\[ \nabla_T \Phi = \frac{1}{2} c \cdot T \cdot \Phi. \]
The restriction \( \varphi = \Phi|_\gamma \) defines a pair of spinors and the covariant derivative of \( \varphi \) along the curve \( \gamma \) is given by the formula
\[ \nabla^\gamma_T(\varphi) = \frac{1}{2} c T \cdot \varphi + \frac{1}{2} \kappa_g T \cdot N \cdot \varphi, \]
where \( T \) and \( N \) are the tangent and the normal vectors of the curve \( \gamma \) and \( \kappa_g \) denotes the curvature of the curve \( \gamma \) in \( X^2(c) \) (see [5]). We compute the 1-dimensional Dirac operator
\[ D(\varphi) = T \cdot \nabla^\gamma_T(\varphi) = -\frac{1}{2} c \varphi - \frac{1}{2} \kappa_g N \cdot \varphi. \]
Let us represent the Clifford multiplication by the normal vector \( N \):
\[ N = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \]
Then we obtain
\[ |D(\varphi)|^2 = \frac{1}{4}(c^2 + \kappa_g^2)|\varphi|^2 = \frac{1}{4}(c^2 + \kappa_g^2). \]
A similar computation for the spinor field \( \psi = f \cdot \varphi \) yields the equation
\[ |D\psi|^2 = |f|^2 + f^2 \left( \frac{c}{4} + \frac{1}{4} \kappa_g^2 \right). \]
Therefore, we obtain
\[ \lambda_k^2(D) \leq \mu_k \left( \frac{d^2}{ds^2} + \frac{c}{4} + \frac{1}{4} \kappa_g^2 \right). \]
Suppose now that the spin structure on $\gamma$ induced by the spin structure of $X^2(c)$ is non-trivial. Then we have $\lambda_{k+1}^2(D) = (4\pi^2/L^2)(k+1/2)^2$ (see [4]) and, in particular, we obtain

$$\frac{4\pi^2}{L^2} \left( k + \frac{1}{2} \right)^2 \leq \mu_{k+1} \left( -\frac{d^2}{ds^2} + \frac{c}{4} + \frac{1}{4} \kappa^2_g \right).$$

**Theorem 1.** Let $\gamma \subset \mathbb{R}^3$ be a plane or spherical curve and denote by $\kappa^2 = c + \kappa^2_g$ the square of its curvature. Suppose that the induced spin structure on $\gamma$ is non-trivial, i.e., the tangent vector field has an odd rotation number. Then

$$\frac{4\pi^2}{L^2} \leq \mu_1 \left( -4 \frac{d^2}{ds^2} + \kappa^2 \right),$$

where $\mu_1$ is the first eigenvalue of the periodic Sturm–Liouville operator on the interval $[0, L]$. Moreover, equality occurs if and only if the curvature is constant.

**Remark.** The purely analytic Maz’ya method yields the inequality

$$\frac{\pi^2}{L^2} \leq \mu \left( -4 \frac{d^2}{ds^2} + \kappa^2 \right)$$

(private communication of M. Shubin). A better geometric lower bound for the Sturm–Liouville operator $-4d^2/ds^2 + \kappa^2$ with potential defined by the square of the curvature $\kappa(s)$ of a closed curve $\gamma$ in Euclidean space seems to be unknown. We conjecture that the estimate given in Theorem 1 holds for any closed curve in $\mathbb{R}^3$. Let us compare this inequality with the well known Fenchel–Milnor inequality

$$2\pi \leq \frac{1}{\gamma} \kappa.$$ 

Thus, by the Cauchy–Schwarz inequality we obtain

$$\frac{4\pi^2}{L^2} \leq \frac{1}{L} \frac{1}{\gamma} \kappa^2.$$ 

Moreover, using the test function $f \equiv 1$, we have

$$\mu_1 \left( -4 \frac{d^2}{ds^2} + \kappa^2 \right) \leq \frac{1}{L} \frac{1}{\gamma} \kappa^2.$$ 

Suppose that $\gamma$ is a simple curve in $\mathbb{R}^3$ and denote by $\varrho$ the minimal number of generators of the fundamental group $\pi_1(\mathbb{R}^3 \setminus \gamma)$. Then we have

$$2\pi \varrho \leq \frac{1}{\gamma} \kappa.$$ 

In the spirit of this remark one should be able to prove the stronger inequality.
\[
\frac{4\pi^2}{L^2} \varrho^2 \leq \mu_1 \left(-4 \frac{d^2}{ds^2} + \kappa^2\right)
\]
in case of a simple curve in \(\mathbb{R}^3\).

**Examples.** We calculated the eigenvalue \(\mu_1\) for some classical curves in \(\mathbb{R}^3\):

(a) *The lemniscate* \(x = \sin(t), y = \cos(t) \sin(t)\):
\[
\frac{4\pi^2}{L^2} = 1.06193, \quad \mu_1 = 3.7315, \quad \frac{1}{L} \int_\gamma \kappa^2 = 4.36004.
\]

(b) *The trefoil* \(x = \sin(3t) \cos(t), y = \sin(3t) \sin(t)\):
\[
\frac{4\pi^2}{L^2} = 0.221, \quad \mu_1 = 5.21, \quad \frac{1}{L} \int_\gamma \kappa^2 = 8.16.
\]

(c) *Viviani’s curve* \(x = 1 + \cos(t), y = \sin(2t), z = 2 \sin(t)\):
\[
\frac{4\pi^2}{L^2} = 0.169071, \quad \mu_1 = 0.5335, \quad \frac{1}{L} \int_\gamma \kappa^2 = 0.567803.
\]

(d) *The torus knot* \(x = (8 + 3 \cos(5t)) \cos(2t), y = (8 + 3 \cos(5t)) \sin(2t), z = 5 \sin(5t)\):
\[
\frac{4\pi^2}{L^2} = 0.00146034, \quad \mu_1 = 0.03232, \quad \frac{1}{L} \int_\gamma \kappa^2 = 0.0333803.
\]

(e) *The spherical spiral* \(x = \cos(t) \cos(4t), y = \cos(t) \sin(4t), z = \sin(t)\):
\[
\frac{4\pi^2}{L^2} = 0.127036, \quad \mu_1 = 1.744, \quad \frac{1}{L} \int_\gamma \kappa^2 = 4.93147.
\]

3. The 2-dimensional Schrödinger operator. For a short curve we prove a similar inequality for the 2-dimensional periodic Schrödinger operator
\[
P_{A,L} = -\left(1 + \frac{A^2}{L^2}\right) \frac{\partial}{\partial t^2} - 4 \frac{\partial^2}{\partial s^2} - \frac{4A}{L} \frac{\partial}{\partial t} \frac{\partial}{\partial s} + \kappa^2(s)
\]
defined on \([0, 2\pi] \times [0, L]\). In case \(t = \text{const}\) one obtains again the inequality for the Sturm–Liouville operator.

**Theorem 2.** Let \(\gamma \subset S^2 \subset \mathbb{R}^3\) be a closed, simple curve of length \(L\) bounding a region of area \(A\), and denote by \(\kappa\) its curvature. Then the spectrum of the 2-dimensional periodic Schrödinger operator \(P_{A,L}\) is bounded by
\[
\frac{4\pi^2}{L^2} \leq \mu_1(P_{A,L}).
\]
Equality holds if and only if the curvature of \(\gamma\) is constant.
In general, let us consider a Riemannian manifold \((Y^n, g)\) of dimension \(n\) as well as an \(S^1\)-principal fibre bundle \(\pi : P \to Y^n\) over \(Y^n\). Denote by \(\vec{V}\) the vertical vector field on \(P\) induced by the action of the group \(S^1\) on the total space \(P\), i.e.,
\[
\vec{V}(p) = \frac{d}{dt}(p \cdot e^{it})_{t=0}, \quad p \in P.
\]
A connection \(Z\) in the bundle \(P\) defines a decomposition of the tangent bundle \(T(P) = T^v(P) \oplus T^h(P)\) into its vertical and horizontal subspace. We introduce a Riemannian metric \(g^*\) on the total space \(P\), requiring that
\[
\begin{align*}
(a) & \quad g^*(\vec{V}, \vec{V}) = 1, \\
(b) & \quad g^*(T^v, T^h) = 0, \\
(c) & \quad \text{the differential } d\pi \text{ maps } T^h(P) \text{ isometrically onto } T(Y^n).
\end{align*}
\]
A closed curve \(\gamma : [0, L] \to Y^n\) of length \(L\) defines a torus \(H(\gamma) := \pi^{-1}(\gamma) \subset P\) and we want to study the isometry class of this flat torus in \(P\). Let \(\alpha = e^{i\Theta} \in S^1\) be the holonomy of the connection \(Z\) along the closed curve \(\gamma\). Consider a horizontal lift \(\hat{\gamma} : [0, L] \to P\) of the curve \(\gamma\). Then
\[
\hat{\gamma}(L) = \hat{\gamma}(0)e^{i\Theta}.
\]
Consequently, the formula
\[
\Phi(t, s) = \hat{\gamma}(s)e^{-i\Theta s/L}e^{it}
\]
defines a parametrization \(\Phi : [0, 2\pi] \times [0, L] \to H(\gamma)\) of the torus \(H(\gamma)\). Since
\[
\frac{\partial \phi}{\partial t} = \vec{V}, \quad \frac{\partial \phi}{\partial s} = dR_{e^{-it}e^{-i\Theta/L}}(\hat{\gamma}(s)) - \frac{\Theta}{L}\vec{V},
\]
we obtain
\[
\begin{align*}
g^*(\frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial t}) & = 1, \\
g^*(\frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial s}) & = -\frac{\Theta}{L}, \\
g^*(\frac{\partial \phi}{\partial s}, \frac{\partial \phi}{\partial s}) & = 1 + \frac{\Theta^2}{L^2},
\end{align*}
\]
i.e., the torus \(H(\gamma)\) is isometric to the flat torus \((\mathbb{R}^2/\Gamma_0, g^*)\), where \(\Gamma_0\) is the orthogonal lattice \(\Gamma_0 = 2\pi \cdot \mathbb{Z} \oplus L \cdot \mathbb{Z}\) and the metric \(g^*\) has the non-diagonal form
\[
g^* = \begin{pmatrix}
1 & -\Theta/L \\
-\Theta/L & 1 + \Theta^2/L^2
\end{pmatrix}.
\]
Using the transformation
\[
x = -\frac{\Theta}{L}s + t, \quad y = s,
\]
we see that \(H(\gamma)\) is isometric to the flat torus \((\mathbb{R}^2/\Gamma, dx^2 + dy^2)\), where the lattice \(\Gamma\) is generated by the two vectors
\[
v_1 = \begin{pmatrix}
2\pi \\
0
\end{pmatrix}, \quad v_2 = \begin{pmatrix}
\Theta \\
L
\end{pmatrix}.
In case the closed curve $\gamma : [0, L] \to Y^n$ is the oriented boundary of an oriented compact surface $M^2 \subset Y^n$, we can calculate the holonomy $\alpha = e^{i\Theta}$ along the curve $\gamma$. Indeed, let $\Omega^Z$ be the curvature form of the connection $Z$. It is a 2-form defined on the manifold $Y^n$ with values in the Lie algebra of the group $S^1$, i.e., with values in $i \cdot \mathbb{R}$. The parameter $\Theta$ is given by the integral

$$\Theta = i \int_{M^2} \Omega^Z.$$ 

Let us consider the Hopf fibration $\pi : S^3 \to S^2$, where

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$$

is the 3-dimensional sphere of radius 1. The connection $Z$ is given by the formula

$$Z = \frac{i}{2} \{z_1 d\bar{z}_1 - z_1 d\bar{z}_1 + z_2 d\bar{z}_2 - z_2 d\bar{z}_2\}$$

and its curvature form $(\omega = z_1/z_2)$

$$\Omega^Z = -\frac{d\omega \wedge d\bar{\omega}}{(1 + |\omega|^2)^2} = -\frac{i}{2} dS^2$$

essentially coincides with half the volume form of the unit sphere $S^2$ of radius 1. However, the differential $d\pi : T^h(S^3) \to T(S^2)$ multiplies the length of a vector by two, i.e., the Hopf fibration is a Riemannian submersion in the sense described before if we fix the metric of the sphere $S^2(1/2) = \{x \in \mathbb{R}^3 : |x| = 1/2\}$ on $S^2$. Consequently, for a closed simple curve $\gamma \subset S^2$ bounding a region of area $A$, the Hopf torus $H(\gamma) \subset S^3$ is isometric to the flat torus $\mathbb{R}^2/\Gamma$ and the lattice $\Gamma$ is generated by the two vectors

$$v_1 = \begin{pmatrix} 2\pi \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} A/2 \\ L/2 \end{pmatrix}.$$ 

The mean curvature $H$ of the torus $H(\gamma) \subset S^3$ coincides with the geodesic curvature $\kappa_g$ of the curve $\gamma \subset S^2 \subset \mathbb{R}^3$ (see [7], [8]). We now apply the inequality

$$\lambda_1^2(D) \leq \mu_1(\Delta + H^2 + 1)$$

to the Hopf torus $H(\gamma) \subset S^3$. Then we obtain the estimate

$$\lambda_1^2(D) \leq \mu_1(P_{A, L}),$$

where $D$ is the Dirac operator on the flat torus $\mathbb{R}^2/\Gamma$ with respect to the induced spin structure. All spin structures of a 2-dimensional torus are classified by pairs $(\varepsilon_1, \varepsilon_2)$ of numbers $\varepsilon_i = 0, 1$. If $\gamma$ is a simple curve in $S^2$, the induced spin structure on the Hopf torus $H(\gamma)$ is non-trivial and given by the pair $(\varepsilon_1, \varepsilon_2) = (0, 1)$. The spectrum of the Dirac operator for all flat tori
is well known (see [4]): The dual lattice $\Gamma^*$ is generated by

$$v_1^* = \left( \frac{1}{2\pi}, \frac{A}{2\pi L} \right), \quad v_2^* = \left( 0, \frac{2}{L} \right)$$

and the eigenvalues of $D^2$ are given by

$$\lambda^2(k,l) = 4\pi^2 \left\| kv_1^* + \left( l + \frac{1}{2} \right) v_2^* \right\|^2 = k^2 + \frac{4\pi^2}{L^2} \left( (2l + 1) - \frac{kA}{2\pi} \right)^2.$$

We minimize $\lambda^2(k,l)$ on the integral lattice $\mathbb{Z}^2$. The isoperimetric inequality $4\pi A - A^2 \leq L^2$ and $A \leq \text{vol}(S^2) = 4\pi$ show that $\lambda^2(k,l)$ attains its minimum at $(k,l) = (0,1)$, i.e.,

$$\frac{4\pi^2}{L^2} \leq \lambda^2(k,l).$$

**Remark.** Suppose now that equality holds for some curve $\gamma \subset S^2$. We consider the corresponding Hopf torus $H(\gamma) \subset S^3$ and then we obtain

$$\lambda^2(D) = \mu_1(\Delta + H^2 + 1).$$

Therefore, the mean curvature $H = \kappa$ is constant (see [1], [5]), i.e., $\gamma$ is a curve on $S^2$ of constant curvature $\kappa$. Consequently, $\gamma$ is a circle in a 2-dimensional plane. Denote by $r$ its radius. Then

$$\kappa^2 = 1/r^2, \quad L = 2\pi r, \quad A = 2\pi(1 - \sqrt{1 - r^2}),$$

and the inequality

$$4\pi^2 / L^2 \leq \kappa^2$$

is an equality for all $r \neq 0$.

**REFERENCES**


Institut für Mathematik
Humboldt-Universität zu Berlin
Rudower Chaussee 25
D-10099 Berlin, Germany
E-mail: friedric@mathematik.hu-berlin.de

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