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COUNTING PARTIAL TYPES IN SIMPLE THEORIES

BY

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Abstract. We continue the work of Shelah and Casanovas on the cardinality of families of pairwise inconsistent types in simple theories. We prove that, in a simple theory, there are at most $\lambda^{<\kappa(T)} + 2^{\mu+|T|}$ pairwise inconsistent types of size μ over a set of size λ. This bound improves the previous bounds and clarifies the role of $\kappa(T)$. We also compute exactly the maximal cardinality of such families for countable, simple theories.

The main tool is the fact that, in simple theories, the collection of nonforking extensions of fixed size of a given complete type (ordered by reverse inclusion) has a chain condition. We show also that for a notion of dependence, this fact is equivalent to Kim–Pillay's type amalgamation theorem; a theory is simple if and only if it admits a notion of dependence with this chain condition, and furthermore that notion of dependence is forking.

1. Introduction. Counting types to understand the complexity of a first order theory was initiated in the 1950s. It has been a recurring theme of model theory since, and became central with Saharon Shelah's stability theory, where the number of types is used to characterize key model-theoretic properties.

One such property is the independence property; it is equivalent to the existence, in each cardinal λ, of a set over which there are 2^λ complete types. Since all simple unstable theories and some nonsimple theories have the independence property, to count types in simple theories, it is necessary to shift the focus from counting complete types to counting partial types. This is done by considering a chain condition on the poset consisting of small partial types over a large set.

Shelah pointed this out when he introduced simple theories already [Sh]. In addition to proving that a theory is simple if and only if forking has local character, he characterized simplicity in terms of a bound on the number of pairwise inconsistent partial types of fixed size over a larger set (Theorem 0.2 of [Sh]). Enrique Casanovas [Ca] extended this and also characterized supersimplicity in this way (see below for precise statements). Moreover, Casanovas showed that this characterization can be used to show the simplicity of a theory. Thus far, in contrast to the case of stable theories, the

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main method of showing the simplicity of a theory had been to use a theorem of Byunghan Kim and Anand Pillay (building on the work of Kim [K]) asserting that a theory is simple if and only if it possesses a dependence relation satisfying a canonical list of nice properties (Theorem 4.2 of [KP]).

This paper continues the work of Shelah and Casanovas. In order to state the results, we need to introduce some notation. For cardinals μ and λ, we let $\mathrm{NT}(\mu,\lambda)$ (NT stands for Number of Types) be the supremum of the cardinality of families of pairwise inconsistent partial types, each of size μ, over a fixed set of cardinality λ. Notice that for $\lambda=\mu\geq|T|$, we retrieve the usual way of counting types in stability theory, so this point of view is an extension of the original way of counting types.

In [Ca] Casanovas proves the following theorems.

Theorem (2.8 of [Ca]). *The following conditions are equivalent*:

(1) *T is simple.*
(2) *For all μ,λ, $\mathrm{NT}(\mu,\lambda)\leq\lambda^{|T|}+2^{\mu}$.*
(3) *For some regular $\mu\geq|T|^{+}$, for all λ, $\mathrm{NT}(\mu,\lambda)\leq\lambda^{|T|}+2^{\mu}$.*
(4) *There are μ,λ such that $\lambda^{<\mu}=\lambda$ and $\mathrm{NT}(\mu,\lambda)<\lambda^{\kappa}$.*

Theorem (3.2 of [Ca]). *The following conditions are equivalent*:

(1) *T is supersimple.*
(2) *For all μ,λ, $\mathrm{NT}(\mu,\lambda)\leq\lambda+2^{\mu+|T|}$.*
(3) *For some μ, for all λ, $\mathrm{NT}(\mu,\lambda)\leq\lambda+2^{\mu+|T|}$.*
(4) *There are μ,λ such that $\mathrm{NT}(\mu,\lambda)<\lambda^{\omega}$.*

In this paper, we prove the following results:

Theorem A. *If T is simple*, *then* $\mathrm{NT}(\mu,\lambda)\leq\lambda^{<\kappa(T)}+2^{\mu+|T|}$ *for all μ,λ.*

This bound makes the role of $\kappa(T)$ explicit. It improves (1)⇒(2) of Theorem 2.8 (since $\kappa(T)\leq|T|^{+}$) and gives (1)⇒(2) of Theorem 3.2 (since for supersimple theories $\kappa(T)=\aleph_0$). The presence of the term $\lambda^{<\kappa(T)}$ plays a similar role in Shelah's stability spectrum theorem (Corollary III 3.8 of [Sh a]). For stable theories, the bound $\mathrm{NT}(\mu,\lambda)\leq\lambda^{<\kappa(T)}+\mu_0$ (μ_0 is the first stability cardinal) follows directly from the stability spectrum theorem (see Proposition 2.2 below).

The proof of Theorem A proceeds very differently from the proof of the upper bound in [Ca]. The idea is to prove that, when the theory is simple, the poset consisting of nonforking extensions of a given size of a fixed complete type (partially ordered by reverse inclusion) has a chain condition (Theorem 2.4). This chain condition is interesting in its own right; it is to the Kim–Pillay type amalgamation theorem [KP] in simple theories, what the bound on the number of nonforking extensions is to stationarity in stable theories. Similar posets were introduced by Shelah in [Sh] for "weak dividing", and in [GIL] for forking. We also show (Theorem 2.6) that a theory is simple if and only if it admits a notion of dependence which satisfies the chain condition; furthermore the notion of dependence coincides with forking. This provides a new way to prove that a theory is simple.

Using the work of Casanovas and Shelah, we can show a converse to Theorem A. The improvement in (2)⇒(1) is that it gives a bound on $\kappa(T)$.

THEOREM B. *For a theory T and a cardinal κ the following conditions are equivalent*:

(1) *T is simple and $\kappa(T) \le \kappa$.*
(2) $\mathrm{NT}(\mu,\lambda) \le \lambda^{<\kappa} + 2^{\mu+|T|}$ *for each μ,λ.*

Finally, Theorem A and lower bounds derived from [Ca] also allow us to complete the computation of the numbers $\mathrm{NT}(\mu,\lambda)$ for infinite μ and λ, when T is simple and countable. For stable, countable T, this was done in [Ca]. Similarly to the stability spectrum (see Section III.5 of [Sh a]) and to [Ke], it seems significantly more difficult to compute $\mathrm{NT}(\mu,\lambda)$ when T is uncountable.

THEOREM C. *Let T be simple and countable.*

(1) *If T is stable*, *then* $\mathrm{NT}(\mu,\lambda) = \lambda^{<\kappa(T)} + \mu_0$ *for all infinite μ,λ (μ_0 is the first stability cardinal).*

(2) *If T is unstable*, *then* $\mathrm{NT}(\mu,\lambda) = \lambda^{<\kappa(T)} + 2^{\mu+|T|}$ *for all infinite μ,λ.*

The notation is standard. T denotes a complete, first order theory. We work inside the monster model, a large sufficiently saturated model of T. All sets, models, sequences, and elements are assumed to be inside the monster model. We use letters a, b to denote finite sequences of elements and occasionally write AB for $A \cup B$. Types are not assumed to be complete, unless specified. A type is over a set A if its parameters come from the set A. The reader is referred to [Sh a] and [KP] for the model-theoretic background.

2. Counting types. Consider the set $S(A,\mu)$ of types q over A such that $|q| \le \mu$ (identify types which are equivalent), partially ordered by $q_1 \le q_2$ if $q_1 \vdash q_2$. We will count types in this poset by considering the size of antichains. In this poset, two types are incompatible if their union is not consistent. Hence, an antichain $\mathcal{A}$ in $S(A,\mu)$ is a family of pairwise inconsistent types in $S(A,\mu)$. We restate the basic definition of [Ca].

DEFINITION 2.1. Let $\mu \le \lambda$ be cardinals. Let

$$\mathrm{NT}(\mu,\lambda) = \sup\{|\mathcal{A}| : \text{there exists } A \text{ of size } \lambda \text{ and } \mathcal{A} \text{ is an antichain in } S(A,\mu)\}.$$

Notice that for $\mu_1 \le \mu_2$ and $\lambda_1 \le \lambda_2$ we have $\mathrm{NT}(\mu_1, \lambda_1) \le \mathrm{NT}(\mu_2, \lambda_2)$. Notice also that for $\lambda \ge |T|$, $\mathrm{NT}(\lambda, \lambda)$ is the supremum of $|S(A)|$ for A of cardinality λ. This allows us to show the following motivating proposition.

PROPOSITION 2.2. *Let T be stable and let μ_0 be the first stability cardinal. Then for every μ, λ we have* $\mathrm{NT}(\mu, \lambda) \le \lambda^{<\kappa(T)} + \mu_0$.

Proof. Let T be stable. Let $\chi = \lambda^{<\kappa(T)} + \mu_0$. Then, by the stability spectrum theorem (Corollary III 3.8 of [Sh a]), T is stable in χ. Hence $\mathrm{NT}(\mu, \lambda) \le \mathrm{NT}(\chi, \chi) \le \chi$. ■

The previous proposition follows from the fact that, because of the stationarity of types over models in a stable theory, a type can have at most 2^μ pairwise inconsistent nonforking extensions of size μ ($\mu \ge |T|$). A similar bound exists in simple theories.

In order to prove our theorems, we introduce another partial order. Fix a cardinal μ, a set A, and a complete type p over a subset of A. Consider the set $\mathrm{NF}_p(A, \mu)$ of all types q over A such that $|q| \le \mu$ and $p \cup q$ is a nonforking extension of p (identify two types that are equivalent over p). Order this set by $q_1 \le q_2$ if $p \cup q_1 \vdash p \cup q_2$. When $\mu \ge |p|$, the notation can be simplified, but it is also interesting to consider the case when $\mu = \aleph_0$. In this partial order, two types q_1 and q_2 are incompatible if and only if $p \cup q_1 \cup q_2$ is not a nonforking extension of p.

DEFINITION 2.3. We say that forking has the *chain condition* if, for every cardinal μ, each set A, and each complete type p over a subset of A, the poset $\mathrm{NF}_p(A, \mu)$ has the $(2^{\mu+|p|})^+$-chain condition, i.e. if $\{p_i \mid i < (2^{\mu+|p|})^+\}$ is such that $|p_i| \le \mu$ and $p \cup p_i$ is a nonforking extension of p for $i < (2^{\mu+|p|})^+$, then there exist $i < j < (2^{\mu+|p|})^+$ such that $p \cup p_i \cup p_j$ is a nonforking extension of p.

The proof is a straightforward extension of Shelah's original argument (appearing only in [GIL] as Theorem 5.8). It uses the basic properties of forking in simple theories.

THEOREM 2.4 (Chain Condition). *If T is simple then forking has the chain condition.*

Proof. Let μ, A, p, and $\{p_i \mid i < (2^{\mu+|p|})^+\}$ be given as in the previous definition. Choose $B \subseteq A$ such that $p \in S(B)$. By increasing the size of each p_i if necessary, we may assume that $\mu \ge |p|$, that each p_i is a nonforking extension of p, and that there exist C_i containing B with $|C_i| \le \mu$ and $p_i \in S(C_i)$ for each $i < (2^\mu)^+$.

Let $\langle M_i \mid i < (2^\mu)^+ \rangle$ be an increasing, continuous chain of models of size 2^μ such that $B \subseteq M_0$ and $C_i \subseteq M_{i+1}$ for each $i < (2^\mu)^+$.

Let $S := \{\delta < (2^\mu)^+ \mid \operatorname{cf} \delta = \mu^+\}$. Then S is a stationary subset of $(2^\mu)^+$. Now define $f : S \to (2^\mu)^+$ by $f(\delta) := \min\{j \mid \operatorname{tp}(C_\delta/M_\delta)$ does not fork over $M_j\}$.

Since T is simple, for every $\delta \in S$ there exists $B_\delta \subseteq M_\delta$ of cardinality at most μ such that $\operatorname{tp}(C_\delta/M_\delta)$ does not fork over B_δ. Since $\operatorname{cf} \delta = \mu^+$, there is $j < \delta$ such that $B_\delta \subseteq M_j$. This shows that $f(\delta) < \delta$ for every $\delta \in S$. Hence, by Fodor's Lemma, there exists a stationary $S^* \subseteq S$ and a fixed $j < (2^\mu)^+$ such that $\operatorname{tp}(C_\delta/M_\delta)$ does not fork over M_j for every $\delta \in S^*$. Without loss of generality, we may assume that $S^* = (2^\mu)^+$ and $j = 0$, i.e., $\operatorname{tp}(C_i/M_i)$ does not fork over M_0 for every $i < (2^\mu)^+$.

By simplicity again, for every $i < (2^\mu)^+$ there exists $N_i \subseteq M_0$ of cardinality μ such that N_i contains B and $\operatorname{tp}(C_i/M_0)$ does not fork over N_i. Hence, by the pigeonhole principle, there exists a subset $S^* \subseteq (2^\mu)^+$ of cardinality $(2^\mu)^+$ and a model $N \subseteq M_0$ of cardinality μ such that $N_i = N$ for every $i \in S^*$. Without loss of generality, we may assume that $S^* = (2^\mu)^+$, i.e., $\operatorname{tp}(C_i/M_0)$ does not fork over N for every $i < (2^\mu)^+$. By transitivity of forking, $\operatorname{tp}(C_i/M_i)$ does not fork over N for each $i < (2^\mu)^+$. Hence, by monotonicity of forking, we have

$(*)$ $\qquad \operatorname{tp}(C_i/NC_j)$ does not fork over N for every $j < i < (2^\mu)^+$.

Since p_i does not fork over B by definition, we can find $q_i \in S(NC_i)$ extending p_i such that q_i does not fork over B for every $i < (2^\mu)^+$. By the pigeonhole principle again, there exists a subset $S^* \subseteq (2^\mu)^+$ of cardinality $(2^\mu)^+$ and a type $q \in S(N)$ such that $q_i \restriction N = q$ for every $i \in S^*$. Without loss of generality, we may assume that $S^* = (2^\mu)^+$, i.e., $q_i \restriction N = q$ for every $i < (2^\mu)^+$.

Thus, by the choice of q_i,

$(**)$ $\qquad q_i$ is a nonforking extension of $q \in S(N)$ for every $i < (2^\mu)^+$.

Hence, by the type amalgamation theorem over models applied to $(*)$ and $(**)$, we see that $q_i \cup q_j$ does not fork over N, for each $j < i < (2^\mu)^+$. Hence, by monotonicity and transitivity, $p_i \cup p_j$ does not fork over B for each $i < j$. ■

Remark 2.5. By applying the argument in the previous proof inductively (in a similar way to the argument in the next proof), one can show that for every integer $n < \omega$, if $\{p_i \mid i < (2^{\mu+|p|})^+\}$ is such that $p \cup p_i$ is a nonforking extension of p and $|p_i| \le \mu$, then there exist $i_1 < \ldots < i_n < (2^{\mu+|p|})^+$ such that $p \cup p_{i_1} \cup \ldots \cup p_{i_n}$ is a nonforking extension of p.

As in [KP], a *notion of dependence* is a relation Γ on triples of sets satisfying invariance, finite character, local character, extension, symmetry, and transitivity (see Definition 4.1 of [KP]). We say $\operatorname{tp}(a/B)$ *Γ-forks* over A if (a, B, A) is in Γ. Theorem 4.2 of [KP] states that a theory is simple if and only if it admits a notion of dependence Γ satisfying the type amalgamation theorem over models. Further, the notion of dependence Γ coincides with forking. We say that a notion of dependence Γ has the *chain condition* if it satisfies Definition 2.3 with "nonforking" replaced by "Γ-nonforking". The next theorem shows that the chain condition has the same consequence as the type amalgamation theorem for a notion of dependence. Hence, it can be used to show that a theory is simple. The proof proceeds similarly to Theorem 4.2 of [KP]. Note that the proof yields a little more than what the theorem states: a theory is simple if it has a notion of dependence which has the chain condition for types p_i over *finitely* many parameters.

THEOREM 2.6. *Let T be an arbitrary theory. T is simple if and only if T admits a notion of dependence which has the chain condition. Furthermore, the dependence relation is forking.*

Proof. If T is simple, it follows from [K] and [Sh] that forking is a notion of dependence. By Theorem 2.4 above, forking has the chain condition.

We now show the converse and the furthermore. It is enough to show that $\mathrm{tp}(a/Ab)$ does not Γ-fork over A if and only if $\mathrm{tp}(a/Ab)$ does not fork over A. Kim and Pillay showed that for a notion of dependence Γ, if $\mathrm{tp}(a/Ab)$ Γ-forks over A then $\mathrm{tp}(a/Ab)$ forks over A. Moreover, if $\mathrm{tp}(a/Ab)$ does not Γ-fork and does not divide, then $\mathrm{tp}(a/Ab)$ does not fork (see [KP], Claim I and Claim III of the proof of Theorem 4.2). Hence, it is enough to show that if $\mathrm{tp}(a/Ab)$ does not Γ-fork over A then $\mathrm{tp}(a/Ab)$ does not divide over A.

Assume that $\mathrm{tp}(a/Ab)$ does not Γ-fork over A. Let $p(x,b)=\mathrm{tp}(a/Ab)$. Let $\langle b_i \mid i<\omega\rangle$ be indiscernible over A. To show that $p(x,b)$ does not divide over A it is enough to show that $\bigcup_{i<\omega} p(x,b_i)$ is consistent. Let $\langle b_i \mid i<(2^{|A|+|T|})^+\rangle$ extending $\langle b_i \mid i<\omega\rangle$ be indiscernible over A. By invariance, $p(x,b_i)$ is a Γ-nonforking extension of $\mathrm{tp}(a/A)$ for $i<(2^{|A|+|T|})^+$. We show by induction on $n<\omega$ that $\bigcup_{i\le n} p(x,b_i)$ is a Γ-nonforking extension of p. To do this, we show by induction on n that

$$p(x,b_{i_0})\cup p(x,b_{i_1})\cup\ldots\cup p(x,b_{i_n}) \text{ is a } \Gamma\text{-nonforking extension of } \mathrm{tp}(a/A)$$

for every $i_0<\ldots<i_n<(2^{|A|+|T|})^+$. For $n=0$ this is the assumption. Assume inductively, that for every $i_0<\ldots<i_n<(2^{|A|+|T|})^+$, the type $p(x,b_{i_0})\cup\ldots\cup p(x,b_{i_n})$ is a Γ-nonforking extension of $\mathrm{tp}(a/A)$. There are $(2^{|A|+|T|})^+$ many types of this form, so by the chain condition the union of two distinct such types is a Γ-nonforking extension of $\mathrm{tp}(a/A)$. Hence, there are $i_0<\ldots<i_n<i_{n+1}<(2^{|A|+|T|})^+$ such that $p(x,b_{i_0})\cup\ldots\cup p(x,b_{i_n})\cup p(x,b_{i_{n+1}})$ is a Γ-nonforking extension of $\mathrm{tp}(a/A)$. By indiscernibility and invariance, this is true for all such $(n+1)$-tuples, which finishes the induction. Thus, $\bigcup_{i\le n} p(x,b_i)$ is a Γ-nonforking (hence consistent) extension of $\mathrm{tp}(a/A)$ for each $n<\omega$. So $\bigcup_{i<\omega} p(x,b_i)$ is consistent. ∎

We can now prove Theorem A.

Proof of Theorem A. Let $\chi = \lambda^{<\kappa(T)} + 2^{\mu+|T|}$. Let A be a set of cardinality λ. Suppose, for a contradiction, that $\{p_i \mid i < \chi^+\}$ is a family of pairwise inconsistent partial types over A, with $|p_i| \leq \mu$. By replacing p_i with an extension of size $\mu + |T|$ if necessary, we may assume that each p_i is in $S(C_i)$ for some $C_i \subseteq A$ with $|C_i| \leq \mu$. By simplicity, for each p_i there exists $B_i \subseteq C_i \subseteq A$ of cardinality less than $\kappa(T)$ such that p_i does not fork over B_i. Since there are at most $\lambda^{<\kappa(T)}$ such subsets of A, by the pigeonhole principle we may assume that there exists B of cardinality less than $\kappa(T)$ such that each p_i does not fork over B. Furthermore, since $|S(B)| \leq 2^{|T|}$, we may assume that there is $p \in S(B)$ such that p_i is a nonforking extension of p for each $i < \mu$. By the chain condition for forking, there exist $i < j$ such that $p_i \cup p_j$ is a nonforking extension of p. Hence, $p_i \cup p_j$ is consistent, a contradiction. ∎

The next fact follows immediately from the proof of Lemma 2.3 of [Ca], or from the proof of Theorem III 7.7, Theorem III 4.1, and Exercise III 4.14 of [Sh a].

FACT 2.7. *Let T be simple. If $\mu < \kappa(T)$, $\lambda^\mu > 2^\mu$, and $\lambda^{<\mu} = \lambda$, then* $\mathrm{NT}(2^\mu, \lambda) \geq \lambda^\mu$.

We can now prove Theorem B.

Proof of Theorem B. (1)⇒(2) follows from Theorem A.

We prove (2)⇒(1): Let $\mu = \kappa$ and $\lambda = \beth_\kappa(|T|)$. Then $\lambda^{<\mu} = \lambda$ and $\lambda^\mu > \lambda$. So, $\mathrm{NT}(\mu, \lambda) \leq \lambda < \lambda^\mu$. Hence, T is simple by Theorem 2.8 of [Ca].

Suppose, for a contradiction, that $\kappa < \kappa(T)$. Since $\lambda^{<\kappa} = \lambda$ and $\lambda^\kappa > 2^\kappa$, we have $\mathrm{NT}(2^\kappa, \lambda) \geq \lambda^\kappa$, by Fact 2.7. Hence, by the assumption of the theorem,

$$\lambda^\kappa \leq \mathrm{NT}(2^\kappa, \lambda) \leq \lambda^{<\kappa} + 2^{2^\kappa + |T|} \leq \lambda.$$

This contradicts König's Lemma. ∎

The next proposition is a converse to Proposition 2.2. Notice that the assumption that $\kappa(T) = \mu_1$ to derive the conclusion on the first stability cardinal is necessary.

PROPOSITION 2.8. *Let κ_1, μ_1 be cardinals. Suppose that* $\mathrm{NT}(\mu, \lambda) \leq \lambda^{<\kappa_1} + \mu_1$ *for each μ, λ. Then T is stable and $\kappa(T) \leq \kappa_1$. If $\kappa_1 = \kappa(T)$ then the first stability cardinal is at most μ_1 .*

Proof. Let $\lambda = 2^{\kappa_1 + \mu_1}$ and $\mu = \lambda$. For each A of cardinality λ, we have $|S(A)| \leq \mathrm{NT}(\lambda, \lambda) \leq \lambda^{<\kappa_1} + 2^{\mu_1} = \lambda$. Hence, T is stable.

The proof that $\kappa(T) \leq \kappa_1$ is as Theorem B, (2)⇒(1). Now suppose that $\mathrm{NT}(\mu, \lambda) \leq \lambda^{<\kappa(T)} + \mu_1$. Let $\lambda = \mu = \mu_0$, the first stability cardinal. Let A be of cardinality μ_0. Then $\lambda^{<\kappa(T)} = \lambda$ and $\mu_0 = |S(A)| \leq \mathrm{NT}(\mu_0, \mu_0) \leq \mu_1$. ∎

We can now prove Theorem C. Equality also holds for supersimple, not necessarily countable, theories (as observed in [Ca] using algebraic types).

Proof of Theorem C. (1) Check that $\lambda^{<\kappa(T)} + \mu_0$ has, in each case, the values computed in Propositions 4.2 and 4.3 of [Ca].

(2) Let μ, λ be infinite. If T is supersimple, then $\kappa(T) = \aleph_0$, so $\mathrm{NT}(\mu, \lambda) \geq \lambda^{<\kappa(T)}$, via the algebraic types. If T is not supersimple, then Theorem 3.2 of [Ca] implies that $\mathrm{NT}(\mu, \lambda) \geq \lambda^{\aleph_0}$. Since $\kappa(T) = \aleph_1$ as T is countable, we have $\mathrm{NT}(\mu, \lambda) \geq \lambda^{<\kappa(T)}$. Hence, since T has the independence property, $\mathrm{NT}(\mu, \lambda) \geq \lambda^{<\kappa(T)} + 2^{\mu}$. Equality follows from Theorem A. ∎

REFERENCES

[Ca] E. Casanovas, *The number of types in simple theories*, Ann. Pure Appl. Logic 98 (1999), 69–86.

[GIL] R. Grossberg, J. Iovino, and O. Lessmann, *A primer of simple theories*, preprint.

[Ke] H. J. Keisler, *Six classes of theories*, J. Austral. Math. Soc. 21 (1976), 257–256.

[K] B. Kim, *Forking in simple unstable theories*, J. London Math. Soc. 57 (1998), 257–267.

[KP] B. Kim and A. Pillay, *Simple theories*, Ann. Pure Appl. Logic 88 (1997), 149–164.

[Sh a] S. Shelah, *Classification Theory and the Number of Nonisomorphic Models*, rev. ed., North-Holland, 1990.

[Sh] —, *Simple unstable theories*, Ann. Math. Logic 19 (1998), 177–203.

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