FUNDAMENTAL SOLUTIONS FOR TRANSLATION AND ROTATION INVARIANT DIFFERENTIAL OPERATORS ON THE HEISENBERG GROUP

BY

PRISCILLA GORELLI (TORINO)

Abstract. Let \( H_1 \) be the three-dimensional Heisenberg group. Consider the left-invariant differential operators of the form \( D = P(-iT, -L) \), where \( P \) is a polynomial in two variables with complex coefficients, \( L \) is the sublaplacian on \( H_1 \) and \( T \) is the derivative with respect to the central direction. We find a fundamental solution of \( D \), whose definition is related to the way the plane curve defined by \( P(x, y) = 0 \) intersects the Heisenberg fan \( F = A \cup B, A = \{(x, y) \in \mathbb{R}^2 : y = (2m + 1)|x|, m \in \mathbb{N}\}, B = \{(x, y) \in \mathbb{R}^2 : x = 0, y > 0\} \). We can write an explicit expression of such a fundamental solution when the curve \( P(x, y) = 0 \) intersects \( F \) at finitely many points, all belonging to \( A \) and, if one of them is the origin, the monomial \( y^k \) has a nonzero coefficient, where \( k \) is the order of zero at the origin. As a consequence, such operators are globally solvable on \( H_1 \).

1. Introduction. In this paper we study problems of solvability of left invariant differential operators on the three-dimensional Heisenberg group \( H_1 \).

Let \( \Omega \) be an open set in a Lie group \( G \). A left-invariant differential operator \( P \) on \( G \) is locally solvable at \( x_0 \in \Omega \) if there exists a neighborhood \( U \) of \( x_0 \) in \( \Omega \) such that for all \( f \in C^\infty(U) \) there exists a distribution \( u \) on \( U \) that satisfies \( Pu = f \) on \( U \).

\( P \) is semiglobally solvable in \( \Omega \) if for all \( f \in \mathcal{D}(\Omega) \) and for all open sets \( U \) relatively compact in \( \Omega \) there exists \( u \in C^\infty \) such that \( Pu = f \) on \( U \).

Finally, \( P \) is globally solvable in \( \Omega \) if \( PC^\infty(\Omega) = C^\infty(\Omega) \). Global solvability is stronger than semiglobal solvability, and the latter implies local solvability.

We shall consider those differential operators that are expressed as polynomials with complex coefficients in \( L \) and \( T \), \( L \) being the sublaplacian and \( T \) the derivative with respect to the central direction. \( L \) and \( T \) commute and generate the algebra of differential operators on \( H_1 \) which are invariant with respect to both left translations and rotations.

Such a problem has already been solved for operators represented by polynomials of degree one. In [9] and [6] it is shown that the operator \(-L + \)

2000 Mathematics Subject Classification: 43A80, 22E30, 35A08.
iaT + c, α, c ∈ C, is locally solvable unless c = 0 and α = 2m + 1, for some integer m. As we shall see, it is natural to formulate the following conjecture: the operator $D = P(-iT, -L)$, where $P$ is a polynomial with complex coefficients, is locally solvable on the Heisenberg group $H_1$ if and only if $P(λ, ξ)$ is not divisible by $ξ - (2m + 1)λ$, for some $m ∈ Z$.

In this work, we show that the above conjecture is correct with certain restrictions on $P$. In the solvable case we in fact construct a fundamental solution. If $G$ is a Lie group, a distribution $E ∈ D^\prime(G)$ is a fundamental solution of an invariant operator $P$ if $PE = δ_0$, $δ_0$ being the Dirac delta at the identity. The existence of a fundamental solution implies semiglobal solvability. Moreover, if $G$ is $P$-convex, then the semiglobal solvability of $P$ implies its global solvability. The Heisenberg group is $P$-convex with respect to all nonzero invariant differential operators (see [4]).

2. Preliminaries. The $(2n + 1)$-dimensional Heisenberg group $H_n$ is the Lie group, diffeomorphic to $\mathbb{R}^{2n+1}$, whose multiplication law is defined as

\[(x, y, t)(x', y', t') = (x + x', y + y', t + t' + 2(x' \cdot y - x \cdot y')),\]

where $x, y, x', y' ∈ \mathbb{R}^n$, $t ∈ \mathbb{R}$ and $x \cdot y$ is the usual inner product on $\mathbb{R}^n$.

A base for its Lie algebra $h_n$ consists of the left invariant vector fields

\[X_j = \frac{∂}{∂x_j} + 2y_j \frac{∂}{∂t}, \quad Y_j = \frac{∂}{∂y_j} - 2x_j \frac{∂}{∂t}, \quad T = \frac{∂}{∂t},\]

where $j = 1, \ldots, n$. The commutation relations are $[X_j, T] = [Y_j, T] = 0$, $[X_j, Y_k] = -4δ_{j,k}T$, for all $j, k = 1, \ldots, n$.

The sublaplacian is the left invariant operator on $H_n$ defined by

\[L = \frac{1}{4} \sum_{j=1}^n (X_j^2 + Y_j^2).\]

If $n = 1$, then $L = \frac{1}{4}(X^2 + Y^2)$. It is a homogeneous operator of degree two with respect to the dilations $δ_r$ on $H_1$, induced by the automorphisms of $h_1$ defined by

\[δ_r X = rX, \quad δ_r Y = rY, \quad δ_r T = r^2 T.\]

Indeed,

\[δ_r L = \frac{1}{4}(δ_r X^2 + δ_r Y^2) = r^2 L.\]

Note that also $T = ∂/∂t$ is homogeneous of degree two.

Consider the spherical functions

\[φ_{λ,m}(x, y, t) = e^{-iλt}l_m(2|λ|(x^2 + y^2)),\]
where \( t_m(x) = e^{-x^2/2}L_m(x) \) and \( L_m(x) = L_m^{(0)}(x) \) is the \( m \)th Laguerre polynomial of index \( \alpha = 0 \), defined by

\[
L_m^{(\alpha)}(x) = \sum_{k=0}^{m} \frac{(m + \alpha)(-x)^k}{m - k}.
\]

The \( \varphi_{\lambda,m} \) are joint bounded radial eigenfunctions of \( L \) and \( T \), and

(2) \( T \varphi_{\lambda,m} = -i\lambda \varphi_{\lambda,m} \),

(3) \( L \varphi_{\lambda,m} = -|\lambda|(2m + 1)\varphi_{\lambda,m} \).

Let \( \Delta \) be the Gelfand spectrum of the Banach algebra \( L^1_{\text{rad}}(H_1) \) of integrable radial functions on \( H_1 \). Then

\[
\Delta = \{ \varphi_{\lambda,m} : \lambda \neq 0, m \in \mathbb{N} \} \cup \{ \varphi_{0,\xi} : \xi \geq 0 \}
\]

where

\[
\varphi_{0,\xi}(x, y, t) = J_0(2\sqrt{\xi(x^2 + y^2)})
\]

and

\[
J_0(t) = \frac{1}{2\pi} \int_0^{2\pi} e^{it \sin \theta} d\theta
\]

is the Bessel function of order 0.

It is shown in [2] that the Gelfand topology on \( \Delta \) coincides with the topology on

\[
F = \{ (\lambda, |\lambda|(2m + 1)) \in \mathbb{R}^2 : \lambda \neq 0, m \in \mathbb{N} \} \cup \{ (0, \xi) \in \mathbb{R}^2 : \xi \geq 0 \}
\]

induced from the Euclidean topology of \( \mathbb{R}^2 \).

The set \( F \) is usually called the Heisenberg fan.

We now state two technical lemmas involving Laguerre functions, which will be useful later on.

**Lemma 2.1.** The Laguerre functions \( t_m^{(\alpha)}(x) = e^{-x^2/2}L_m^{(\alpha)}(x) \) satisfy the following estimates:

(4) \[ |t_m^{(\alpha)}(x)| \leq 1, \]
Proof. Estimate (4) follows from the properties of Laguerre polynomials (see, for instance, Section 10.12 in [5]), while (5) is an immediate consequence of the following property:

\[
\frac{d^j}{dx^j} f_m^{(\alpha)}(x) = \sum_{h=0}^{j} c_h m(m-1) \cdots (m-h+1) f_{m-h}^{(\alpha+h)}(x),
\]

which can be proved by induction from the identity

\[
\frac{d}{dx} f_m^{(\alpha)}(x) = -\frac{1}{2} f_m^{(\alpha)}(x) + \frac{m}{\alpha + 1} f_{m-1}^{(\alpha+1)}(x)
\]

(see [5], formula (15) of Section 10.12).

Lemma 2.2. For all \( \lambda \neq 0 \),

\[
\left| \frac{\partial^j}{\partial \lambda^j} \varphi_{-\lambda,m}(x, y, t) \right| \leq C_j \left[ |t| + (m + 1)(x^2 + y^2) \right]^j.
\]

Proof. By the estimates of Lemma 2.1 we have

\[
\left| \frac{\partial^j}{\partial \lambda^j} \varphi_{-\lambda,m}(x, y, t) \right| = \left| \frac{\partial^j}{\partial \lambda^j} (e^{i\lambda t} f_m^{(0)}(2|\lambda|(x^2 + y^2))) \right|
\]

\[
\leq \sum_{h=0}^{j} \binom{j}{h} \left| \frac{\partial^{j-h}}{\partial \lambda^{j-h}} e^{i\lambda t} \right| \left| \frac{\partial^h}{\partial \eta^h} f_m^{(0)}(2|\lambda|(x^2 + y^2)) \right|_{\eta=2|\lambda|(x^2+y^2)}
\]

\[
\leq \sum_{h=0}^{j} \binom{j}{h} |t|^{j-h} C_h (m + 1)^h (2(x^2 + y^2))^h
\]

\[
\leq C_j \sum_{h=0}^{j} \binom{j}{h} |t|^{j-h} [(m + 1)(x^2 + y^2)]^h
\]

\[
= C_j \left[ |t| + (m + 1)(x^2 + y^2) \right]^j.
\]

3. Solvability of polynomials in \( L \) and \( T \). We will give some techniques that enable us to find a fundamental solution of operators of the form

\[
D = P(-iT, -L),
\]

where \( P \) is a polynomial in two variables with complex coefficients, \( L \) is the sublaplacian, \( T \) is the derivative with respect to \( t \).
Proposition 3.1. Let $D_1$ and $D_2$ be operators of the form (8). Then 
$D = D_1D_2$ is locally solvable if and only if $D_1$ and $D_2$ are locally solvable.

Proof. Suppose $D$ is locally solvable; then there exist a neighborhood $U$ and a distribution $u \in \mathcal{D}(U)$ such that for all $f \in C^\infty(U)$ one has $Du = f$ in $U$. Since $D_1$ and $D_2$ commute, we have

$$D_1(D_2u) = f = D_2(D_1u)$$
on $U$, that is, $D_1$ and $D_2$ are locally solvable. Let us see that the converse is also true.

If $D_1$ is locally solvable, then there exists an open set $U_1$ such that for all $f \in C^\infty(U_1)$ (in particular $f \in \mathcal{S}(U_1)$) there exists $u \in \mathcal{D}'(U_1)$ which is a solution of $D_1u = f$ in $U_1$. Since $D_2$ is locally solvable, there exist a neighborhood $U_2$ and a distribution $v \in \mathcal{D}'(U_2)$ such that $D_2v = u$ in $U_2$. Therefore $D$ is locally solvable, for $Dv = D_1D_2v = D_1u = f$ in $U_1 \cap U_2$.

Corollary 3.2. (a) If $P(\lambda, \xi)$ is identically zero on some oblique ray of the fan, then $D$ is not locally solvable.

(b) If $P(\lambda, \xi)$ is identically zero on the vertical ray of the fan, i.e. $D = T^h D_1$, then $D$ is locally solvable if and only if $D_1$ is locally solvable.

Proof. (a) By hypothesis $P(\lambda, \xi)$ is divisible by $\xi - (2m + 1)\lambda$, for some $m \in \mathbb{Z}$. Then $D = D_1D_2$, where $D_1 = -L + i(2m + 1)T$. Such an operator is not locally solvable (see [6]). By Proposition 3.1, $D$ is not locally solvable.

(b) $T^h$ is known to be locally solvable. Indeed, solving the problem $T^h u = u$, where $u \in \mathcal{D}'(U)$, is equivalent to finding an $h$th primitive of $u$ in the variable $t$. Such a primitive always exists (see Theorem IV, Ch. II, Sec. 5 in [8]). The statement follows from Proposition 3.1.

We will therefore restrict our investigation to those operators such that $P(\lambda, \xi)$ does not vanish identically on any ray of the fan.

Theorem 3.3. If $P$ is a homogeneous polynomial, then $D = P(-iT, -L)$ is solvable if and only if $P(\lambda, \xi)$ is not divisible by $\xi - (2m + 1)\lambda$, for some $m \in \mathbb{Z}$. Moreover, in this case $D$ is globally solvable.

Proof. It is well known that if $P$ is a homogeneous polynomial in two variables, then it factors as a product of terms of degree one. Since the operator $-L + i(2m + 1)T$, corresponding to the polynomial $\xi - (2m + 1)\lambda$, is not locally solvable for all $m \in \mathbb{Z}$, the assertion follows from Proposition 3.1.

The last statement is true because $D$ is homogeneous with respect to the dilations $\delta_\epsilon$ on $H_1$ defined before.

Let us describe the irreducible unitary representations of $H_n$. For every $\lambda \neq 0$, we have the Schrödinger representation $\pi_\lambda$, which is unique up to
equivalence and is defined in the following way. Given \( f \in L^2(\mathbb{R}^n) \),
\[
(\pi_x(x, y, t)f)(\xi) = e^{-i\lambda(t+2x \cdot y-4\xi \cdot y)}f(\xi - x).
\]
To the value \( \lambda = 0 \) there correspond the one-dimensional representations
\[
\pi_{\xi, \eta}(x, y, t) = e^{-i(\xi \cdot x + \eta \cdot y)},
\]
where \( \xi, \eta \in \mathbb{R}^n \).

Such representations are pairwise inequivalent and every irreducible unitary representation of \( H_n \) is equivalent to one of them.

The Fourier transform of a function \( f \in L^1(H_n) \) is the collection of all operators
\[
\pi(f) = \int_{H_n} f(x, y, t)\pi(x, y, t) \, dx \, dy \, dt
\]
where \( \pi \) ranges over the set of unitary irreducible representations of \( H_n \) described above. The inversion formula
\[
f(x, y, t) = \frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R}} \text{tr}(\pi_{\lambda}(f)\pi_{\lambda}(x, y, t)^*)|\lambda|^n \, d\lambda
\]
holds in a dense subspace of \( L^1(H_n) \), in particular for Schwartz functions. If we choose an orthonormal basis of \( L^2(\mathbb{R}^n) \), we can compute the trace explicitly. Put \( n = 1 \) and fix the normalized Hermite basis \( \{h^\lambda_m\}_{m \in \mathbb{N}} \) of \( L^2(\mathbb{R}) \), where
\[
h^\lambda_m(\xi) = \frac{|\lambda|^{1/4}}{2^{(m-1)/2} \sqrt{m!} \pi^{1/4}} \phi_m(\sqrt{|\lambda|} \xi)
\]
with \( \phi_m(\xi) = D_m^0 e^{-2\xi^2} \) and \( D_\lambda = \frac{1}{2}(d/d\xi - 4\lambda \xi) \). In this basis, (9) can be rewritten as
\[
f(x, y, t) = \frac{1}{(2\pi)^{n+1}} \sum_m \int_{\mathbb{R}} \langle \pi_{\lambda}(f)h^\lambda_m, \pi_{\lambda}(x, y, t)h^\lambda_m \rangle |\lambda|^n \, d\lambda,
\]
where the inner product is taken in \( L^2(\mathbb{R}) \).

We have
\[
\varphi_{\lambda, m}(v) = \langle \pi_{\lambda}(v)h^\lambda_m, h^\lambda_m \rangle,
\]
where \( v = (x, y, t) \). If we put \( \tilde{g}(\lambda, m, n) = \langle \pi_{\lambda}(g)h^\lambda_n, h^\lambda_m \rangle \) for all \( g \in \mathcal{S}(H_1) \), then
\[
\tilde{g}(\lambda, m, m) = \int_{H_1} \varphi_{\lambda, m}(v)g(v) \, dv.
\]
Therefore, since \( \|\varphi_{\lambda, m}\|_\infty = 1 \) for all \( m \) and \( \lambda \),
\[
|\tilde{g}(\lambda, m, m)| \leq \|g\|_{L^1(H_1)}.
\]

Let \( f \) be a Schwartz function on \( H_1 \) and assume that \( Du = f \). By formally applying the Fourier transform to both sides, we get
\[
\pi_{\lambda}(Du)h^\lambda_m = \pi_{\lambda}(f)h^\lambda_m.
\]
If \( V \) is a left invariant vector field, we have
\[
\pi_\lambda(Vu) = -\pi_\lambda(u)d\pi_\lambda(V).
\] (13)

Take \( D \) as in (8). By (13) we get
\[
\pi_\lambda(Du) = \pi_\lambda(u)d\pi_\lambda(tD).
\] (14)

Moreover,
\[
d\pi_\lambda(T)h^\lambda_m = -i\lambda h^\lambda_m, \quad d\pi_\lambda(L)h^\lambda_m = -(2m+1)|\lambda|h^\lambda_m.
\]

Therefore
\[
d\pi_\lambda(tD)h_m^\lambda = d\pi_\lambda(P(iT,-L))h_m^\lambda = P(d\pi_\lambda(iT), d\pi_\lambda(-L))h_m^\lambda = P(\lambda, |\lambda|(2m+1))h_m^\lambda,
\]

and, by (14),
\[
\pi_\lambda(Du)h_m^\lambda = \pi_\lambda(u)P(\lambda, |\lambda|(2m+1))h_m^\lambda.
\] (15)

Formula (15) can be viewed as an analogue of the identity
\[
(p(-i\partial)u)^\lambda(\xi) = p(\xi)\hat{u}(\xi),
\]

holding on \( \mathbb{R}^n \) for a differential operator with constant coefficients. The polynomial \( p(\xi) \) appearing in (16) is called the symbol of the operator \( p(-i\partial) \).

For this reason we call \( P(\lambda, \xi) \) the symbol of \( D \).

From (12) and (15) it follows that
\[
\pi_\lambda(Du)h_m^\lambda = \pi_\lambda(u)P(\lambda, |\lambda|(2m+1))h_m^\lambda = \pi_\lambda(f)h_m^\lambda,
\]

therefore
\[
\pi_\lambda(u)h_m^\lambda = \frac{\pi_\lambda(f)h_m^\lambda}{P(\lambda, |\lambda|(2m+1))}.
\] (17)

From the inversion formula (10) and from (17) we get the following formal expression for \( u \):
\[
u(v) = \frac{1}{(2\pi)^2} \int \sum_{m=0}^{\infty} \frac{\langle \pi_\lambda(f)h_m^\lambda, \pi_\lambda(v)h_m^\lambda \rangle}{P(\lambda, |\lambda|(2m+1))} |\lambda| d\lambda
\]
\[
= \frac{1}{(2\pi)^2} \int \sum_{m=0}^{\infty} \int_{H_1} f(w) \frac{\langle \pi_\lambda(w)h_m^\lambda, \pi_\lambda(v)h_m^\lambda \rangle}{P(\lambda, |\lambda|(2m+1))} dw |\lambda| d\lambda
\]
\[
= \frac{1}{(2\pi)^2} \int \sum_{m=0}^{\infty} \int_{H_1} f(w) \frac{h_m^\lambda, \pi_\lambda(w^{-1}v)h_m^\lambda}{P(\lambda, |\lambda|(2m+1))} dw |\lambda| d\lambda
\]
\[
= \frac{1}{(2\pi)^2} \int \sum_{m=0}^{\infty} \int_{H_1} f(w) \phi_{-\lambda,m}(w^{-1}v) \frac{dw}{P(\lambda, |\lambda|(2m+1))} |\lambda| d\lambda
\]
\[
= \frac{1}{(2\pi)^2} \int \sum_{m=0}^{\infty} f * \phi_{-\lambda,m}(v) \frac{\lambda}{P(\lambda, |\lambda|(2m+1))} |\lambda| d\lambda.
\]
Therefore, if we can define
\[ K(v) = \frac{1}{(2\pi)^2} \int \sum_{m=0}^{\infty} \frac{\varphi_{-\lambda,m}(v)}{P(\lambda,|\lambda|(2m+1))} |\lambda| \, d\lambda \]

as a distribution, it follows that \( u = f \ast K \), so that a fundamental solution of \( D \) is the tempered distribution defined by
\[ \langle K, g \rangle = \frac{1}{(2\pi)^2} \int_{H_1} \int_{\mathbb{R}} \sum_{m=0}^{\infty} \frac{\varphi_{-\lambda,m}(v)g(v)}{P(\lambda,|\lambda|(2m+1))} |\lambda| \, d\lambda \, dv \]
\[ = \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \sum_{m=0}^{\infty} \frac{\hat{g}(-\lambda,m,m)}{P(\lambda,|\lambda|(2m+1))} |\lambda| \, d\lambda, \]

for all \( g \in S(H_1) \). Note that only the radial coefficients \( \hat{g}(-\lambda,m,m) \) occur in this formula, so \( K \) is radial.

For a generic polynomial \( P \), (18) does not converge absolutely in general. The series may not converge, and the integral has singularities when the algebraic curve defined by \( P(\lambda,\xi) = 0 \) intersects the Heisenberg fan. Thus, we are going to face our problem by considering separately different cases, according to the mutual position of the algebraic curve \( P(\lambda,\xi) = 0 \) and the fan. For each case, we define a fundamental solution of \( D \), modifying (18) in a suitable way, in order to get a well defined tempered distribution.

As we have already said above in this section, we are reduced to considering algebraic curves \( P(\lambda,\xi) = 0 \) that intersect each ray of the fan in at most finitely many points.

4. First case: no intersections. The simplest situation occurs when \( P(\lambda,\xi) \) is never zero on \( F \). To solve this problem we use the following fact (see [7], Appendix A, Example 2.7).

**Lemma 4.1.** If \( P \in \mathbb{R}[x_1,\ldots,x_n] \) and \( P(x) > 0 \) for all \( x \in \mathbb{R}^n \), then there exist \( C > 0 \) and \( N \in \mathbb{N} \) such that
\[ P(x) > C(1 + |x|^2)^{-N} \quad \forall x \in \mathbb{R}^n. \]

A consequence of this lemma is the following

**Lemma 4.2.** If \( P \in \mathbb{C}[x,y] \) and \( P(x,y) \neq 0 \) in the closed domain of the plane defined by \( y \geq |x|(2m+1) \), then in this region we have the estimate
\[ |P(x,y)| > C(1 + x^2 + y^2)^{-N}, \]

for some \( C > 0 \) and \( N \in \mathbb{N} \).

**Proof.** By changing coordinates we can reduce to the case \( P(x,y) \neq 0 \) in the first quarter of the plane. Therefore assume that for all \( x \geq 0, y \geq 0 \) we have \( |P(x,y)| > 0 \).
If \( P(x, y) = P_1(x, y) + iP_2(x, y) \) with \( P_1(x, y), P_2(x, y) \in \mathbb{R}[x, y] \), then \( |P(x, y)| = \sqrt{P_1(x, y)^2 + P_2(x, y)^2} \) and \( Q(x, y) = P_1(x, y)^2 + P_2(x, y)^2 \in \mathbb{R}[x, y] \). Since \( Q \) is positive for all \( x \geq 0 \) and \( y \geq 0 \), the polynomial \( R(x, y) = Q(x^2, y^2) \) is positive for all \( (x, y) \in \mathbb{R}^2 \).

By Lemma 4.1 there exist \( C_1 > 0 \) and \( N \in \mathbb{N} \) such that
\[
R(x, y) > C_1(1 + x^2 + y^2)^{-N}.
\]

For \( x \geq 0 \) and \( y \geq 0 \), \( Q(x, y) = R(\sqrt{x}, \sqrt{y}) \), therefore
\[
Q(x, y) > C_1(1 + x + y)^{-N} > C_2(1 + x^2 + y^2)^{-N/2}
\]
and so
\[
|P(x, y)| > C(1 + x^2 + y^2)^{-N/4}.
\]

Consider an operator \( D \) whose symbol \( P \) is such that \( P(\lambda, \xi) = 0 \) defines an algebraic curve that does not intersect \( F \), i.e. \( P(\lambda, |\lambda|(2m + 1)) \neq 0 \) for all \( m \in \mathbb{N}, \lambda \in \mathbb{R} \) and \( P(0, \xi) \neq 0 \) for all \( \xi > 0 \).

**Theorem 4.3.** Take \( D = P(-iT, -L) \) such that \( P(\lambda, \xi) \) is not zero on \( F \).

Define the distribution \( K \) by
\[
\langle K, g \rangle = \frac{1}{(2\pi)^2} \sum_{m=0}^{\infty} \frac{\hat{g}(-\lambda, m)}{P(\lambda, |\lambda|(2m + 1))} |\lambda| \, d\lambda, \quad g \in \mathcal{S}(H_1),
\]
where the integral on the right-hand side is absolutely convergent. Then \( K \) is a fundamental solution of \( D \).

**Proof.** The algebraic curve \( P(\lambda, \xi) = 0 \) in the \( \lambda, \xi \) plane has a finite number of connected components (see [3], Theorems 2.3.6 and 2.4.5). Since it does not intersect \( F \), there exists an integer \( k \in \mathbb{N} \) such that \( P(\lambda, \xi) \neq 0 \) in the closed region defined by \( \xi \geq (2k + 1)|\lambda| \).

By Lemma 4.2, for all \( m \geq k \) one has
\[
|P(\lambda, |\lambda|(2m + 1))| > C(1 + \lambda^2(2m + 1))^{-N},
\]
for some \( C > 0 \) and \( N \in \mathbb{N} \). Moreover, for fixed \( m < k \), we define
\[
\mu_m = \min_{\lambda \in \mathbb{R}} |P(\lambda, |\lambda|(2m + 1))| > 0.
\]

Let \( M \) be a positive constant such that \( M < \min\{\mu_m : m = 1, \ldots, k - 1\} \).

Hence \( |P(\lambda, |\lambda|(2m + 1))| > M \) for \( m < k \). Putting these two estimates together shows that there exist a positive constant \( C \) and a natural number \( N \) such that
\[
|P(\lambda, |\lambda|(2m + 1))| > C(1 + \lambda^2(2m + 1))^2 \quad \text{for every } m.
\]
Since the symbol of \( t^D \) is \( P(-\lambda, |\lambda|(2m + 1)) \), it follows from (15) that, for all \( g \in \mathcal{S}(H_1) \),
\[
\pi_{\lambda}(g) h^\lambda_m = \frac{\pi_{\lambda}(t^D g) h^\lambda_m}{P(-\lambda, |\lambda|(2m + 1))},
\]
whence
(20)
\[
\pi_{-\lambda}(g) h^\lambda_m = \frac{\pi_{-\lambda}(t^D g) h^\lambda_m}{P(\lambda, |\lambda|(2m + 1))}
\]
and, recalling (11),
(21)
\[
|\hat{g}(-\lambda, m, m)| = \frac{|(t^D g)^\lambda(-\lambda, m, m)|}{|P(\lambda, |\lambda|(2m + 1))|} \leq C \frac{\|t^D g\|_{L^1}}{|P(\lambda, |\lambda|(2m + 1))|}.
\]
Set \( A = I + L^2 \); then \( t^A = A \) and, by replacing \( D \) with \( A^{N+2} \) in (20),
we get
\[
\langle K, g \rangle = \frac{1}{(2\pi)^2} \sum_{m=0}^{\infty} \frac{\hat{g}(-\lambda, m, m) |\lambda|}{P(\lambda, |\lambda|(2m + 1))} \, d\lambda
\]
Moreover, by (19), we get
\[
|\langle K, g \rangle| \leq \frac{\|A^{N+2} g\|_{L^1}}{(2\pi)^2} \sum_{m=0}^{\infty} \frac{|\lambda| \cdot |P(\lambda, |\lambda|(2m + 1))|^{-1}}{(1 + \lambda^2(2m + 1)^2)^{N+2}} \, d\lambda
\]
\[
\leq C \|A^{N+2} g\|_{L^1} \sum_{m=0}^{\infty} \frac{|\lambda|(1 + \lambda^2(2m + 1)^2)^N}{(1 + \lambda^2(2m + 1)^2)^{N+2}} \, d\lambda
\]
\[
\leq C \|A^{N+2} g\|_{L^1} \sum_{m=0}^{\infty} \frac{1}{(2m + 1)^2} \int_0^{\infty} \frac{t}{(1 + t^2)^{N+2}} \, dt
\]
\[
\leq C' \|A^{N+2} g\|_{L^1} \leq C'' \|g\|_{(n)}
\]
where \( \| \cdot \|_{(n)} \) is a continuous Schwartz norm. Therefore \( K \) is a tempered distribution. Let us show that it is a fundamental solution. We verify that \( DK = \delta \), by testing both sides of the identity on a Schwartz function \( f \) and applying (20):
\[
\langle DK, f \rangle = \langle K, t^D f \rangle = \frac{1}{(2\pi)^2} \sum_{m=0}^{\infty} \frac{\langle t^D f, (t^D g)^\lambda(-\lambda, m, m) \rangle}{P(\lambda, |\lambda|(2m + 1))} \, d\lambda
\]
\[
= \frac{1}{(2\pi)^2} \sum_{m=0}^{\infty} \frac{\langle \pi_{-\lambda}(t^D f) h^\lambda_m, h^\lambda_m \rangle}{P(\lambda, |\lambda|(2m + 1))} \, d\lambda
\]
Therefore a solution of the problem $Du = f$ is

$$u(x, y, t) = (f * K)(x, y, t).$$

5. Second case: a finite number of intersections, all away from the vertical ray. We now turn to the case in which the algebraic curve $P(\lambda, \xi) = 0$ intersects the Heisenberg fan in a finite number of points, all of them belonging to the oblique rays and different from the origin.

Let us begin, for simplicity, by assuming that $\{(\lambda, \xi) \in \mathbb{R}^2 : P(\lambda, \xi) = 0\}$ intersects the fan with multiplicity $h \geq 1$ in one single point, lying on the $k$th ray. We can assume that this point has the form $(\alpha, |\alpha|(2k + 1))$, with $\alpha > 0$. Therefore, there exists a polynomial $Q(\lambda)$ such that, for $\lambda \geq 0$,

$$P(\lambda, |\lambda|(2k + 1)) = (\lambda - \alpha)^h Q(\lambda) \quad \text{and} \quad Q(\lambda) \neq 0; \quad \text{for} \lambda < 0, \quad P(\lambda, |\lambda|(2m + 1)) \neq 0 \quad \text{for} \ m \neq k \text{ and } \lambda \in \mathbb{R}.

Given a $C^\infty$ function $\varphi(x)$, define

$$R_{h, \alpha}(\varphi(x)) = \varphi(x) - \sum_{j=0}^{h-1} \frac{\varphi^{(j)}(\alpha)}{j!} (x - \alpha)^j.$$

If $g(x)$ is a rational function with a pole of order $h$ at $\alpha$ and $I$ is an interval containing $\alpha$, then

$$\varphi \mapsto \int_I R_{h, \alpha}(\varphi) g(x) \, dx$$

is a well defined distribution, which is a modified version of Hadamard’s finite part (see [8], Ch. 2, Sec. 2, Example 2).

Note that

$$(22) \quad R_{h, \alpha}((x - \alpha)^h g(x)) = (x - \alpha)^h g(x).$$

**THEOREM 5.1.** Consider $D = P(-iT, -L)$ and suppose that $P$ is as above. Then $D$ has a fundamental solution $K \in \mathcal{S}'(H_1)$, defined as follows: for all $g \in \mathcal{S}(H_1)$,

$$\langle K, g \rangle = \frac{1}{(2\pi)^2} \sum_{m=0}^{\infty} \langle K_m, g \rangle$$
where

\[
(K_m, g) = \int_{\mathbb{R}} \frac{\hat{g}(-\lambda, m, m)\lambda}{P(\lambda, |\lambda|(2m+1))} d\lambda \quad \text{for} \ m \neq k,
\]

\[
(K_k, g) = \int_{\mathbb{R}\setminus[0,2\alpha]} \frac{\hat{g}(-\lambda, k, k)\lambda}{P(\lambda, |\lambda|(2k+1))} d\lambda + \int_{0}^{2\alpha} \frac{R_{h,\alpha}(\hat{g}(-\lambda, k, k))\lambda}{P(\lambda, |\lambda|(2k+1))} d\lambda.
\]

**Proof.** All of the integrals converge absolutely. For all \( m \neq k \) and for all \( \lambda \in \mathbb{R} \), \( P(\lambda, |\lambda|(2m+1)) \neq 0 \), so we can argue as in the proof of Theorem 4.3 to show that

\[
\left| \sum_m (K_m, g) \right| \leq C \|g\|_N, \quad \text{for some} \ N \gg 0.
\]

If \( m = k \), we have

\[
(K_k, g) = \int_{\mathbb{R}\setminus[0,2\alpha]} \frac{\hat{g}(-\lambda, k, k)\lambda}{P(\lambda, |\lambda|(2k+1))} d\lambda + \int_{0}^{2\alpha} \int_{H_1}^{H_2} \frac{d^h}{d\lambda^h} \varphi_{-\lambda, k}(v) \lambda |g(v)| h!Q(\lambda) d\lambda dv,
\]

where \( \xi \) is strictly between \( \alpha \) and \( \lambda \). The first term is absolutely convergent because again \( P(\lambda, |\lambda|(2k+1)) \neq 0 \) in \( \mathbb{R} \setminus [0,2\alpha] \). If we apply estimate (5) to the derivatives of the functions \( \varphi_{-\lambda, m}(v) \) we can show that also the second integral is absolutely convergent, so we deduce that \( K \) is a tempered distribution. Let us show that it is a fundamental solution of \( D \). Using (22) we have

\[
(K_k, tDf) = \int_{\mathbb{R}\setminus[0,2\alpha]} \hat{f}(-\lambda, k, k)\lambda d\lambda + \int_{0}^{2\alpha} \frac{R_{h,\alpha}(P(\lambda, |\lambda|(2k+1))\hat{f}(-\lambda, k, k))\lambda}{P(\lambda, |\lambda|(2k+1))} d\lambda = \int_{\mathbb{R}\setminus[0,2\alpha]} \hat{f}(-\lambda, k, k)\lambda d\lambda + \int_{0}^{2\alpha} \frac{P(\lambda, |\lambda|(2k+1))\hat{f}(-\lambda, k, k)|\lambda|}{P(\lambda, |\lambda|(2k+1))} d\lambda.
\]

Therefore

\[
(K, tDf) = \frac{1}{(2\pi)^2} \sum_{m=0}^{\infty} \int_{\mathbb{R}} \hat{f}(-\lambda, m, m)\lambda d\lambda = f(0,0,0).
\]
This result extends in an obvious way to those operators $D = P(-iT, -L)$ such that $P(\lambda, \xi) = 0$ intersects the fan with finite multiplicity in finitely many points, all of them belonging to the oblique rays and different from the origin.

**Corollary 5.2.** Let $D = P(-iT, -L)$ be such that $P(\lambda, |\lambda|(2m + 1)) = 0$ only at finitely many points, say $(\lambda_{j,h}, |\lambda_{j,h}||(2m_j + 1))$, $j = 1, \ldots, r$, $h = 1, \ldots, r_j$, each of them lying on the curve $\xi = |\lambda|(2m_j + 1)$ and having multiplicity $\mu_{j,h}$. Suppose also that $P(0, \xi) \neq 0$ for all $\xi \geq 0$. Let $I_{j,h}$ be intervals centered at $\lambda_{j,h}$ such that $I_{j,h} \cap I_{j,h'} = \emptyset$ if $h \neq h'$. Then $D$ has a fundamental solution

$$\langle K, g \rangle = \frac{1}{(2\pi)^2} \left( \sum_{j=1}^{r} \langle K_{m_j}, g \rangle + \sum_{m \notin \{m_1, \ldots, m_r\}} \langle K_m, g \rangle \right), \quad g \in S(H_1),$$

where

$$\langle K_m, g \rangle = \int_{\mathbb{R}} \frac{\hat{g}(-\lambda, m, m)|\lambda|}{P(\lambda, |\lambda|(2m + 1))} d\lambda \quad \text{if} \quad m \notin \{m_1, \ldots, m_r\},$$

$$\langle K_{m_j}, g \rangle = \int_{\bigcup I_{j,h}} \frac{R_{\mu_{j,h}, \lambda_{j,h}}(\hat{g}(-\lambda, m_j, m_j)|\lambda|)}{P(\lambda, |\lambda|(2m_j + 1))} d\lambda$$

$$+ \int_{\mathbb{R} \setminus \bigcup I_{j,j}} \frac{\hat{g}(-\lambda, m_j, m_j)|\lambda|}{P(\lambda, |\lambda|(2m_j + 1))} d\lambda.$$
for all \( x, y \geq 0 \) with \( \sqrt{x^2 + y^2} \geq 1 \).

**Proof.** Define \( Q(x, y) = P(x + 1/\sqrt{2}, y) \). Then \( Q(x, y) \neq 0 \) for all \( x \geq 0, y \geq 0 \), therefore by Lemma 4.2 we have the estimate

\[
|Q(x, y)| > C_1(1 + x^2 + y^2)^{-N_1}
\]

for all \( x \geq 0, y \geq 0 \). Hence

\[
|P(x, y)| = |Q(x - 1/\sqrt{2}, y)| > C_2(1 + x^2 + y^2)^{-N_1}
\]

for all \( x \geq 1/\sqrt{2}, y \geq 0 \). In the same way we can show that there exist \( C_3 > 0 \) and \( N_2 \in \mathbb{N} \) such that, for all \( x \geq 0, y \geq 1/\sqrt{2},
\]

\[
|P(x, y)| > C_3(1 + x^2 + y^2)^{-N_2}.
\]

If we take \( C = \max(C_2, C_3) \) and \( N = \min(N_1, N_2) \), we get the estimate (24) for all \( x, y \geq 0 \) with \( \sqrt{x^2 + y^2} \geq 1 \).

We begin with the case where the origin is the only zero.

**Theorem 6.2.** Suppose that

\[
P(\lambda, \xi) = c_\xi \xi^k + \sum_{|\alpha| = k, \alpha \neq \xi} c_\alpha \xi^\alpha \lambda^{\alpha_2} + \sum_{|\alpha| > k} c_\alpha \xi^\alpha \lambda^{\alpha_2}, \quad c_\alpha \in \mathbb{C}, \quad c_\xi \neq 0,
\]

and that \( P(\lambda, |\lambda|(2m + 1)) \neq 0 \) for all \( \lambda \neq 0, m \in \mathbb{N} \). Define the distribution \( K \), for all \( g \in \mathcal{S}(H_1) \), by

\[
\langle K, g \rangle = \frac{1}{(2\pi)^2} \sum_{m=0}^{\infty} \left\{ \int_{|\lambda| \geq \delta/(2m+1)} \frac{\tilde{g}(-\lambda, m, m)|\lambda|}{P(\lambda, |\lambda|(2m+1))} d\lambda + \int_{|\lambda| < \delta/(2m+1)} \frac{R_{k+N_m-1,0}(\tilde{g}(-\lambda, m, m)|\lambda|}{P(\lambda, |\lambda|(2m+1))} d\lambda \right\}
\]

where \( N_m \in \mathbb{N} \) is zero except for finitely many \( m \in \mathbb{N} \) and \( \delta \) is a suitable positive constant. Then \( K \) is a fundamental solution of \( D = P(-iT, -L) \).

**Proof.** Let \( \sigma \) be a positive constant. On the line \( \xi = \lambda/\sigma, P(\lambda, \xi) \) takes the value

\[
P_\sigma(\xi) = P(\sigma \xi, \xi) = \xi^k \left( c_\xi + \sum_{|\alpha| = k, \alpha \neq \xi} c_\alpha \sigma^{\alpha_2} \right) + \sum_{|\alpha| > k} c_\alpha \sigma^{\alpha_2} \xi^{|\alpha|}.
\]

Note that \( c_\xi + \sum_{|\alpha| = k, \alpha \neq \xi} c_\alpha \sigma^{\alpha_2} \) tends to \( c_\xi \) as \( \sigma \to 0 \). Therefore, if \( \sigma \) is small enough, the quantity \( |c_\xi + \sum_{|\alpha| = k, \alpha \neq \xi} c_\alpha \sigma^{\alpha_2}| \) is not zero and can be bounded from below by a positive constant. Thus, there exists \( \sigma_0 \) such that for all \( \sigma \leq \sigma_0 \), \( |c_\xi + \sum_{|\alpha| = k, \alpha \neq \xi} c_\alpha \sigma^{\alpha_2}| \geq C_1 > 0 \).

Since

\[
\sum_{|\alpha| > k} c_\alpha \sigma^{\alpha_2} \xi^{|\alpha|} = o(\xi^k) \quad \text{as} \quad \xi \to 0,
\]

(24) holds for all \( x, y \geq 0 \) with \( \sqrt{x^2 + y^2} \geq 1 \).
if \( \xi \) is small enough, then \( \sum_{|\alpha|>k} c_\alpha \sigma^{\alpha_2} \xi^{\alpha} \) is negligible with respect to \( \xi^k \). Therefore there exists \( \delta_0 > 0 \) such that if \( \xi \leq \delta_0 \) and \( \sigma \leq \sigma_0 \), then

\[
|P_\sigma(\xi)| \geq \left| C_1 \xi^k - \sum_{|\alpha|>k} c_\alpha \sigma^{\alpha_2} \xi^{\alpha}\right| \geq C_2 |\xi|^k.
\]

Thus, in the triangle

\[
\mathcal{E} = \{(\lambda, \xi) \in \mathbb{R}^2 : |\lambda|/\sigma_0 \leq \xi \leq \delta_0 \}
\]

we have \( |P(\lambda, \xi)| \geq C |\xi|^k \).

Hence,

\[
|P(\lambda, \lambda/((2m + 1))] | \geq C (2m + 1)^k \lambda^k
\]

for all \( m \geq 1/(2\sigma_0) - 1/2 \) and all \( \lambda \) such that \( |\lambda| \leq \delta_0 \sigma_0 \).

For finitely many \( m < 1/(2\sigma_0) - 1/2 \), it may happen that the sum \( c_\alpha + \sum_{|\alpha|=k, \alpha \neq \pi} c_\alpha / (2m + 1)^{\alpha_2} \) is zero. Therefore, for all \( m < 1/(2\sigma_0) - 1/2 \), there exist \( N_m \in \mathbb{N} \) and \( \delta_1 > 0 \) such that, if \( |\lambda| < \delta_1 \), then

\[
|P(\lambda, \lambda/((2m + 1))] | > M \lambda^{k+N_m}.
\]

Put \( \delta = \min(\sigma_0, \delta_0, \delta_1) \) and let us show that \( K \) in (25) is a tempered distribution. Note that

\[
\langle K, g \rangle = \frac{1}{(2\pi)^2} \sum_{m=0}^{\infty} (I_1^m + I_2^m),
\]

where

\[
I_1^m = \int_{|\lambda| \geq \delta/(2m+1)} \frac{\tilde{g}(\lambda, m, m)|\lambda|}{P(\lambda, \lambda/((2m + 1))] } d\lambda
\]

\[
I_2^m = \sum_{|\lambda| < \delta/(2m+1)} \left[ \frac{d^{k+N_m-1}}{d \lambda^{k+N_m-1}} \tilde{g}(\lambda, m, m) \right]_{\lambda = \lambda_m} \lambda^{k+N_m-1} |\lambda| \frac{\lambda^{k+N_m-1} |\lambda|}{(k+N_m-1)! P(\lambda, \lambda/((2m + 1))] } d\lambda
\]

and \( \lambda_m \) in \( I_2^m \) is a value between 0 and \( \lambda \), for all \( m \).

Take \( m < 1/(2\sigma_0) - 1/2 \). Then \( I_1^m \) is absolutely convergent because \( P(\lambda, \lambda/((2m + 1))] \neq 0 \) for all \( \lambda \) such that \( |\lambda| \geq \delta/(2m+1) \). By estimate (27) we get

\[
\left| \frac{d^{k+N_m-1}}{d \lambda^{k+N_m-1}} \phi_{-\lambda, m}(v) \right|_{\lambda = \lambda_m} \leq C \left| \phi_{-\lambda, m} \right|_{\infty} \leq C,
\]

where \( \phi_{-\lambda, m}(v) = \tilde{g}(\lambda, m, m) \frac{\lambda^{k+N_m-1} |\lambda|}{(k+N_m-1)! P(\lambda, \lambda/((2m + 1))] } \frac{d^{k+N_m-1}}{d \lambda^{k+N_m-1}} \phi_{-\lambda, m}(v) \).
Therefore the integrals occurring in \( K \), corresponding to \( m < 1/(2\sigma_0) - 1/2 \), are absolutely convergent.

Consider now the infinitely many terms in \( K \) labeled by \( m \geq 1/(2\sigma_0) - 1/2 \). Recall that \( N_m = 0 \) for such \( m \). By applying (7) to the derivatives of \( \varphi_{-\lambda,m}(v) \), and (26) to the polynomial \( P \), we get

\[
\left| \int_{|\lambda| \leq \delta/(2m+1)} \frac{d^{k-1}}{d\lambda^{k-1}} \varphi_{-\lambda,m}(v) \right|_{|\lambda| = \lambda_m} \frac{\lambda^k}{(k-1)!(P(\lambda,|\lambda|(2m+1))} \leq C \| g \|_1.
\]

It follows that, for all \( m \geq 1/(2\sigma_0) - 1/2 \),

\[
|I_m^2| \leq \frac{C_2}{(m+1)^2} \int_{H_1} \left( (|t| + (x^2 + y^2))^{k-1} |g(x,y,t)| \right) dx dy dt
\]

\[
= \frac{C_2}{(m+1)^2} \|((|t| + (x^2 + y^2))^{k-1} g) \|_1 \leq \frac{C_3}{(m+1)^2} \| g \|_{(N)},
\]

for some \( N \gg 0 \).

By hypothesis, estimate (24) holds for \( P \). Let \( h = N \) be the exponent appearing in (24) and \( A = -L(I + L^2)^{h+1} \). Then, by (21), we have

\[
\left| \int_{|\lambda| \leq \delta/(2m+1)} \frac{\tilde{g}(-\lambda,m,m)}{P(\lambda,|\lambda|(2m+1))} \right| d\lambda
\]

\[
\leq \frac{C_1}{2m+1} \| \tilde{g} \|_1 \| A \|_1
\]

\[
\leq \frac{C_1}{(2m+1)^2} \int_{|t| \geq \delta} \frac{dt}{1+t^2} \leq \frac{C_2}{(2m+1)^2} \| \tilde{g} \|_1.
\]

Therefore \( K \in S'(H_1) \). Let us show that it is a fundamental solution of \( D \):
\[ \langle K, iDf \rangle = \langle K, P(-iT, -L)f \rangle \]

\[
= \frac{1}{(2\pi)^2} \sum_{m=0}^{\infty} \left\{ \begin{array}{c}
\langle \hat{f}(-\lambda, m, m) | \lambda \rangle d\lambda \\
|\lambda| \geq \delta/(2m + 1)
\end{array} \right.

+ \left\{ \begin{array}{c}
\frac{R_{k+N_m-1,0} (iDf)^\wedge (-\lambda, m, m) | \lambda \rangle}{P(\lambda, |\lambda|(2m + 1))} d\lambda \\
|\lambda| < \delta/(2m + 1)
\end{array} \right. 
\]

\[
= \frac{1}{(2\pi)^2} \sum_{m=0}^{\infty} \left\{ \begin{array}{c}
\langle \hat{f}(-\lambda, m, m) | \lambda \rangle d\lambda \\
|\lambda| \geq \delta/(2m + 1)
\end{array} \right.

+ \left\{ \begin{array}{c}
\frac{R_{k+N_m-1,0} (P(\lambda, |\lambda|(2m + 1)) \hat{f}(-\lambda, m, m) | \lambda \rangle}{P(\lambda, |\lambda|(2m + 1))} d\lambda \\
|\lambda| < \delta/(2m + 1)
\end{array} \right. 
\]

\[
= \frac{1}{(2\pi)^2} \sum_{m=0}^{\infty} \left\{ \begin{array}{c}
\hat{f}(-\lambda, m, m) | \lambda \rangle d\lambda = f(0, 0, 0) \\
|\lambda| \geq \delta/(2m + 1)
\end{array} \right.

+ \left\{ \begin{array}{c}
\frac{R_{k+N_m-1,0} (P(\lambda, |\lambda|(2m + 1)) \hat{f}(-\lambda, m, m) | \lambda \rangle}{P(\lambda, |\lambda|(2m + 1))} d\lambda \\
|\lambda| < \delta/(2m + 1)
\end{array} \right. 
\]

since

\[
\frac{R_{k+N_m-1,0} (P(\lambda, |\lambda|(2m + 1)) \hat{f}(-\lambda, m, m))}{P(\lambda, |\lambda|(2m + 1))} = P(\lambda, |\lambda|(2m + 1)) \hat{f}(-\lambda, m, m). \]

Putting together the results obtained up to now, we can generalize Theorem 6.2, allowing \( P(\lambda, \xi) \) to have zeros on \( F \) also outside the origin.

**Theorem 6.3.** Suppose \( P(\lambda, \xi) \) has the form (23) and let \( P(\lambda, \xi) \) vanish on \( F \) only at the origin and at finitely many points, \( (\lambda_j, |\lambda_j|(2m_j + 1)), j = 1, \ldots, r, \) with multiplicity \( \mu_j \). Choosing a sufficiently small positive constant \( \delta \), let \( I_j \) be intervals centered at \( \lambda_j \) such that

\[
I_j \cap \left( -\frac{\delta}{2m_j + 1}, \frac{\delta}{2m_j + 1} \right) = \emptyset
\]

and \( I_j \cap I_{j'} = \emptyset \) if \( m_j = m_{j'} \). Put also

\[
B_m = \mathbb{R} \setminus \left[ \left( -\frac{\delta}{2m_j + 1}, \frac{\delta}{2m_j + 1} \right) \cup \bigcup_{m=m_j} I_j \right].
\]

Define the distribution \( K \), for all \( g \in S(H_1) \), by

\[
\langle K, g \rangle = \frac{1}{(2\pi)^2} \sum_{m=0}^{\infty} \left\{ \begin{array}{c}
\int_{B_m} \hat{g}(-\lambda, m, m) |\lambda\rangle d\lambda \\
|\lambda| \geq \delta/(2m + 1)
\end{array} \right.

+ \left\{ \begin{array}{c}
\frac{R_{k+N_m-1,0} (\hat{g}(-\lambda, m, m) | \lambda \rangle)}{P(\lambda, |\lambda|(2m + 1))} d\lambda \\
|\lambda| < \delta/(2m + 1)
\end{array} \right. 
\]
\[ + \frac{1}{(2\pi)^2} \sum_{j=1}^{r} \frac{R_{\mu_j,\lambda}(\hat{g}(-\lambda, m_j, m_j)) \lambda! d\lambda}{P(\lambda, |\lambda|(2m_j + 1))}, \]

where \( N_m = 0 \) except for finitely many \( m \in \mathbb{N} \). Then \( K \) is a fundamental solution of \( D \).

REFERENCES