COLLOQUIUM MATHEMATICUM

VOL. 83 2000 NO. 2

ON WEAK TYPE INEQUALITIES FOR RARE MAXIMAL FUNCTIONS

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Abstract. The properties of rare maximal functions (i.e. Hardy–Littlewood maximal functions associated with sparse families of intervals) are investigated. A simple criterion allows one to decide if a given rare maximal function satisfies a converse weak type inequality. The summability properties of rare maximal functions are also considered.

1. Introduction. For a locally integrable function $f:\mathbb{R}^d\to\mathbb{R}$ the classic Hardy–Littlewood maximal function Mf is defined as

$$Mf(x)=\sup_{I\ni x}\frac{1}{|I|}\int_I |f(y)|\,dy$$

where the supremum is taken over all bounded cubic intervals $I\subseteq\mathbb{R}^d$ containing x. It is well known that the Hardy–Littlewood maximal function does not map from L to L, but only from L to weak L. In particular, the following weak type inequality holds for every $f\in L$ and arbitrary positive λ:

$$\frac{c_d}{\lambda}\int_{\{x:Mf(x)>\lambda\}}|f|\le|\{x:Mf(x)>\lambda\}|\le\frac{C_d}{\lambda}\int_{\{x:Mf(x)>\lambda\}}|f|. \tag{1}$$

A well known theorem of Hardy and Littlewood [2] states that if f is supported on the unit cube $\mathbb{I}^d$ and $f\in L\log^+L(\mathbb{I}^d)$, then $Mf\in L(\mathbb{I}^d)$. Later, Stein [3] proved that the converse of this theorem also holds: if $f\in L$ and $Mf\in L(\mathbb{I}^d)$, then $f\in L\log^+L(\mathbb{I}^d)$.

The proofs of these results are based on the weak type inequalities (1) stated above, and these, in turn, are proved by using the Vitali and Whitney covering lemmas. Of course, the covering lemmas depend on properties of the family of sets to which the argument is applied, thus it is natural to consider a rare maximal function where the supremum is taken over a restricted set of intervals, and ask whether these inequalities and Stein's phenomenon remain true.

2000 *Mathematics Subject Classification*: Primary 42B25.
Research of K. Hare supported in part by NSERC.

We shall see that this need not be the case. Indeed, in this note we characterize the rare maximal functions which satisfy the weak type inequality. We also show that Stein's phenomenon need not hold for individual functions f, but that rare maximal functions never map an entire Orlicz class larger than $L\log^+ L$ into L.

2. Weak type inequalities. For simplicity we restrict ourselves to the one-dimensional case which is entirely typical. Let $l = \{l_k\}$ where $l_k \le 1$, $l_k \downarrow 0$, and let

$$\mathbf{I} = \{\text{intervals } I \subset \mathbb{R} : |I| \in l\}.$$

We define the *rare maximal function* $M_l f$ by the formula

$$M_l f(x) = \sup_{I\in\mathbf{I},\, I\ni x} \frac{1}{|I|}\int_I |f(y)|\,dy.$$

Define $E_\lambda \equiv \{x : M_l f(x) > \lambda\}$. For every $\lambda > 0$, E_λ is an open set and is the union of the intervals I such that

$$|I|^{-1}\int_I |f| > \lambda.$$

Applying a Vitali covering argument to these intervals yields the weak type inequality

$$|E_\lambda| \le \frac{2}{\lambda}\int_{E_\lambda} |f|.$$

The situation with the converse inequality is quite different.

THEOREM 1. *Let $l = \{2^{-m_k}\}$ with $m_k \in \mathbb{R}^+$. The rare maximal function $M_l f$ satisfies the weak type inequality*

$$\frac{1}{\lambda}\int_{\{x:M_l f(x)>\lambda\}} |f| \le C|\{x : M_l f(x) > \lambda\}|$$

for some constant C and every $f \in L$ if and only if

$$\sup_k (m_{k+1} - m_k) < \infty.$$

Proof. First, we will prove that if $\sup_k (m_{k+1} - m_k) = \infty$ then there exists a summable function f such that

$$\sup_\lambda \frac{\lambda^{-1}\int_{\{M_l f>\lambda\}} |f|}{|\{x : M_l f(x) > \lambda\}|} = \infty.$$

To do this we use the assumption that $\sup_k (m_{k+1} - m_k) = \infty$ to inductively define a subsequence $\{m_{k_n}\}$ as follows: Choose k_1 such that $m_{k_1+1} - m_{k_1} > 1$. Given $m_{k_1}, \ldots, m_{k_{n-1}}$, select k_n so that $m_{k_n} > 1 + m_{k_{n-1}+1}$ and

$m_{k_n+1} - m_{k_n} > n$. Let $\{\alpha_j\} = \{m_{k_1}, m_{k_1+1}, m_{k_2}, m_{k_2+1}, \ldots\}$ and set

$$f(x) = \sum_{k=1}^{\infty} a_k \chi_{[0;2^{-\alpha_k}]}(x)$$

where $a_k > 0$ will be specified later. Observe that $\{2^{-\alpha_j}\}$ is a lacunary sequence. Since f is a decreasing function for positive x, which vanishes for negative x, $M_l f(-|x|) \le M_l f(|x|)$ and $M_l f(x)$ decreases for positive x. In particular, for $x \in [2^{-\alpha_{n+1}}; 2^{-\alpha_n}]$ we have $M_l f(x) \ge M_l f(2^{-\alpha_n})$. Also, notice that

$$|[2^{-m_{k_j+1}}; 2^{1-m_{k_j+1}}]| \ge 2^{-m_n}$$

for all $n > k_j$, and that f is constant on the interval $[2^{-m_{k_j+1}}; 2^{-m_{k_j}}]$. It follows that if $x \in [2^{1-m_{k_j+1}}; 2^{-m_{k_j}}]$ then

$$M_l f(x) = M_l f(2^{-m_{k_j}}) = 2^{m_{k_j}} \int_0^{2^{-m_{k_j}}} |f(y)|\, dy.$$

These observations imply that

$$[0; 2^{-m_{k_j+1}}] \subset \{x > 0 : M_l f(x) > M_l f(2^{-m_{k_j}})\} \subset [0; 2^{1-m_{k_j+1}}],$$

and hence

$$|\{x : M_l f(x) > M_l f(2^{-m_{k_j}})\}| \le 2^{2-m_{k_j+1}}.$$

One can show by direct calculation (using the lacunarity of $\{2^{-\alpha_j}\}$) that

$$\int_0^{2^{-\alpha_n}} |f| \sim \sum_{s<n} a_s 2^{-\alpha_n} + \sum_{s \ge n} a_s 2^{-\alpha_s}.$$

Thus if n is chosen such that $m_{k_j+1} = \alpha_n$ then

$$\int_{\{M_l f(x) > M_l f(2^{-m_{k_j}})\}} |f| \ge \int_0^{2^{-m_{k_j+1}}} |f| \ge C \sum_{s \ge n} a_s 2^{-\alpha_s}.$$

Furthermore, if we let $\lambda = M_l f(2^{-m_{k_j}})$, then

$$\lambda \sim \sum_{s=1}^{n-1} a_s + 2^{m_{k_j}} \sum_{s=n}^{\infty} a_s 2^{-\alpha_s}.$$

Hence

$$\begin{aligned}
\frac{\lambda^{-1} \int_{\{M_l f > \lambda\}} |f|}{|\{x : M_l f(x) > \lambda\}|} &\ge \frac{C \sum_{s \ge n} a_s 2^{-\alpha_s}}{(\sum_{s<n} a_s + 2^{m_{k_j}} \sum_{s \ge n} a_s 2^{-\alpha_s}) 2^{-m_{k_j+1}}} \\
&\ge \frac{C \sum_{s \ge n} a_s 2^{-\alpha_s} (2^{m_{k_j+1} - m_{k_j}})}{\sum_{s<n} 2^{-m_{k_j}} a_s + \sum_{s \ge n} a_s 2^{-\alpha_s}}.
\end{aligned}$$

Set now $a_s = 2^{\alpha_s}/s^2$. Then

$$\sum_{s=n}^{\infty} a_s 2^{-\alpha_s} = \sum_{s=n}^{\infty} \frac{1}{s^2} \sim \frac{1}{n}.$$

Since $\alpha_{n-1} = m_{k_j}$ we have

$$2^{-m_{k_j}} \sum_{s=1}^{n-1} a_s = 2^{-m_{k_j}} \sum_{s=1}^{n-1} \frac{2^{\alpha_s}}{s^2} \sim 2^{-m_{k_j}} \frac{2^{\alpha_{n-1}}}{n^2} = \frac{1}{n^2} = o\left(\frac{1}{n}\right).$$

Consequently, for all $j \in \mathbb{N}$,

$$\frac{\lambda^{-1}\int_{\{M_l f > \lambda\}} |f|}{|\{x : M_l f(x) > \lambda\}|} \ge C2^{m_{k_j+1}-m_{k_j}} \ge C2^j,$$

and hence

$$\sup_{\lambda} \frac{\lambda^{-1}\int_{\{M_l f > \lambda\}} |f|}{|\{x : M_l f(x) > \lambda\}|} = \infty.$$

Now assume $\sup(m_{k+1} - m_k) \equiv c_0 < \infty$. Note that E_λ is a disjoint union of intervals J, so $|E_\lambda| = \sum |J|$. For every such J there is an index k and intervals J^* and J_* such that

$$J_* \subset J \subset J^*, \quad J^* \ne J, \quad |J_*| = 2^{-m_k}, \quad |J^*| = 2^{-m_{k+1}}.$$

But $J^* \not\subset E_\lambda$, hence

$$|J^*|^{-1} \int_{J^*} |f| \le \lambda.$$

Since $m_{k+1} - m_k \le c_0$ for every k,

$$|E_\lambda| \ge \sum |J_*| \ge 2^{-c_0} \sum |J^*| \ge 2^{-c_0} \sum \frac{1}{\lambda} \int_{J^*} |f|$$
$$\ge 2^{-c_0} \sum \frac{1}{\lambda} \int_J |f| = 2^{-c_0} \frac{1}{\lambda} \int_{\bigcup J} |f| = 2^{-c_0} \frac{1}{\lambda} \int_{E_\lambda} |f|$$

and this is the desired inequality. ∎

3. Stein's phenomenon. Standard arguments show that any rare maximal function satisfying the weak type inequality of Theorem 1 has Stein's property (cf. [1, 6.1]). In contrast, our next result shows that suitable rare maximal functions do not.

THEOREM 2. *There exists a sequence l such that for some $f \in L(\mathbb{I})$, $f \notin L\log^+ L(\mathbb{I})$ but $M_l f \in L(\mathbb{I})$.*

Proof. We will demonstrate that if $\{m_k\} \subset \mathbb{N}$ is a strictly increasing sequence satisfying

$$\sup_{k\in\mathbb{N}} m_k/k = \infty$$

and $l = \{2^{-(m_k+3)}\}$, then Stein's phenomenon does not hold for the rare maximal function $M_l f$.

To see this, set $E_k = [0; 2^{-m_k}]$ and $G_k = E_k \backslash E_{k+1} = (2^{-m_{k+1}}; 2^{-m_k}]$. Also, set

$$G_k^+ = (2^{-m_{k+1}}; (2^{-m_{k+1}} + 2^{-m_k})/2], \qquad G_k^- = ((2^{-m_{k+1}} + 2^{-m_k})/2; 2^{-m_k}].$$

Notice $|G_k| \geq 2^{-m_k-1}$, while $|G_k^+| = |G_k^-| = \frac{1}{2}|G_k|$. Choose an increasing subsequence m_{n_i} of m_k such that $m_{n_i} \geq 2^i n_i$. Set $a_k = 0$ if $k \notin \{n_i\}$ and $a_k = 1/m_{n_i}$ if $k = n_i$. Define

$$f(x) = \sum_{k\geq 1} 2^{m_k} a_k \chi_{G_k^+}(x).$$

The function f belongs to $L(\mathbb{I})$ since

$$\sum_{k\geq 1} 2^{m_k} a_k |G_k^+| \leq \sum_{k\geq 1} a_k \leq \sum_{i\geq 1} 2^{-i} < \infty.$$

Next we show that $f \notin L\log^+ L(\mathbb{I})$. For this we first observe that

$$\sum_{a_k\neq 0} |a_k \log a_k| \leq \sum_{i\geq 1} \frac{\log(2^i n_i)}{2^i n_i} < \infty.$$

Now

$$\begin{aligned} \|f\log^+ f\| &= \sum_{a_k\neq 0} 2^{m_k} a_k \log(2^{m_k} a_k)|G_k^+| \\ &\geq \frac{1}{4}\sum_{a_k\neq 0} a_k m_k \log 2 + \frac{1}{4}\sum_{a_k\neq 0} a_k \log a_k. \end{aligned}$$

But clearly the first series diverges while the second is convergent. So $f \notin L\log^+ L(\mathbb{I})$.

Now we estimate $M_l f$. Let $x \in G_k$ and I be any interval of length $|I| = 2^{-m_n-3}$ containing x. If $I \cap G_{k+1}^+ \neq \emptyset$, then since the interval G_k is separated from G_{k+1}^+ by G_{k+1}^-, it follows that $|I| \geq |G_{k+1}^-| \geq 2^{-m_{k+1}-2}$. As $|I| = 2^{-m_n-3}$ for some n and $|I| > 2^{-m_{k+1}-3}$ this means that $|I| \geq 2^{-m_k-3}$. Thus

$$\begin{aligned} \frac{1}{|I|}\int_I f &= \frac{1}{|I|}\Big(\sum_{j>k,\, j\in\{n_i\}} \int_{I\cap G_j^+} f + \sum_{j\leq k,\, j\in\{n_i\}} \int_{I\cap G_j^+} f\Big) \\ &\leq 2^{m_k+3} \sum_{j>k,\, j\in\{n_i\}} 2^{m_j} a_j |G_j^+| + \sum_{j\leq k} 2^{m_j} a_j \frac{|G_j^+ \cap I|}{|I|} \end{aligned}$$

$$\le 2^{m_k+3} \sum_{j>k,\, j\in\{n_i\}} a_j + \max_{j\le k,\, j\in\{n_i\}} 2^{m_j} a_j$$
$$\le 2^{m_k+3} \sum_{j>k,\, j\in\{n_i\}} a_j + 2^{m_{\beta(k)}} a_{\beta(k)}$$

where $\beta(k) = n_i$ if $k \in [n_i; n_{i+1})$ (with the final inequality holding because $\{2^{m_{n_i}} a_{n_i}\}$ is an increasing sequence).

Otherwise $I \cap G_j^+ = \emptyset$ for all $j > k$. Then we have

$$\frac{1}{|I|}\int_I f = \frac{1}{|I|} \sum_{j\le k,\, j\in\{n_i\}} \int_{I\cap G_j^+} f \le \sum_{j\le k,\, j\in\{n_i\}} 2^{m_j} a_j \le \max_{j\le k,\, j\in\{n_i\}} 2^{m_j} a_j.$$

In either case, if $x \in G_k$ then

$$M_l f(x) \le 2^{m_k+3} \sum_{j>k,\, j\in\{n_i\}} a_j + 2^{m_{\beta(k)}} a_{\beta(k)}.$$

Thus

$$\|M_l f\| = \sum_{k\ge 1} \int_{G_k} M_l f \le \sum_{k\ge 1} \Big(2^{m_k+3} \sum_{j>k,\, j\in\{n_i\}} a_j + 2^{m_{\beta(k)}} a_{\beta(k)}\Big)|G_k|$$
$$\le 8 \sum_{k\ge 1} \sum_{j>k,\, j\in\{n_i\}} a_j + \sum_{k\ge 1} 2^{m_{\beta(k)}} a_{\beta(k)} 2^{-m_k}.$$

By switching the order of summation one can see that

$$\sum_{k\ge 1} \sum_{j>k,\, j\in\{n_i\}} a_j \le \sum_{j\in\{n_i\}} j a_j = \sum_{j\ge 1} n_j a_{n_j} < \infty.$$

Moreover,

$$\sum_{k\ge 1} 2^{m_{\beta(k)}} a_{\beta(k)} 2^{-m_k} = \sum_{i\ge 1} 2^{m_{n_i}} a_{n_i} \sum_{k\in[n_i;n_{i+1})} 2^{-m_k}$$
$$\le 2 \sum_{i\ge 1} 2^{m_{n_i}} a_{n_i} 2^{-m_{n_i}} < \infty.$$

Hence $M_l f \in L(\mathbb{I})$. ∎

Let us note that the assumption $\sup_k (m_{k+1} - m_k) = \infty$ is weaker than $\sup_k m_k/k = \infty$. It would be very interesting to investigate whether the condition $\sup_k m_k/k = \infty$ is sharp in Theorem 2. Unfortunately, we are not able to answer this question. If the answer were affirmative it would be a very unexpected fact.

Finally, we show that in the scale of Orlicz classes $\Phi(L)$ it is possible to prove a weak form of Stein's phenomenon. Indeed, we have the following theorem.

THEOREM 3. *Let* $l_k \downarrow 0$ *and* $\Phi : [0;\infty) \to [0;\infty)$ *be some increasing function such that* $\Phi(L) \subset L(\mathbb{I})$. *If* $M_l f \in L(\mathbb{I})$ *for all functions* $f \in \Phi(L)(I)$, *then* $\Phi(L) \subset L \log^+ L(\mathbb{I})$.

Proof. Assume that $\Phi(L) \not\subset L(\log^+ L)$. This is equivalent to the assumption that for

$$\psi(t) \equiv \frac{\Phi(t)}{t \log t}$$

there exist $b_k \uparrow \infty$ as $k \to \infty$ such that

$$\psi(b_k) \downarrow +0 \quad \text{as } k \to \infty.$$

We will show how to construct a function $f \in \Phi(L)$ with $M_l f \notin L$.

Without loss of generality we may assume that $l_k = 2^{-m_k}$ with $m_k \in \mathbb{N}$ and $m_k \uparrow \infty$. Let $r_j(t)$, $j = 0, 1, \ldots$, denote the standard Rademacher functions and define

$$E_k = \{t \in [0;1] : r_{m_j}(t) = 1;\ j = 0, 1, \ldots, k\}.$$

Notice that E_k is a union of dyadic-rational intervals of length 2^{-m_k}, $|E_k| = 2^{-k}$ and $E_k \supset E_{k+1}$. Let $G_k = E_k \backslash E_{k+1}$. By construction the sets G_k are pairwise disjoint. Clearly each G_k is a union of dyadic-rational intervals of length $2^{-m_{k+1}}$ and has measure 2^{-k-1}.

Finally, we define the function

$$f(x) = \sum_{k=1}^{\infty} a_k \chi_{G_k}(x)$$

where a_k are positive numbers which will be specified later. It is obvious that

$$\int \Phi(f) = \sum_{k \geq 1} \Phi(a_k)|G_k| = \sum_{k \geq 1} \psi(a_k) a_k \log(a_k) 2^{-k-1}.$$

We will now estimate from below the rare maximal function. Let $x \in G_k$ and let I be the unique dyadic-rational interval which contains x and has length $|I| = 2^{-m_k}$. Notice I is contained in E_k. The crucial fact is that due to the dyadic structure of G_j the value of the fraction

$$\frac{|G_j \cap I_1|}{|I_1|} = 2^{-j-1+k}$$

does not depend on the concrete choice of m_k for $j \geq k$. Thus

$$\frac{1}{|I|} \int_I f(y)\,dy = \sum_{j=1}^{\infty} \frac{1}{|I|} \int_I a_j \chi_{G_j}(y)\,dy \geq \sum_{j \geq k} a_j \frac{|G_j \cap I|}{|I|} = \sum_{j \geq k} a_j 2^{-j-1+k}.$$

This means
$$M_l f(x) \geq \sum_{j\geq k} a_j 2^{-j-1+k}\chi_{G_k}(x),$$
and hence
$$\|M_l f\| \geq \sum_{k\geq 1}\sum_{j\geq k} a_j 2^{-j-1+k}|G_k| \geq \sum_{k\geq 1}\sum_{j\geq k} a_j 2^{-j-1+k}2^{-k-1}.$$
Changing the order of summation we have
$$\|M_l f\| \geq \sum_{j\geq 1}\sum_{k\leq j} a_j 2^{-j-2} = \sum_{j\geq 1} a_j 2^{-j-2} j.$$
These calculations show that if we can find $a_k \geq 1$ such that
$$\sum_{k\geq 1} a_k 2^{-k} k = \infty$$
and
$$\sum_{k\geq 1} \psi(a_k) a_k \log(a_k) 2^{-k} < \infty,$$
then $f \in \Phi(L)$ but $M_l f \notin L(I)$.

Without loss of generality we may assume that $\psi(b_k) \leq 2^{-k}$ and $b_{k+1} \geq 2b_k$. The second assumption ensures that there exists a strictly increasing sequence $\{n_j\}$ of positive integers such that
$$\frac{2^{n_j}}{n_j} \leq b_j < \frac{2^{n_j+1}}{n_j+1}.$$
Set $a_k = b_j$ if $n_j \leq k < n_{j+1}$. Then it is easy to check that $\sum a_k 2^{-k} k$ diverges. Furthermore,
$$\sum_{k\geq 1} \psi(a_k) a_k \log(a_k) 2^{-k} \leq \sum_{j\geq 1} \psi(b_j) b_j \log(b_j) \sum_{n_j\leq k} 2^{-k}.$$
But
$$b_j \log b_j \leq C\frac{2^{n_j}}{n_j}\log\left(\frac{2^{n_j}}{n_j}\right) \leq C2^{n_j}$$
and $\psi(b_j) \leq 2^{-j}$, thus
$$\sum_{k\geq 1} \psi(a_k) a_k \log(a_k) 2^{-k} \leq C\sum_{j\geq 1} \psi(b_j) < \infty. \ \blacksquare$$

COROLLARY. *Let $l_k \downarrow 0$ and α be a positive number. If $M_l f \in L$ for all functions $f \in L(\log^+ L)^\alpha$, then $\alpha \geq 1$.*

The theorem above shows that there are no conditions in terms of the growth of the individual function f, except the trivial condition $f \in L\log^+ L$, which guarantee the summability of the rare maximal operator. However, it is easy to see that such a condition may be found in terms of the integral smoothness of f.

Namely, assume that the function f is defined on the unit torus and introduce the modulus of continuity of f in the standard way:

$$\omega(f;h) = \sup_{|t|\le h} \|f(\cdot + t) - f(\cdot)\|.$$

Then

$$\begin{aligned} M_l f(x) &\le \sup_{k\ge 1} \frac{1}{l_k} \int_{-l_k}^{l_k} |f(x+t) - f(x)|\,dt + f(x) \\ &\le \sum_{k\ge 1} \frac{1}{l_k} \int_{-l_k}^{l_k} |f(x+t) - f(x)|\,dt + f(x). \end{aligned}$$

Thus

$$\begin{aligned} \|M_l f\| &\le \sum_{k\ge 1} \left\| \frac{1}{l_k} \int_{-l_k}^{l_k} |f(\cdot + t) - f(\cdot)|\,dt \right\| + \|f\| \\ &\le 2\sum_{k\ge 1} \frac{1}{l_k} \int_{0}^{l_k} \|f(\cdot + t) - f(\cdot)\|\,dt + \|f\| \le 2\sum_{k\ge 1} \omega(f;l_k) + \|f\|. \end{aligned}$$

Recall that the case $l_k = 2^{-k}$ corresponds to the Hardy–Littlewood maximal function Mf. So the condition

$$\sum_{k\ge 1} \omega(f;2^{-k}) < \infty \tag{2}$$

is sufficient for the summability of Mf for the individual function f. Thus (2) implies that $f \in L\log^+ L$ and this condition is sharp in the sense that for an arbitrary modulus of continuity $\omega(\delta)$ with $\sum \omega(2^{-k}) = \infty$, there exists a function f such that $\omega(f;\delta) \le \omega(\delta)$, while $f \notin L\log^+ L$ (for details see [4]).

Thus there is no improvement of the class of summability for the Hardy–Littlewood maximal operator of smooth functions. However, if l_k is a more rare sequence, such that (2) is not true, but

$$\sum_{k\ge 1} \omega(f;l_k) < \infty, \tag{3}$$

then (3) is a sufficient condition for the summability of the rare maximal function for the individual function, which is weaker than the inclusion of f in $L\log^+ L$.

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Received 5 May 1999 (3812)