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WHITNEY MAPS—A NON-METRIC CASE

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Abstract. It is shown that there is no Whitney map on the hyperspace 2^X for non-metrizable Hausdorff compact spaces X. Examples are presented of non-metrizable continua X which admit and ones which do not admit a Whitney map for $C(X)$.

Given a Hausdorff compact space X, we consider the space 2^X of all non-empty compact subsets of X equipped with the Vietoris topology. Any subspace $H(X)$ of the space 2^X is called a *hyperspace* of X. In particular $F_n(X)$ stands for the family of all non-empty subsets of X of cardinality at most n (where $n \in \mathbb{N}$), and $C(X)$ denotes the hyperspace of subcontinua of X (i.e., of connected members of 2^X). The reader is referred to [4] and [5] for needed information on hyperspaces.

A continuum X containing two points a and b is called an *arc* (from a to b) provided that each point of $X \setminus \{a, b\}$ separates a and b in X. We write ab to denote an arc with end points a and b. Note that an arc is metrizable if and only if it is homeomorphic to the closed unit interval $[0, 1]$.

Given a Hausdorff compact space X and its hyperspace $H(X)$, by a *Whitney map* for $H(X)$ we mean a mapping $\mu : H(X) \to ab$ such that

(0.1) $\mu(\{x\}) = a$ for each point $x \in X$;

(0.2) $A \subsetneq B$ implies $\mu(A) < \mu(B)$.

When X is a compact metric space, then a Whitney map for 2^X or $C(X)$ does always exist, and several constructions of such mappings are known: see e.g. [5, 0.50.1–0.50.3, pp. 25–26] or [4, Theorem 13.4, p. 107; Exercises 13.5–13.8, pp. 108–109].

The following theorem answers a question of Robert Heath asked during a private conversation with the second named author.

Theorem 1. *The following conditions are equivalent for a Hausdorff compact space X:*

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(1.1) *X is metrizable*;

(1.2) *there exists a Whitney map from the hyperspace 2^X onto an arc*;

(1.3) *there exists a Whitney map from the hyperspace $F_2(X)$ onto an arc.*

Proof. As stated above, the implication from (1.1) to (1.2) is well known. The one from (1.2) to (1.3) is obvious. It remains to show that (1.3) implies (1.1). So, assume (1.3). Let $\mu : F_2(X) \to ab$ be a Whitney map. Let $\{x_1, x_2, \ldots\}$ be a countable infinite subset of X. Then there is a cluster point x_0 of this set. Continuity of the mapping $x \mapsto \mu(\{x_0, x\})$ implies that the family $B_n = \{[a\mu(\{x_0, x_n\})) : x_n \neq x_0\}$, $n \in \mathbb{N}$, of half-open intervals is a countable local basis for ab at a. Define $f : X \times X \to ab$ by $f(\langle x, y\rangle) = \mu(\{x, y\})$. Then the diagonal Δ of $X \times X$ is a G_δ-subset since $\Delta = \bigcap\{f^{-1}(B_n) : n \in \mathbb{N}\}$. Therefore X is metrizable by [1, Exercise 4.2.B, p. 264]. ■

Corollary 2. *If a Hausdorff compact space is non-metrizable*, *then there is no Whitney map for the hyperspace 2^X.*

For any *dendron* (i.e., a Hausdorff continuum such that any two of its distinct points are separated by a third one) X there is a canonical embedding of $F_2(X)$ in $C(X)$ (which maps any pair $\{x, y\}$ with $x \neq y$ to the unique arc xy). Therefore we get the following.

Observation 3. *For any non-metrizable dendron X there is no Whitney map from the hyperspace $C(X)$. In particular*, *there is no Whitney map from the hyperspace $C(X)$ for any non-metrizable arc X.*

The assumption that X is a dendron is essential in the above observation. To see this recall the following example K (see [2] and [3]).

Let D be a totally disconnected Hausdorff compact space without isolated points (in the metric case only the Cantor set has these properties), and let $f : D \to D$ be a homeomorphism such that for each $x \in D$ the orbits of x, i.e., the sets $\{f^n(x) : n \in \{0\} \cup \mathbb{N}\}$ and $\{f^{-n}(x) : n \in \{0\} \cup \mathbb{N}\}$ are dense in D. In the product $D \times [0, 1]$ identify $\langle x, 0\rangle$ with $\langle f(x), 1\rangle$ for each $x \in D$. Let K be the quotient space. It is shown in [3] that each proper subcontinuum of K is a metric arc and that covering dimension of K is one.

Observe that using the natural flow on K one can easily define, for any arc A in K, the length $\ell(A)$ of A. Then a Whitney map $\mu : C(K) \to [0, 1]$ can be defined as follows:

$$\mu(A) = \frac{\ell(A)}{1 + \ell(A)} \quad \text{for } A \in C(K) \setminus \{K\}, \text{ and } \mu(K) = 1.$$

Problem 4. Characterize non-metrizable continua X for which there exists a Whitney map for $C(X)$.

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