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MATRICES OVER UPPER TRIANGULAR BIMODULES AND Δ-FILTERED MODULES OVER QUASI-HEREDITARY ALGEBRAS

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Abstract. Let Λ be a directed finite-dimensional algebra over a field k, and let B be an upper triangular bimodule over Λ. Then we show that the category of B-matrices $\mathrm{mat}\, B$ admits a projective generator P whose endomorphism algebra $\mathrm{End}\, P$ is quasi-hereditary. If $\mathcal{A}$ denotes the opposite algebra of $\mathrm{End}\, P$, then the functor $\mathrm{Hom}(P, -)$ induces an equivalence between $\mathrm{mat}\, B$ and the category $\mathcal{F}(\Delta)$ of Δ-filtered $\mathcal{A}$-modules. Moreover, any quasi-hereditary algebra whose category of Δ-filtered modules is equivalent to $\mathrm{mat}\, B$ is Morita equivalent to $\mathcal{A}$.

1. Introduction. The aim of this note is to interpret matrices over upper triangular bimodules as Δ-filtered modules over certain quasi-hereditary algebras. We therefore fix a finite-dimensional algebra Λ over a field k, and consider a finite-dimensional Λ-Λ-bimodule B.

The category $\mathrm{mat}\, B$ of matrices over B can be defined as follows: Let $1 = e_1 + \ldots + e_t$ be a decomposition of the unit element of Λ into pairwise orthogonal primitive idempotents. Then the bimodule B decomposes as k-vector space into a direct sum $B = \bigoplus_{i,j} e_iBe_j$. A *matrix over* B is a pair (d, M) where $d = (d_1, \ldots, d_t) \in \mathbb{N}^t$ is a dimension vector and $M = (M_{ij})_{i,j\in\{1,\ldots,t\}}$ is a (square) block matrix whose blocks M_{ij} are matrices of size $d_j \times d_i$ with entries in e_iBe_j.

A morphism in $\mathrm{mat}\, B$ from (d, M) to (d', M') is a block matrix $H = (H_{ij})_{i,j\in\{1,\ldots,t\}}$ whose blocks H_{ij} are matrices of size $d'_j \times d_i$ with entries in $e_i\Lambda e_j$ such that $HM = M'H$.

In our main result, we require that the algebra Λ is directed and the bimodule B is upper triangular over the directed algebra Λ, i.e. there is an ordering of the idempotents $e_1, \ldots, e_t$ such that $e_i(\mathrm{rad}\, \Lambda)e_j = 0$ and $e_iBe_j = 0$ whenever $i \geq j$. Here we denote by $\mathrm{rad}\, \Lambda$ the Jacobson radical of Λ.

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THEOREM 1.1. *Let B be an upper triangular bimodule over a directed algebra Λ. Then the category of B-matrices* $\operatorname{mat} B$ *admits a projective generator P whose endomorphism algebra* $\operatorname{End} P$ *is quasi-hereditary. If $\mathcal{A}$ denotes the opposite algebra of* $\operatorname{End} P$, *then the functor* $\operatorname{Hom}(P, -)$ *induces an equivalence between* $\operatorname{mat} B$ *and the category $\mathcal{F}(\Delta)$ of Δ-filtered $\mathcal{A}$-modules. Moreover, any quasi-hereditary algebra whose category of Δ-filtered modules is equivalent to* $\operatorname{mat} B$ *is Morita equivalent to $\mathcal{A}$.*

In the particular case when the category $\operatorname{mat} B$ consists of subspaces of a directed vector space category, this theorem is proven in [Ba]. Our motivation, however, was to obtain a general result that applies to all upper triangular bimodules, for instance those bimodules stemming from the action of a parabolic group R on unipotent normal subgroups of R: Let Λ be the path algebra of the directed Dynkin quiver of type $\mathbb{A}_t$ and let the bimodule B be the radical of Λ. For a fixed dimension vector $d = (d_1, \ldots, d_t)$, denote by $R(d)$ the opposite group of the group of invertible block matrices $H = (H_{ij})_{i,j \in \{1,\ldots,t\}}$ whose blocks H_{ij} are matrices of size $d_j \times d_i$ with entries in $e_i \Lambda e_j$. Then the group $R(d)$ is a parabolic group, the space of matrices over B with dimension d is the Lie algebra of the unipotent radical of the parabolic $R(d)$, and the action is the adjoint action of the conjugation on the Lie algebra (this has been observed in [D] already). More generally, if $R(d)_u^{(l)}$ denotes the lth member of the descending central series of $R(d)$, we obtain the orbits of R on the Lie algebra $\mathfrak{r}(d)_u^{(l)}$ of $R(d)_u^{(l)}$ as isomorphism classes of matrices over the bimodule $B = (\operatorname{rad} \Lambda)^{l+1}$ with Λ as before. One can generalize this even more to arbitrary unipotent subgroups of $R(d)$; see [BH2].

In a series of papers [HR1], [HR2] and [BH1], recently all instances of parabolic subgroups R in $\mathrm{GL}_n(k)$ acting with a finite number of orbits on $\mathfrak{r}_u^{(l)}$ were classified. One main step in the proof of these results relates (by an *ad hoc* construction) the orbits of the action of R on $\mathfrak{r}_u^{(l)}$ to a classification problem of Δ-filtered modules over a certain quasi-hereditary algebra $\mathcal{A}$, where $\mathcal{A}$ depends only on the number of blocks of R/R_u and on l.

In the present note, we give a general approach and thus explain the occurrence of quasi-hereditary algebras in [HR1], [HR2] and [BH1]. Moreover, this approach allows results concerning Δ-filtered modules over quasi-hereditary algebras (such as the existence of almost split sequences, properties of the Euler form, degeneration of modules) to be applied to the various categories $\operatorname{mat} B$ stemming from subspace categories of directed vector space categories, orbits of parabolic groups R on some unipotent normal subgroup U and other upper triangular situations.

Given an upper triangular bimodule B, we describe how to construct a projective generator P of the category $\operatorname{mat} B$. There is, however, no hope to get the quiver and relations of the endomorphism algebra $\mathcal{A}$ of P directly from B in general. But it is possible to obtain such a description of $\mathcal{A}$ for the problem outlined above: a parabolic subgroup R acting on a unipotent normal subgroup U. This example is considered in detail in [BH2]. In fact, in these instances, the abstract equivalence in Theorem 1.1 preserves not only the orbits, but much more structure. For results related to the geometry of the orbits, we also refer to [BHRZ].

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2. Matrices over bimodules and standardization

2.1. *Basic notation.* We denote by k a fixed field. Our algebras are always finite-dimensional k-algebras with unit, but in general non-commutative. Modules are supposed to be finitely generated left modules. In particular, $\operatorname{mod} k$ is the category of finite-dimensional k-vector spaces and $\operatorname{mod} \Lambda$ is the category of finitely generated left modules over an algebra Λ. All categories are k-categories and an equivalence preserves the underlying k-structure.

For more details on quasi-hereditary algebras, we refer to [DR], and our basic reference for representations of quivers is [R]. We also mention the textbook [GR] for an introduction to representation theory, in particular to matrix problems. For more information on representation theory of posets and vector space categories, we also refer to [S].

2.2. *Bimodules.* It is well known that modules over an algebra Λ correspond to k-linear functors $\operatorname{mod} \Lambda \to \operatorname{mod} k$, likewise for bimodules. For the proof of our main theorem, it turns out to be more convenient to introduce bimodules as bifunctors over arbitrary additive categories. Thus, let Γ be a finite-dimensional k-algebra and $X_1, \ldots, X_t$ be a finite number of pairwise non-isomorphic indecomposable Γ-modules. We denote by $\operatorname{add}\{X_i\}$ the additive hull of the objects $X_1, \ldots, X_t$ in $\operatorname{mod} \Gamma$.

Definition. A *bimodule B over* $\operatorname{add}\{X_i\}$ is a k-bilinear functor

$$(\operatorname{add}\{X_i\})^{\mathrm{op}} \times \operatorname{add}\{X_i\} \to \operatorname{mod} k.$$

Remarks. 1. Let Λ be the opposite algebra of $\operatorname{End}_\Gamma(\bigoplus_{i=1}^t X_i)$. If the algebra Γ is representation finite and $\{X_i\}$ is a set of representatives of the isomorphism classes of indecomposable Γ-modules, then Λ is the Auslander algebra of Γ. More generally, for any vector $d = (d_1, \ldots, d_t)$ of non-negative integers we denote the opposite algebra of $\operatorname{End}_\Gamma(\bigoplus_{i=1}^t X_i^{d_i})$ by $\Lambda(d)$. Now, having a k-bilinear functor $(\operatorname{add}\{X_i\})^{\mathrm{op}} \times \operatorname{add}\{X_i\} \to \operatorname{mod} k$ is equivalent to having a bimodule B over Λ.

2. The bifunctors $\mathrm{Hom}_\Gamma(-,-), \mathrm{Ext}^i_\Gamma(-,-)$ and $\mathrm{rad}_\Gamma(-,-)$ provide natural examples of bimodules over $\mathrm{add}\{X_i\}$. Here $\mathrm{rad}_\Gamma(X,Y)$ denotes the k-vector space of radical morphisms from X to Y (cf. [GR, §3.2] or [R, §2.2]). With these examples in mind, we use for any bimodule B over $\mathrm{add}\{X_i\}$ the following notation: If $\phi : X' \to X$ and $\psi : Y \to Y'$ are morphisms in $\mathrm{add}\{X_i\}$ and m is an element of $B(X,Y)$, then we denote the element $B(\phi,\psi)(m)$ of $B(X',Y')$ by $\phi m \psi$.

Definition. Let B be a bimodule over $\mathrm{add}\{X_i\}$. A *matrix over* B is a pair (X,m), where $X \in \mathrm{add}\{X_i\}$ and $m \in B(X,X)$. The matrices over B form an exact category $\mathrm{mat}\, B$ (see Lemma 2.1) whose morphism set from (X,m) to (X',m') consists of all morphisms $\phi : X \to X'$ in $\mathrm{add}\{X_i\}$ satisfying $\phi m = m'\phi$.

Note that the equality $\phi m = m'\phi$ is defined in the vector space $B(X,X')$. The notation $m\phi$ for $B(X,\phi)(m)$ suggests viewing this equation as a commutative diagram:

$$\begin{array}{ccc} X & \stackrel{m}{\to} & X \\ \phi\downarrow & & \downarrow\phi \\ X' & \stackrel{m'}{\to} & X' \end{array}$$

keeping in mind that ϕ is a morphism, whereas m and m' are elements of $B(X,X)$ and $B(X',X')$, respectively.

2.3. *Exact structures.* We fix a bimodule B over $\mathrm{add}\{X_i\}$ and denote by $\mathcal{E}$ the class of pairs (i,d) of composable morphisms in $\mathrm{mat}\, B$

$$0 \to (X',m') \stackrel{i}{\to} (X,m) \stackrel{d}{\to} (X'',m'') \to 0$$

such that the induced sequence of Γ-modules

$$0 \to X' \stackrel{i}{\to} X \stackrel{d}{\to} X'' \to 0$$

is split exact in $\mathrm{add}\{X_i\}$.

Lemma 2.1 ([T, §2]). *The pair* $(\mathrm{mat}\, B, \mathcal{E})$ *is an exact category in the sense of* [Q].

Note that $(\mathrm{mat}\, B, \mathcal{E})$ is not an abelian category in general. There is, however, an exact embedding into an abelian category:

Proposition 2.2 ([Q], [GR, §9.1]). *If* $(\mathcal{C},\mathcal{E})$ *is a skeletally small exact category, then there is an equivalence* $G : \mathcal{C} \to \mathcal{D}$ *onto a full subcategory* $\mathcal{D}$ *of an abelian category such that* $\mathcal{D}$ *is closed under extensions and that* $\mathcal{E}$ *is formed by the composable pairs* (i,d) *inducing exact sequences*

$$0 \to GX \stackrel{Gi}{\to} GY \stackrel{Gd}{\to} GZ \to 0.$$

We refer to [K, App. A] for a proof.

The previous proposition allows us to use most of the concepts known from categories of modules, thus we can form extension groups in $\operatorname{mat} B$ with respect to the exact structure defined by $\mathcal{E}$ etc. We note that $\operatorname{Ext}^2(-,-)$ vanishes for all objects in $\operatorname{mat} B$ (see e.g. [T]). Thus, we just write Ext for the first extension group in the exact category $(\operatorname{mat} B, \mathcal{E})$.

2.4. *Triangular bimodules.* We now impose a triangularity condition on the bimodule B in order to obtain the existence of a projective generator of the category $\operatorname{mat} B$.

DEFINITION. The bimodule B over $\operatorname{add}\{X_i\}$ is *triangular* if $B(X_i, X_j) = 0$ whenever $i \geq j$.

Note that this definition depends on the chosen numeration of the X_i's. When B is a triangular bimodule, we identify the Γ-module X_i with the B-matrix $(X_i, 0)$. Let $\mathcal{F}(X)$ denote the full subcategory of all B-matrices which have a filtration by elements in $\{X_1, \ldots, X_t\}$.

LEMMA 2.3. *Let B be a triangular bimodule over* $\operatorname{add}\{X_i\}$. *Then*

(i) $\operatorname{mat} B = \mathcal{F}(X)$;

(ii) $\operatorname{Ext}(X_i, X_j) = 0$ *for* $i \geq j$;

(iii) $\operatorname{mat} B$ *has enough projectives.*

Proof. (i) We only have to show $\operatorname{mat} B \subset \mathcal{F}(X)$. Therefore, let (X, m) be a B-matrix with $X \neq 0$. Let $i \in \{1, \ldots, t\}$ be the smallest index such that X_i is a direct summand of X. Then (X, m) is isomorphic to a B-matrix

$$\left(X' \oplus X_i, \begin{bmatrix} m' & * \\ 0 & 0 \end{bmatrix}\right),$$

where $m' \in B(X', X')$ and $* \in B(X_i, X')$. Therefore we obtain a short exact sequence

$$0 \to (X', m') \xrightarrow{i} (X, m) \xrightarrow{d} (X_i, 0) \to 0.$$

The desired filtration of (X, m) is constructed by induction on the number of direct summands of X. Consequently, $\operatorname{mat} B \subset \mathcal{F}(X)$.

(ii) Recall that $\operatorname{Ext}(X_i, X_j)$ is formed by short exact sequences

$$0 \to (X_j, 0) \xrightarrow{i} (X, m) \xrightarrow{d} (X_i, 0) \to 0.$$

As discussed in part (i), the B-matrix (X, m) is isomorphic to

$$\left(X_j \oplus X_i, \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix}\right),$$

where $* \in B(X_i, X_j)$.

(iii) Recall that an object P is projective if $\operatorname{Ext}(P, X) = 0$ for all objects X. Since we have an additive exact category with simple objects $X_i = (X_i, 0)$, this is equivalent to showing $\operatorname{Ext}(P, X_i) = 0$ for all simple objects

X_i. We construct an object P_i inductively in the following way: $X_i^0 := X_i$ and X_i^l is the universal extension of X_i^{l-1} by simple objects:

$$0 \to \bigoplus_r \operatorname{Ext}(X_i^{l-1}, X_r)^* \otimes X_r \to X_i^l \to X_i^{l-1} \to 0.$$

If we apply $\operatorname{Hom}(-, X_j)$ to this sequence, then we obtain a long exact sequence

$$0 \to \operatorname{Hom}(X_i^{l-1}, X_j) \to \operatorname{Hom}(X_i^l, X_j) \to \bigoplus_r \operatorname{Hom}(X_r, X_j) \otimes \operatorname{Ext}(X_i^{l-1}, X_r)$$
$$\to \operatorname{Ext}(X_i^{l-1}, X_j) \to \operatorname{Ext}(X_i^l, X_j)$$
$$\to \bigoplus_r \operatorname{Ext}(X_r, X_j) \otimes \operatorname{Ext}(X_i^{l-1}, X_r) \to 0.$$

By construction of the universal extension, the connecting homomorphism is surjective, thus $\operatorname{Ext}(X_i^l, X_j) \simeq \bigoplus_r \operatorname{Ext}(X_r, X_j) \otimes \operatorname{Ext}(X_i^{l-1}, X_r)$. Applying induction we obtain $\operatorname{Ext}(X_i^l, X_j) = 0$ whenever $i \geq l + j$. Thus the process of extending X_i^l must stop and $X_i^l = X_i^{l_0}$ for some index l_0 and all $l \geq l_0$. In fact, $l_0 \leq t$. We denote the object $X_i^{l_0}$ by P_i'. It is a projective object by construction, but *not* indecomposable in general. By P_i we denote the unique indecomposable direct summand which contains X_i in its filtration. Thus P_i is indecomposable and projective.

It remains to show that we have enough projectives. Let (X, f) be a matrix. Then $(X, 0) \simeq (\bigoplus X_i^{a_i}, 0)$ for certain natural numbers a_i. There exists a map $\bigoplus P_i^{a_i} \to (X, f)$ sending the top of P_i to $(X_i, 0)$; this map yields a surjection. ∎

We note that the last statement was already proven in [T]. However, since we use a different language in this note, we prove this statement again for the convenience of the reader.

Since we have enough projective objects in $\operatorname{mat} B$ when the bimodule B is triangular, we can construct a finite-dimensional algebra $\mathcal{A}$ so that $\operatorname{mat} B$ is a full subcategory of the category $\operatorname{mod} \mathcal{A}$ of left finitely generated $\mathcal{A}$ modules. The category $\operatorname{mat} B$ consists of those modules which have a filtration by the images of $(X_i, 0)$ for the various i.

LEMMA 2.4. *Let $\mathcal{A}$ be the opposite algebra of the endomorphism algebra of a projective generator of* $\operatorname{mat} B$. *Then* $\operatorname{mat} B$ *is a full subcategory of* $\operatorname{mod} \mathcal{A}$.

Proof. This is a standard result. We fix a projective generator P. Then the functor $X \mapsto \operatorname{Hom}(P, X)$ defines a full exact embedding of $\operatorname{mat} B$ into $\operatorname{mod} \operatorname{End}(P)^{\mathrm{op}}$. ∎

In this way, we obtain a particular module category in which we can embed our exact category, not just an abstract existence result as in Proposition 2.2. It is natural to ask what is the relation between the algebra $\Lambda = (\mathrm{End}_\Gamma(\bigoplus_{i=1}^t X_i))^{\mathrm{op}}$ and the algebra $\mathcal{A}$ constructed above. In many examples it turns out that Λ is a subalgebra of $\mathcal{A}$, but in general it is only a subquotient:

Proposition 2.5. *There exists a subalgebra $\overline{\Lambda}$ of $\mathcal{A}$ such that Λ is a quotient of $\overline{\Lambda}$.*

Proof. We first construct the algebra $\overline{\Lambda}$. Let $P \to \bigoplus_{i=1}^t X_i$ be a surjection of the projective generator P onto the direct sum of simple objects in $\mathrm{mat}\, B$, and let U be the kernel. Thus we have an exact sequence

$$0 \to U \to P \to \bigoplus_{i=1}^{t} X_i \to 0.$$

We define $\overline{\Lambda}$ to be the opposite algebra of the algebra of all endomorphisms f of P satisfying $f(U) \subset U$. Recall that Λ is the opposite algebra of $\mathrm{End}(\bigoplus_{i=1}^t X_i)$. Since $f \in \overline{\Lambda}$ maps U into itself, it induces an endomorphism of $\bigoplus_{i=1}^t X_i$. In this way, we obtain an algebra homomorphism $\overline{\Lambda} \to \Lambda$. We claim that this algebra homomorphism is surjective: let g be an element of $\mathrm{End}(\bigoplus_{i=1}^t X_i)$. Then g induces a map $g' \in \mathrm{Hom}(P, \bigoplus_{i=1}^t X_i)$ which lifts to a map in $\mathrm{End}(P)$ since P is projective and $P \to \bigoplus_{i=1}^t X_i$ is surjective. This shows the claim. ■

2.5. *Upper triangular bimodules and quasi-hereditary algebras*

Definition. The bimodule B over $\mathrm{add}\{X_i\}$ is *upper triangular* if the algebra Λ is directed (i.e. $\mathrm{rad}_\Lambda(X_i, X_j) = 0$ whenever $i \geq j$) and if $B(X_i, X_j) = 0$ whenever $i \geq j$.

Remark. Lemma 2.3 can be reformulated as follows: if the bimodule B is upper triangular, then $X_1, \ldots, X_t$ is standardizable in the sense of [DR, §3] and $\mathrm{mat}\, B = \mathcal{F}(X)$. Thus each upper triangular bimodule B defines a quasi-hereditary algebra $\mathcal{A} = \mathcal{A}(B)$, unique up to Morita equivalence. The aim of this part is to make this correspondence more precise (Theorem 2.7). We need the following result.

Theorem 2.6 ([DR, §3]). *Let $X_1, \ldots, X_t$ be a standardizable set of objects of an abelian category $\mathcal{C}$. Then there exists a quasi-hereditary algebra $\mathcal{A}$, unique up to Morita equivalence, such that the subcategory $\mathcal{F}(X)$ of $\mathcal{C}$ and the category of all Δ-filtered $\mathcal{A}$-modules are equivalent.*

If we combine Proposition 2.2 and Theorem 2.6, then we immediately obtain our main result, which is a restatement of Theorem 1.1. We denote by $\Delta(i)$ the images of X_i under the embedding of Proposition 2.2. Moreover,

$\mathcal{F}(\Delta)$ denotes the category of Δ-filtered $\mathcal{A}$-modules, that is, the category of those modules which have a filtration by the various $\Delta(i)$.

THEOREM 2.7. *Let B be an upper triangular bimodule over* $\operatorname{add}\{X_i\}$ *and $\mathcal{A}$ be the opposite algebra of the endomorphism algebra of a projective generator P of* $\operatorname{mat} B$. *Then the algebra $\mathcal{A}$ is quasi-hereditary and the functor* $\operatorname{Hom}(P,-)$ *induces an equivalence between* $\operatorname{mat} B$ *and* $\mathcal{F}(\Delta)$. *Moreover, any quasi-hereditary algebra whose category of Δ-filtered modules is equivalent* (*as an exact category*) *to* $\operatorname{mat} B$ *is Morita equivalent to $\mathcal{A}$.*

Proof. The proof of the existence and uniqueness up to Morita equivalence of the algebra $\mathcal{A}$ follows from Proposition 2.2 and Theorem 2.6: We embed the category $\operatorname{mat} B$ in an abelian category $\mathcal{D}$, as stated in Proposition 2.2. Then we apply the theorem above. Assume we have two different embeddings into categories $\mathcal{D}$ and $\mathcal{D}'$ as in Proposition 2.2. Then the corresponding images are equivalent exact categories. By construction (see the proof of Theorem 2.4 in [DR]), we obtain Morita equivalent quasi-hereditary algebras $\mathcal{A}$ and $\mathcal{A}'$. The particular form of $\mathcal{A}$ follows by construction. ∎

REMARK. Thus we have a functor $\operatorname{Hom}(P,-)$ which identifies $\operatorname{mat} B$ with $\mathcal{F}(\Delta) \subset \operatorname{mod} \mathcal{A}$. This functor maps the simple object X_i to the simple object $\Delta(i)$ of $\mathcal{F}(\Delta)$ and maps the indecomposable projective object $P(i)$ to the corresponding projective $\mathcal{A}$-module.

We note that it is natural to work with $\operatorname{proj} \Lambda$ instead of $\operatorname{add} X_i$. Then Λ is a quotient of a subalgebra of $\mathcal{A}$ by Proposition 2.5. In several examples (see [BH2]), we obtain Λ as a subalgebra of $\mathcal{A}$, and, moreover, the category of Δ-filtered modules coincides with those modules which are projective over Λ. However, we do not have a characterization of those bimodules, or algebras $\mathcal{A}$, with this property.

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