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ACTIONS OF PARABOLIC SUBGROUPS IN GL_n *ON UNIPOTENT NORMAL SUBGROUPS AND QUASI-HEREDITARY ALGEBRAS*

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Abstract. Let R be a parabolic subgroup in GL_n. It acts on its unipotent radical R_{u} and on any unipotent normal subgroup U via conjugation. Let Λ be the path algebra $k\mathbb{A}_t$ of a directed Dynkin quiver of type $\mathbb{A}$ with t vertices and B a subbimodule of the radical of Λ viewed as a Λ-bimodule. Each parabolic subgroup R is the group of automorphisms of an algebra $\Lambda(d)$, which is Morita equivalent to Λ. The action of R on U can be described using matrices over the bimodule B. The advantage of this description is that each bimodule B gives rise to an infinite number of those actions simultaneously: for each d in $\mathbb{N}^t$ we obtain a parabolic group $R(d)$, which is the group of invertible elements in $\Lambda(d)$, together with a unipotent normal subgroup $U(d)$ in $R(d)$.

All those bimodules B are upper triangular with respect to the natural order of Λ. Then, according to [BH2], Theorem 1.1, there exists a quasi-hereditary algebra $\mathcal{A}$ such that the orbits of $R(d)$ on $U(d)$ are in bijection to the isomorphism classes of Δ-filtered $\mathcal{A}$-modules of dimension vector d. We compute the quiver and relations of the quasi-hereditary algebra $\mathcal{A}$ corresponding to the action of the parabolic group $R(d)$ on $U(d)$. Moreover, we show that the Lie algebra of $R(d)$ can be identified with the algebra $\Lambda(d)$, and the Lie algebra of $U(d)$ is isomorphic to a bimodule $B(d)$ over $\Lambda(d)$.

1. Introduction. Let k be a fixed ground field. All instances of parabolic subgroups R in $\mathrm{GL}_n(k)$ acting with a finite number of orbits on its unipotent radical R_{u} were classified in [HR], Theorem 1.1. More generally, let $R_{\mathrm{u}}^{(l)}$ be the lth member of the descending central series; we can also consider the action of R on the Lie algebra $\mathfrak{r}_{\mathrm{u}}^{(l)}$ of $R_{\mathrm{u}}^{(l)}$. Also, all instances of pairs (R,l), where R acts with a finite number of orbits on $\mathfrak{r}_{\mathrm{u}}^{(l)}$, were classified in [BH1], Theorem 1.2. Moreover, we introduced the notion of a tame and of a wild action and classified also those instances. One main step in the proof of the results above relates the orbits of the action of R on $\mathfrak{r}_{\mathrm{u}}^{(l)}$ to a classification problem of modules over a certain quasi-hereditary algebra $\mathcal{A}$, where $\mathcal{A}$ depends only on the number of blocks of R/R_{u} and on l.

The aim of this paper is to obtain the analogous result for the action of R on any unipotent normal subgroups U (Theorems 4.1 and 5.1).

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Recall that a *Borel subgroup* in GL_t is the stabilizer of a proper maximal flag $0 = V_0 \subset V_1 \subset \ldots \subset V_{t-1} \subset V_t = V$ in a t-dimensional vector space V. We choose a fixed decomposition $V_i = V_{i-1} \oplus W_i$ for each $i = 1, \ldots, t$. Let $d = (d_1, \ldots, d_t)$ be a dimension vector, that is, d_i is an integer with $d_i \geq 0$ for $i = 1, \ldots, t$. Let M_i be a d_i-dimensional vector space. We denote by $V(d)$ the vector space $\bigoplus_{i=1}^t W_i \otimes M_i$ and by $R(d)$ the stabilizer of its flag

$$0 \subset V_1(d) := W_1 \otimes M_1 \subset \ldots \subset V_{t-1}(d) := \bigoplus_{i=1}^{t-1} W_i \otimes M_i \subset V(d) = \bigoplus_{i=1}^{t} W_i \otimes M_i.$$

Conversely, given a parabolic subgroup R in $\mathrm{GL}_n(k)$, there exists a unique sincere dimension vector d, that is, $d_i \geq 1$, with R isomorphic to $R(d)$.

Further, let I be the set of roots of a unipotent normal subgroup U in B. We identify I with a subset in $\{(i,j) \mid 1 \leq i < j \leq t\}$. Note that U stabilizes the flag of V and its Lie algebra is

$$\mathfrak{n} = \{f \in \mathrm{End}(V) \mid f(V_j) \subset V_{i-1} \text{ whenever } (i,j) \notin I\}.$$

Then we define $U(d) = U_I(d)$ to be the unipotent normal subgroup in $R(d)$, whose Lie algebra is

$$\mathfrak{n}(d) = \{f \in \mathrm{End}(V(d)) \mid f(V_j(d)) \subset V_{i-1}(d) \text{ whenever } (i,j) \notin I\}.$$

Given a parabolic subgroup R in GL_n and a unipotent normal subgroup U of R, there exist a set I as above and a sincere dimension vector d with R isomorphic to $R(d)$ and U isomorphic to $U_I(d)$. Moreover, we can choose these isomorphisms equivariant with respect to the action via conjugation.

In Section 4 we define a quasi-hereditary algebra $\mathcal{A} := \mathcal{A}(I)$ which depends only on the root ideal I. We denote the category of Δ-filtered $\mathcal{A}$-modules by $\mathcal{F}(\Delta)$. The Δ-*dimension vector* d of an object X in $\mathcal{F}(\Delta)$ is defined as follows: d_i is the multiplicity of $\Delta(i)$ in a Δ-filtration of X. Further, we fix I for the rest of the paper, and whenever we consider the category $\mathcal{F}(\Delta)$, we refer to the subcategory of Δ-filtered modules over the quasi-hereditary algebra $\mathcal{A} := \mathcal{A}(I)$. Also, the groups $R(d)$ and $U(d)$ are the groups defined by I as above. Our principal result of this paper is Theorem 1.1. It follows from Lemma 3.1 and Theorem 4.1; the details of the proof are given in Section 6. Note that the elements in $U(d)$ form an algebraic variety. Also, there exists an algebraic variety which parametrizes Δ-filtered $\mathcal{A}$-modules of a fixed dimension vector (see Section 6 for a precise definition). Thus we can speak about degenerations and families of orbits in $U(d)$ and about degenerations and families of objects in $\mathcal{F}(\Delta)$ respectively.

THEOREM 1.1. *There exists a natural bijection between the orbits of* $R(d)$ *on* $U(d)$ *and the set of isomorphism classes of objects in the category of* Δ-

filtered $\mathcal{A}$-modules $\mathcal{F}(\Delta)$ of Δ-dimension vector d. This bijection is induced by a morphism of algebraic varieties. In particular, it preserves degenerations and families.

We mention certain applications of this result.

Let I be a fixed root ideal as above and assume $\mathcal{A} := \mathcal{A}(I)$ is Δ-finite, that is, there exist only finitely many isomorphism classes of indecomposable Δ-filtered $\mathcal{A}$-modules. This implies that $R(d)$ acts with a finite number of orbits on $U(d)$ for any d. By recent results in [BHRZ] a classification of the indecomposable objects in $\mathcal{F}(\Delta)$ allows us to compute the orbits of the action and the Auslander–Reiten quiver of $\mathcal{F}(\Delta)$ determines all the Hasse diagrams (or equivalently, the Bruhat–Chevalley order of the action of $R(d)$ on $U(d)$) for the various d in one strike.

Assume for a moment that I is the set of all elements $\{(i,j) \mid 1 \leq i < j \leq t\}$ and let $\mathcal{A}$ be the corresponding quasi-hereditary algebra. By results in [BHRR] Richardson's dense orbit theorem [Ri] is equivalent to the existence of a unique module in $\mathcal{F}(\Delta)$ without self-extensions for each d. Note that in general $R(d)$ does not act with a dense orbit on $U(d)$; for examples we refer to the families in [BH1], Section 4. In fact, it is an open problem to classify all instances when $R(d)$ acts with a dense orbit on $U(d)$.

The paper is organized as follows. In Section 2 we consider actions of linear algebraic groups $R(d)$ defined by a bimodule B over a finite-dimensional basic algebra Λ and a dimension vector d. We also prove certain basic results concerning those actions. In Section 3 we recall some basic facts on matrices over bimodules and quasi-hereditary algebras. We also recall the main result of [BH2] (Theorem 1.1) which relates the orbits of the action of $R(d)$ on $B(d)$ to Δ-filtered modules over a certain quasi-hereditary algebra $\mathcal{A}$ associated with B. The crucial problem is to compute the algebras $\mathcal{A}$ for a certain given class of bimodules B.

The remaining parts of this paper determine $\mathcal{A}$ for two classes of actions, respectively bimodules: the action of $R(d)$ on $U(d)$ and the action of $R(d^+) \times R(d^-)$ on an abelian normal unipotent subgroup $U(J)$ of $R(d^+, d^-)$. Here J is considered as a subset of $\{(i,j) \mid 1 \leq i \leq t^+,\ 1 \leq j \leq t^-\}$ (see Section 5 for a precise definition). Then in Section 4 we define a quasi-hereditary algebra depending on I and state the main result concerning the first action (Theorem 4.1). In Section 5 we define a quasi-hereditary algebra depending on J and obtain an analogous result for the second action (Theorem 5.1). Finally, in the last section we prove the results using categories of flags.

We note that this technique does not generalize to arbitrary upper triangular bimodules. It uses special properties of the path algebra of a directed Dynkin quiver of type $\mathbb{A}$.

Basic notation. We denote by k a fixed field. Our algebras are always finite-dimensional k-algebras with unit, but in general non-commutative. Modules are supposed to be finitely generated left modules, in particular, $\operatorname{mod} k$ is the category of finite-dimensional k-vector spaces and $\operatorname{mod} \Lambda$ is the category of finitely generated left modules over an algebra Λ. Bimodules are also finite-dimensional. We identify Λ-bimodules with left Λ^{env}-modules (see also Section 2). All groups are linear algebraic groups defined over k, where we only consider the k-rational points. If we speak about algebraic varieties and morphisms we allow also ground fields which are not algebraically closed. Also, all categories are k-categories and an equivalence preserves the underlying k-structure. The set I is always a root ideal in the positive roots of GL_t. We identify I with a subset in $\{(i,j) \mid 1 \le i < j \le t\}$. The set J is in addition contained in the set of positive roots of a maximal abelian unipotent normal subgroup H of the Borel subgroup of GL_t. We restrict the action of B to B/H and identify J with a subset of $\{(i,j) \mid 1 \le i \le t^+,\ 1 \le j \le t^-\}$, where $t = t^+ + t^-$.

For basic properties of quasi-hereditary algebras, we refer to [DR]. Our basic reference for representations of quivers and finite-dimensional algebras is [R]. For an introduction to linear algebraic groups we mention [S] and basic facts on roots can be found in [B].

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2. Bimodules and actions of linear algebraic groups. In this section, with any bimodule B over a basic algebra Λ and any dimension vector d for Λ we associate a linear action of an algebraic group $R(d)$ on a vector space $B(d)$. For examples we refer to the next section. Note that Λ is basic precisely if $\Lambda/\operatorname{rad}\Lambda$ is a product of division algebras over k, where $\operatorname{rad}\Lambda$ denotes the radical of Λ.

Let Λ be a finite-dimensional basic algebra and B a finite-dimensional Λ-bimodule. We denote by $\{P_\Lambda(i)\}$ a set of representatives of the isomorphism classes of indecomposable projective Λ-modules, where $1 \le i \le t$. Moreover, let $\{e_i\}$ for $i = 1, \ldots, t$ be a complete set of pairwise orthogonal idempotents of Λ so that $P_\Lambda(i)$ is isomorphic to Λe_i. For any dimension vector $d = (d_1, \ldots, d_t)$, that is, d_i a non-negative integer for $i = 1, \ldots, t$, we denote by $\Lambda(d)$ the opposite algebra of $\operatorname{End}(\bigoplus_{i=1}^t P_\Lambda(i)^{d_i})$. It is well known that $\Lambda(d)$ is Morita equivalent to Λ for any sincere dimension vector d, that is, d_i is positive for $i = 1, \ldots, t$, and $\Lambda(1, \ldots, 1)$ is isomorphic to Λ. Further, we denote by $\Lambda^{\mathrm{env}}(d)$ the algebra $\Lambda(d) \otimes \Lambda(d)^{\mathrm{op}}$, where $\Lambda(d)^{\mathrm{op}}$

denotes the opposite algebra of $\Lambda(d)$. It is well known that $\Lambda(d)$-bimodules can be considered as $\Lambda^{\mathrm{env}}(d)$-modules and vice versa. In particular, a complete set of representatives of the simple Λ^{env}-modules can be obtained as tensor products of simple Λ-modules with simple Λ^{op}-modules and we obtain its minimal projective covers $P_{\Lambda^{\mathrm{env}}}(i,j)$, for $1 \leq i,j \leq t$. These modules $P_{\Lambda^{\mathrm{env}}}(i,j)$ form a complete set of representatives of the isomorphism classes of indecomposable projective Λ^{env}-modules. We can define a bimodule $B(d)$ over $\Lambda(d)$ via

$$B(d) := \mathrm{Hom}_{\Lambda^{\mathrm{env}}}\Big(\bigoplus_{i,j} P(i,j)^{d_i d_j}, B\Big),$$

where we use the identification of Λ-bimodules with Λ^{env}-modules. In this way, for a given bimodule B over Λ and any dimension vector d we obtain a bimodule $B(d)$ over the Morita-equivalent algebra $\Lambda(d)$. Let M_i be a vector space of dimension d_i as in the introduction and denote by $B_{i,j}$ the vector space $e_i B e_j$. Then

$$\Lambda(d)_{i,j} \simeq M_i^* \otimes \Lambda_{i,j} \otimes M_j \quad \text{and} \quad B(d)_{i,j} \simeq M_i^* \otimes B_{i,j} \otimes M_j.$$

Using the natural pairing of a vector space with its dual, we get the formulas for the multiplication in $\Lambda(d) := \bigoplus_{i,j} \Lambda(d)_{i,j}$ and the bimodule structure of $B(d) := \bigoplus_{i,j} B(d)_{i,j}$. Here λ is an element of Λ, b is an element of B, m is an element of M, and μ is an element of M^*:

$$\mu \otimes \lambda \otimes m \cdot \mu' \otimes \lambda' \otimes m' = \mu'(m)\mu \otimes \lambda\lambda' \otimes m'$$

and

$$\mu \otimes \lambda \otimes m \cdot \mu' \otimes b' \otimes m' \cdot \mu'' \otimes \lambda'' \otimes m'' = \mu'(m)\mu''(m')\mu \otimes \lambda b' \lambda'' \otimes m''.$$

Now we fix Λ and denote by $R(d)$ the subgroup of units in $\Lambda(d)$. This group acts on $B(d)$ via conjugation. Note that $B(d)$ has the structure of a vector space and this action is a linear action of $R(d)$ on this vector space $B(d)$. Our first result states some properties of the group $R(d)$.

PROPOSITION 2.1. *The group $R(d)$ is a linear algebraic group, it is open and dense in $\Lambda(d)$. The Lie algebra of $R(d)$ is isomorphic to $\Lambda(d)$ with Lie bracket defined by $[x,y] := xy - yx$. The Lie algebra of the unipotent radical $R(d)_{\mathrm{u}}$ of $R(d)$ is isomorphic to the Jacobson radical of $\Lambda(d)$. For a sincere dimension vector d the algebraic group $R(d)$ is reductive precisely when Λ is semisimple, that is, $\Lambda = \Lambda(1,\ldots,1)$ is a product of division algebras and $R(1,\ldots,1)$ is the product of the multiplicative groups of these division algebras.*

Proof. First note that $\Lambda(d)$ acts on itself via the left regular representation. In this way we obtain an embedding of $\Lambda(d)$ into the matrix ring $\mathrm{End}(\Lambda(d))$. This embedding is an algebra homomorphism and it identifies

$\Lambda(d)$ with a Zariski closed subvariety of $\mathrm{End}(\Lambda(d))$, in fact with a linear subspace. Further, one checks directly that $R(d)$ is just the intersection of the image of $\Lambda(d)$ with $\mathrm{GL}(\Lambda(d))$. Consequently, $R(d)$ is a linear algebraic group, it is open and dense in $\Lambda(d)$. The density follows since $R(d)$ is non-empty and open in the irreducible variety $\Lambda(d)$. Because $R(d)$ is open and dense in the vector space $\Lambda(d)$, its Lie algebra is isomorphic to $\Lambda(d)$. Since it is a closed algebraic subgroup of $\mathrm{GL}(\Lambda(d))$, the Lie bracket of the Lie algebra of $R(d)$ is the restriction of the Lie bracket in $\mathrm{End}(\Lambda(d))$. The latter is known to be $[x, y] = xy - yx$. Because the embedding $\Lambda(d) \subset \mathrm{End}(\Lambda(d))$ is an algebra homomorphism we obtain the Lie bracket in $\Lambda(d)$ by the same formula.

Next we determine the unipotent radical of $R(d)$. Note that an element $1 + x$ for $x \in \operatorname{rad}\Lambda$ is unipotent. Conversely, let $y \in R(d) \subset \Lambda(d)$ be unipotent, that is, $y - 1$ is nilpotent. Consider the image $\overline{y}$ of $1 - y$ in $\Lambda(d)/\operatorname{rad}\Lambda(d)$ and assume $\overline{y} \neq 0$. Then any power $\overline{y}^l$ is also non-zero and consequently $(1 - y)^l$ is non-zero, which contradicts the assumption. Consequently, $1 - y \in \operatorname{rad}\Lambda(d)$. The group $1 + \operatorname{rad}\Lambda(d)$ is normal in $R(d)$ and maximal unipotent. The Lie algebra of $1 + \operatorname{rad}\Lambda(d)$ is $\operatorname{rad}\Lambda(d)$. Thus, the Lie algebra of the unipotent radical of $R(d)$ is isomorphic to $\operatorname{rad}\Lambda(d)$.

Recall that $R(d)$ is reductive precisely when $R(d)_{\mathrm{u}}$ is trivial. By the arguments before, this happens if $\operatorname{rad}\Lambda(d) = 0$. For a sincere dimension vector d this is by Morita theory equivalent to $\operatorname{rad}\Lambda = 0$. Moreover, $\operatorname{rad}\Lambda(d) = 0$ if and only if $\Lambda(d)$ is semisimple. ■

Example. Before we define in the next section matrices over bimodules over arbitrary finite-dimensional algebras we restrict our attention for a moment to particular semisimple algebras. So assume $\Lambda \simeq \bigoplus^t k$. A bimodule over Λ is just a bigraded vector space $B = \bigoplus_{i,j=1}^t B_{i,j}$. Let e_i for $i = 1, \dots, t$ be a standard basis of Λ which is compatible with the chosen identification of Λ with $\bigoplus^t k$. Then B is a bimodule via $e_i b_{i',j'} e_j = \delta_{i,i'}\delta_{j,j'} b_{i,j}$ for any $b_{i,j} \in B_{i,j}$. Here δ denotes the Kronecker symbol. We assign to the bimodule B a quiver Q as follows. The set of vertices Q_0 is $\{1, \dots, t\}$ and we have $d_{i,j}$ arrows from i to j precisely if $\dim B_{i,j} = d_{i,j}$. Note that we can recover B from Q by the formula above. Let $\mathcal{A}$ be the path algebra kQ of the quiver Q. It is well known that isomorphism classes of $\mathcal{A}$-modules of dimension vector d can be parameterized by the representation space, which is isomorphic to $B(d)$. The group $R(d)$ is isomorphic to $\prod_{i=1}^t \mathrm{GL}_{d_i}$, which acts via conjugation on $B(d)$. The isomorphism classes of $\mathcal{A}$-modules of dimension vector d are in one-to-one correspondence with the orbits of this action.

Note that $\mathcal{A}$ is finite-dimensional if and only if B is directed, that is, the quiver Q is directed. Then $\mathcal{A}$ is quasi-hereditary with any order, in particular with the natural one given by Q, that is, $\mathcal{F}(\Delta)$ and mod-$\mathcal{A}$ coincide. This is a very particular example for which Theorem 3.2 holds.

Finally, note that we can define a weighted quiver for any bimodule over a product of division algebras over k.

3. Matrices over upper triangular bimodules and quasi-hereditary algebras. The aim of [BH2] is to generalize the example above to the following situation. Let Λ be any directed algebra. That is, the quiver of $\operatorname{rad}\Lambda$ viewed as $\Lambda/\operatorname{rad}\Lambda$-bimodule is directed. Let B be an upper triangular bimodule over Λ, that is, the quiver of B viewed as a $\Lambda^{\mathrm{env}}/\operatorname{rad}\Lambda^{\mathrm{env}}$-bimodule has arrows $i \to j$ only for $i < j$. First we have to define the analogous category for $\mathcal{A}$-mod. We define two such categories which are equivalent. The first one is the category of matrices over B. Note that the category of matrices can be defined in more general situations, e.g. we do not need an upper triangular bimodule and Λ need not be directed.

For a dimension vector d we already fixed vector spaces M_i of dimension d_i. For a second dimension vector c we now also fix vector spaces N_i of dimension c_i. Then we define vector spaces $\Lambda(d,c)$ and $B(d,c)$ as follows:

$$\Lambda(d,c)_{i,j} := M_i^* \otimes \Lambda_{i,j} \otimes N_j \quad \text{and} \quad B(d,c)_{i,j} := M_i^* \otimes B_{i,j} \otimes N_j.$$

DEFINITION. A *matrix over* B is a pair (d,b) consisting of a dimension vector d and an element b in $B(d)$. A morphism $f : (d,b) \to (c,a)$ is an element f in $\Lambda(d,c)$ which satisfies $fb = af$ in $B(d,c)$. We denote the category of all matrices over B by $\operatorname{mat} B$. It is an additive category which has a natural exact structure (see [BH2], Section 2.3).

First note that the definition is slightly different from the one given in [BH2], Section 2.2, but equivalent: the difference here is that we have fixed certain representatives $P_\Lambda(i)$ and $P_{\Lambda^{\mathrm{env}}}(i,j)$, and identify the pair (d,b) with the pair $(\bigoplus P_\Lambda(i) \otimes M_i, b)$, where b can be considered as an element in $B(\bigoplus P_\Lambda(i) \otimes M_i, \bigoplus P_\Lambda(i) \otimes M_i)$.

LEMMA 3.1. *Two matrices (d,b) and (c,a) are isomorphic in* $\operatorname{mat} B$ *precisely if $d = c$ and there exists an element f in $R(d) \subset \mathrm{GL}(\Lambda(d))$ such that $a = fbf^{-1}$, that is, b and a are in the same $R(d)$-orbit in $B(d)$.*

Proof. First note that the existence of such an f implies that f and $g := f^{-1}$ are morphisms with $fg = 1$ and $gf = 1$. Thus f is an isomorphism in $\operatorname{mat} B$. Conversely, let (d,b) and (c,a) be isomorphic objects in $\operatorname{mat} B$. Then there exist f and g with $fb = af$ and $fg = 1$ and $gf = 1$. In particular, $d = c$ and $f^{-1} = g$. Thus we obtain $a = fbf^{-1}$. ■

Next we need some basic facts on quasi-hereditary algebras. Let $\mathcal{A}$ be a finite-dimensional basic algebra and $(Q_0, \le)$ a total order on the vertices of the quiver of $\mathcal{A}$. Let $P_{\mathcal{A}}(i)$ for $i \in Q_0$ be a complete set of representatives of the indecomposable projective $\mathcal{A}$-modules. We define modules $\Delta(i) := P_{\mathcal{A}}(i)/\eta_i$, where η_i is the image of $\bigoplus_{j<i} \operatorname{Hom}(P_{\mathcal{A}}(j), P_{\mathcal{A}}(i)) \otimes P_{\mathcal{A}}(j)$ in $P_{\mathcal{A}}(i)$.

We denote by $\mathcal{F}(\Delta)$ the category of those $\mathcal{A}$-modules which admit a filtration with factors isomorphic to some $\Delta(j)$ for $j = 1, \ldots, t$. The algebra $\mathcal{A}$ is called *quasi-hereditary* with the given order if all modules $P_{\mathcal{A}}(i)$ for $i = 1, \ldots, t$ are in $\mathcal{F}(\Delta)$ and the modules $\Delta(i)$ have a trivial endomorphism ring.

The theorem below (see [BH2], Theorem 1.1) identifies the category $\operatorname{mat} B$ with the category $\mathcal{F}(\Delta)$ for a certain quasi-hereditary algebra $\mathcal{A}$. The algebra $\mathcal{A}$ can be constructed explicitly for any bimodule B, but this construction is quite complicated and only some very particular examples are explicitly known (see e.g. [BH1]). Moreover, it is in general not clear that the equivalence between $\operatorname{mat} B$ and $\mathcal{F}(\Delta)$ preserves the geometric structure as claimed in our situation in Theorem 1.1.

THEOREM 3.2 ([BH2]). *Let B be an upper triangular bimodule. Then there exists a quasi-hereditary algebra $\mathcal{A}$ so that the category* $\operatorname{mat} B$ *of B-matrices and the category $\mathcal{F}(\Delta)$ of Δ-filtered $\mathcal{A}$-modules are equivalent. Moreover, the algebra $\mathcal{A}$ is unique up to Morita equivalence.*

4. Actions of parabolic subgroups via conjugation and matrix problems. Recall that a standard parabolic subgroup in GL_n has an upper triangular block structure. We denote the size of the blocks by $d = (d_1, \ldots, d_t)$ for $d_i \geq 1$ and all $i = 1, \ldots, t$. Thus for $d = (1, \ldots, 1)$, the corresponding standard parabolic subgroup $R(d)$ is isomorphic to the automorphism group of the algebra $\Lambda = k\mathbb{A}_t$, where $\mathbb{A}_t$ is the directed Dynkin quiver. For a general d, the group $R(d)$ is isomorphic to the automorphism group of the Morita equivalent upper triangular block matrix algebra

$$\Lambda(d) = \begin{array}{|c|c|c|c|c|}\hline * & * & * & \cdots & * \\ \hline 0 & * & * & \cdots & * \\ \hline 0 & 0 & * & \cdots & * \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & 0 & \cdots & 0 \\ \hline \end{array} \begin{array}{c} d_1 \\ d_2 \\ d_3 \\ \\ d_t \end{array}$$
$$\quad d_1 \quad d_2 \quad d_3 \qquad\qquad d_t$$

Now we consider the action of $R(d)$ on a unipotent normal subgroup U via conjugation. The group U also has a block structure: the entries on the diagonal are always unit matrices, and the set of blocks containing non-zero entries is contained in the upper part and is closed under left and upper shift. Hence, U is determined by a sincere dimension vector d and a subset $I \subset \{(i,j) \mid i < j;\ i,j = 1, \ldots, t\}$, where for all $(i,j) \in I$ we have $(i-1,j), (i,j+1) \in I$. The group U will also be denoted by $U_I(d)$ if necessary. Note that this notation coincides with the definition of Section 1 and also with the notation in Section 2 for the particular algebra above.

We associate with I a set I^{gen} consisting of all blocks $(i,j) \in I$ which are minimal with respect to the shift to the left and to the bottom: that is, $(i,j) \in I^{\text{gen}}$ precisely when neither $(i+1,j)$ nor $(i,j-1)$ is in J. This set I^{gen} is the minimal set of generators as a root ideal. Moreover, we define a set $I' \subset I$ consisting of $(i,t) \in I$ for i maximal with (i,t) in I and all (i,j) satisfying $(i+1,j+1) \in I$ but $(i+1,j) \notin I$. We also define the index j' satisfying $(1,j') \notin J$ but $(1,j'+1) \in J$. We illustrate these sets in the following figure; the elements in I are denoted by "$*$":

	*	*	*	*	*	*
				*	*	*
				*	*	*
					*	*

$$\begin{aligned} I &= \{(1,2),(1,3),(1,4),(1,5),(1,6),(1,7),(2,5),\\ &\qquad (2,6),(2,7),(3,5),(3,6),(3,7),(4,6),(4,7)\},\\ I^{\text{gen}} &= \{(1,2),(3,5),(4,6)\},\\ I' &= \{(1,4),(3,5),(4,7)\}, \qquad j'=1. \end{aligned}$$

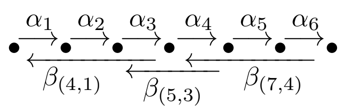
For the unipotent subgroup $U = P_{\mathrm{u}}^{(l)}$ considered in [BH1], for instance, the set I is generated by

$$I^{\text{gen}} = \{(i, i+l+1) \mid i = 1,\dots,t-l-1\} = I'.$$

Note that $I^{\text{gen}} = I'$ precisely if U is isomorphic to $P_{\mathrm{u}}^{(l)}$.

Let Λ be the basic algebra $\Lambda(1,\dots,1)$ as defined above. We define a bimodule $B(I)$ over Λ as follows. A k-basis of $B(I)$ consists of elements $b_{(i,j)}$ for $(i,j) \in I$. Choose a multiplicative basis of Λ consisting of elements $\lambda_{(i,j)}$ for $1 \le i \le j \le t$. We define a bimodule structure on the vector space $B(I)$ via $\lambda_{(i,j)} b_{(i',j')} \lambda_{(i'',j'')} = \delta_{j,i'}\delta_{j',i''} b_{(i,j'')}$. Thus $B(I)$ is the natural bimodule defined by ordinary matrix multiplication of the matrix of upper triangular matrices on the set of block-matrices with entries in the (i,j)-block for (i,j) in I. Note that $B(I)$ is generated as a bimodule over Λ by the elements $b_{(i,j)}$ for $(i,j) \in I^{\text{gen}}$. Moreover, the Lie algebra of $U_I(d)$ is isomorphic to $B(d)$. Thus the action of R on U coincides with the action defined in Section 2 for the particular algebra Λ and the bimodule B as above.

Next we define a quasi-hereditary algebra $\mathcal{A} = \mathcal{A}(I)$. This algebra does not depend on the entries in the dimension vector d, it only depends on I. First we define the quiver $Q(I)$ of $\mathcal{A}(I)$: it has vertices $Q_0 = \{1,\dots,t\}$, and arrows $\alpha_i : i \to i+1$ for $i = 1,\dots,t-1$, and $\beta_{(j,i)} : j \to i$ whenever $(i,j) \in I'$. Next we define relations $\alpha^r\beta = \beta\alpha^s$ whenever possible and a zero relation $\alpha^r\beta$ starting at j' and including the first possible arrow β. Then we define the algebra $\mathcal{A}$ to be the path algebra of the quiver $Q(I)$ modulo the ideal generated by these relations. Again we illustrate the construction in a figure containing the quiver and the relations for the problem in the previous figure:



$$\alpha_1\alpha_2\alpha_3\beta_{(4,1)} = 0,$$
$$\alpha_4\beta_{(5,3)} = \beta_{(4,1)}\alpha_1\alpha_2\alpha_3,$$
$$\alpha_5\alpha_6\beta_{(7,4)} = \beta_{(5,3)}\alpha_3.$$

Thus, for a given subset I, we have defined a bimodule $B = B(I)$ over Λ and a quasi-hereditary algebra $\mathcal{A} = \mathcal{A}(I)$. The next result states that this quasi-hereditary algebra $\mathcal{A}$ is exactly the quasi-hereditary algebra of Theorem 3.2 associated with the bimodule B. We note that $\mathcal{A}$ has a unique structure of a quasi-hereditary algebra, so that the standard modules $\Delta(i)$ are exactly the projective Λ-modules $P_\Lambda(i)$ via the natural embedding $\Lambda \subset \mathcal{A}$. Consequently, an $\mathcal{A}$-module X is in $\mathcal{F}(\Delta)$ precisely when all linear maps $X(\alpha)$ are injective.

THEOREM 4.1. *The category of modules with Δ-filtration over $\mathcal{A}(I)$ is equivalent to the category of matrices over $B(I)$. This equivalence is induced by sending a matrix $(\bigoplus_i P(i) \otimes M_i, f)$ to the following representation*: *α_i is the obvious inclusion $\bigoplus_{j=1}^{i} M_j \to \bigoplus_{j=1}^{i+1} M_j$ and $\beta_{(j,i)}$ is the restriction of f to $\bigoplus_{l=1}^{j} M_l$.*

The proof uses the category of flags, which we introduce in Section 6.

5. Actions of parabolic subgroups via multiplication. In this section we consider the simultaneous action of two parabolic groups $R(d^+)$ and $R(d^-)$ on an invariant vector space V contained in the tensor product of the two natural representations W^+ and W^- of $R(d^+)$ and $R(d^-)$ respectively. In other words, we consider the simultaneous action of a parabolic group $R(d^+)$ via left multiplication and of a parabolic group $R(d^-)$ via right multiplication on the vector space of those block matrices $M(J)$ which have non-zero entries only in the (i,j)-blocks for (i,j) in J, where J is considered as a subset of $\{1,\dots,t^+\}\times\{1,\dots,t^-\}$. As in the previous section, J must be closed under shift to the top and to the right, that is, whenever $(i,j) \in J$, then $(i-1,j)$ and $(i,j+1)$ are also in J. Again we want to construct a bimodule $B = B(J)$ over $\Lambda(d^+)$ from the left and $\Lambda(d^-)$ from the right such that the orbits of the action are in natural one-to-one correspondence with the isomorphism classes of matrices over B and the action of the group is of the form as introduced in Section 3.

Note that the orbits of this action of $R(d^+)\times R(d^-)$ on $V = V(J)$ coincide with the orbits of the action of the parabolic group $R(d^+,d^-)$, that is, just take the dimension vector $(d^+,d^-) = (d^+_1,\dots,d^+_{t^+},d^-_1,\dots,d^-_{t^-})$ on a normal subgroup $U(I)$ of $U(\{(i,j) \mid i = 1,\dots,t^+,\ j = t^+ + 1,\dots,t^+ + t^-\})$ for $(i,j+t^+) \in I$ precisely when $(i,j) \in J$, where we use for a moment the notation of the previous section. If we require, in addition, that the stabilizer of a point $x \in V$ of the action is exactly the automorphism group of the endomorphism algebra of the corresponding matrix, then we get different bimodules for the two different actions considered above (see Theorems 4.1 and 5.1), but the isomorphism classes of matrices coincide. Thus in the rest of this section we consider the action of $R(d^+) \times R(d^-)$ on $V(J)$, the other problem has already been considered before.

We first fix some notation. Let $d^+ = (d_1^+, \ldots, d_{t^+}^+)$ and $d^- = (d_1^-, \ldots, d_{t^-}^-)$ be two dimension vectors of length t^+ and t^-, respectively, and let J be a subset in $\{1, \ldots, t^+\} \times \{1, \ldots, t^-\}$ which is closed under the right and upper shift. Similarly to the previous section we define subsets J^{gen} and J' of J, and an element j' in J. Note that we can use the same definition as in the previous section if we identify J with a subset I of $\{(i,j) \mid i = 1, \ldots, t^+,\ j = t^+ + 1, \ldots, t^+ + t^-\}$.

Example. Let $t^+ = 3$, $t^- = 2$ and $J = \{(1,1), (1,2), (2,1), (2,2), (3,2)\}$. Then $J' = \{(3,2), (2,1)\}$, and $j' = 0$.

The group $R(d^+) \times R(d^-)$ is the group of invertible elements in $\Lambda(d^+) \oplus \Lambda(d^-)$, where we use the definition from the beginning of the previous section. We define a bimodule $B = B(J)$ as follows: a basis of B consists of elements $b_{(i,j)}$ for $(i,j) \in J$. The $\Lambda(d^+) \oplus \Lambda(d^-)$-bimodule structure is defined via left multiplication of $\Lambda((1, \ldots, 1)^+)$ and right multiplication of $\Lambda((1, \ldots, 1)^-)$: $\lambda^+_{(i,j)} b_{(i',j')} \lambda^-_{(i'',j'')} = \delta_{(i',j)} \delta_{(i'',j')} b_{(i,j'')}$, where δ denotes the Kronecker delta. Note that we have a natural inclusion of $\Lambda(d^+) \oplus \Lambda(d^-)$ into $\Lambda(d^+, d^-)$. Under this inclusion the bimodule $B(J)$ just coincides with $B(I)$ (here we again use the identification of J and I as explained above).

We define a quiver $Q(J)$ as follows:

$$Q_0 = \{(+,1), \ldots, (+,t^+), (-,1), \ldots (-,t^-)\} \quad \text{and}$$
$$Q_1 = \{(a,i) \xrightarrow{\alpha_{(a,i)}} (a,i+1) \mid a = +,-\} \cup \{(+,i) \xrightarrow{\beta_{(i,j)}} (-,j) \mid (i,j) \in J'\}.$$

Moreover, we consider the relations $\alpha^r \beta = \beta \alpha^s$, whenever possible, and a zero relation $\alpha^r \beta$ starting at j' and including the first possible arrow β. We define the algebra $\mathcal{A}(J)$ to be the path algebra of this quiver subject to the relations defined above.

Note that $\mathcal{A}(J)$ is quasi-hereditary with respect to several orders, yet we only consider the order defined by $(a,i) \leq (b,j)$ precisely when $a = -$ and $b = +$, or $a = b$ and $i \leq j$. This is equivalent to the property that the category of Δ-filtered modules coincides with the category of all those modules X with $X(\alpha_{(a,i)})$ injective.

Theorem 5.1. *The category of matrices over $B(J)$ is equivalent to the category of modules with Δ-filtration over the quasi-hereditary algebra $\mathcal{A}(J)$. This equivalence is induced by sending a matrix $(\bigoplus_i P(+,i) \otimes M_{(+,i)} \oplus \bigoplus_j P(-,j) \otimes M_{(-,j)}, f)$ to the following representation: $\alpha_{(a,i)}$ is the inclu-*

sion into the first summand: $\bigoplus_{l=1}^{i} M_{(a,l)} \to \bigoplus_{l=1}^{i+1} M_{(a,l)}$ *for* $a = +, -$, *and* $\beta_{(i,j)}$ *for* $(i,j) \in J'$ *is the restriction of* f *to* $\bigoplus_{l=1}^{i} M_{(+,l)}$.

The proof is analogous to that of Theorem 4.1 in the following section and uses the category of separated flags associated with J.

6. Categories of flags and the proof of the main results. In this section we introduce the category of flags $\mathcal{FL}(I)$ associated with a set I, and the category of separated flags $\mathcal{FL}^{\rm s}(J)$ associated with a set J, where I and J are as introduced in Sections 4 and 5. Using this technique we first prove Theorem 4.1; the proof of Theorem 5.1 is quite analogous and we omit the details.

Let V be a finite-dimensional vector space together with a flag $\{0\} = V_0 \subset V_1 \subset \ldots \subset V_t = V$ of length t and an endomorphism $f : V \to V$ satisfying $f(V_j) \subset V_i$ whenever $(i,j) \in I'$ and $f(V_{j'}) = 0$. The objects in the category $\mathcal{FL}(I)$ are those pairs (V,f). A morphism $\phi : (V,f) \to (V',f')$ is a linear map $\phi : V \to V'$ which preserves the flag, that is, $\phi(V_i) \subset V_i'$, and commutes with the given endomorphisms, that is, $f'\phi = \phi f$. We note that for $I^{\rm gen} = \{(i, i+l+1) \mid i = 1, \ldots, t-l-1\}$, this category was already considered in [BH1] and for $l = 0$ in [HR].

LEMMA 6.1. *The map assigning to a flag* (V,f) *the representation* $(V_i, \alpha_i, \beta_{(i,j)})$, *where* α *is the injection* $V_i \subset V_{i+1}$ *and* $\beta_{(i,j)}$ *is the restriction of* f, *induces an equivalence between the category* $\mathcal{FL}(I)$ *and the category* $\mathcal{F}(\Delta)$ *of* Δ*-filtered* $\mathcal{A}(I)$*-modules.*

Proof. Note that an $\mathcal{A}(I)$-module has a Δ-filtration precisely when all maps $\alpha_{(i,j)}$ are injective. Then the proof is elementary and left to the reader (it is proven for special cases in [BH1] and [HR] and the proof of this lemma does not need any additional argument). ■

In the next step, we prove that the category of matrices $\operatorname{mat} B(I)$ is also equivalent to $\mathcal{FL}(I)$. Recall that an object (d,b) in $\operatorname{mat} B(I)$ may be considered as a projective Λ-module $\bigoplus P_\Lambda(i) \otimes M_i$ together with an element b, where $\dim M_i = d_i$. We can identify the module $\bigoplus P_\Lambda(i) \otimes M_i$ with a representation of the directed quiver $\mathbb{A}_t$ with injective maps, also denoted by α_i. That is, we can identify $\bigoplus P_\Lambda(i) \otimes M_i$ with the flag $\{0\} = V_0 \subset V_1 = M_1 \subset \ldots \subset V_t = V = \bigoplus_{l=1}^{t} M_l$. The element b consists of a set of homomorphisms $b_{i,j} : M_i \to M_j$ for $i,j = 1, \ldots, t$ satisfying the condition $b_{i,j} = 0$ whenever $B_{(i,j)} = 0$, that is, $(i,j) \notin I$. Thus we obtain for each matrix $(\bigoplus P_\Lambda(i) \otimes M_i, b)$ a flag (V,f), where we identify b with f. This map obviously defines a functor from $\operatorname{mat} B(I)$ to $\mathcal{FL}(I)$.

LEMMA 6.2. *The category* $\operatorname{mat} B(I)$ *is equivalent to the category of flags* $\mathcal{FL}(I)$.

P r o o f. The assignment above defines a flag for each matrix. Conversely, for each flag we can choose a corresponding matrix, so the functor is dense. It is straightforward to check that the conditions $f(V_j) \subset V_i$ whenever $(i,j) \in I$ and $f(V_{j'}) = 0$ on the endomorphism f in the flag are satisfied precisely when the conditions $b_{(i,j)} = 0$ whenever $(i,j) \notin I$ on the matrix b are satisfied. Moreover, the functor is fully faithful. ∎

Proof of Theorem 4.1. From the two lemmata above, we obtain the equivalence of the categories $\operatorname{mat} B$ and $\mathcal{F}(\Delta)$ in $\operatorname{mod} \mathcal{A}(I)$. Since $\mathcal{A}(I)$ is quasi-hereditary and the matrix $(P_\Lambda(i), 0)$ corresponds to $\Delta(i)$ under the equivalences in the two lemmata, we obtain our result by Theorem 3.2. ∎

Proof of Main Theorem 1.1. We start with the proof of the first claim. By Lemma 3.1 the orbits of the action of $R(d)$ on $U_I(d)$ are in natural bijection with the isomorphism classes of matrices $(P_\Lambda(i) \otimes M_i, b)$ over $B(I)$, where $\dim M_i = d_i$. By Theorem 4.1 the isomorphism classes of matrices over $B(I)$ correspond to the isomorphism classes of Δ-filtered modules, where the dimension vector is obviously preserved as stated in the claim.

To prove the second part of the theorem we need some notation. Denote by $\mathcal{R}(\Delta)(d)$ the representation space of all Δ-filtered $\mathcal{A}$-modules of Δ-dimension vector d. The set $\mathcal{R}(\Delta)(d)$ is an algebraic variety and the functors in Lemmata 6.1 and 6.2 define a morphism of algebraic varieties $\phi : U_I(d) \simeq B(I)(d) \to \mathcal{R}(\Delta)(d)$. The morphism ϕ is equivariant and admits a local inverse. We can identify $\mathcal{R}(\Delta)(d)$ with the space of all flags in $\mathcal{FL}(I)$ of dimension vector d with fixed vector spaces V_i. We choose a basis of these vector spaces V_i which is compatible with the inclusions. Then there exists a unique element g in $G(d)/P(d)$, where $G(d) = \prod \mathrm{GL}_{d_i}$ and $P(d)$ is a subgroup of $G(d)$ such that gV is the standard flag. Choose locally a section $\pi : G(d)/P(d) \to G(d)$. Then $(V, f) \mapsto (\pi(g)V, \pi(g) f \pi(g^{-1}))$ defines locally a section of ϕ. Consequently, degenerations and families of orbits in $\mathcal{R}(\Delta)(d)$ and $U_I(d)$ coincide. ∎

For the proof of Theorem 5.1 we need a modification of the category of flags: we consider the category of separated flags $\mathcal{FL}^{\mathrm{s}}(J)$, corresponding to a set J, introduced at the beginning of Section 6. The objects are triples (V, W, f) consisting of two flags $\{0\} = V_0 \subset V_1 \subset \ldots \subset V_{t^+} = V$ and $\{0\} = W_0 \subset W_1 \subset \ldots \subset W_{t^-} = W$, and a linear map $f : V \to W$ satisfying the conditions $f(V_j) \subset W_i$ for $(i,j) \in J'$ and $f(V_{j'}) = 0$. A morphism ϕ between two objects $(V, W, f) \to (V', W', f')$ consists of two linear maps $\phi_1 : V \to V'$ and $\phi_2 : W \to W'$ which preserve the flags and commute with the given linear maps, that is, $\phi_2 f = f' \phi_1$.

Given a separated flag (V, W, f), it is straightforward to define a representation of the algebra $\mathcal{A}(J)$ defined in Section 4: the vector space in $(+, i)$ is V_i and the vector space in $(-, i)$ is W_i. The maps α are the inclusions of the flag, and the maps β are the restrictions of the map f. As before, it is easy to check the following lemma.

LEMMA 6.3. *The natural map defined above induces an equivalence between the category* $\mathcal{FL}^{\mathrm{s}}(J)$ *and the category of* Δ*-filtered* $\mathcal{A}(J)$*-modules.*

Also the second lemma proven for the category of flags generalizes with appropriate modifications to the category of separated flags.

LEMMA 6.4. *The category* $\mathrm{mat}\, B(J)$ *is equivalent to the category* $\mathcal{FL}^{\mathrm{s}}(J)$.

Proof of Theorem 5.1. The proof is analogous to that of Theorem 4.1: the categories $\mathrm{mat}\, B(J)$ and $\mathcal{F}(\Delta)$ are equivalent, $\mathcal{A}(J)$ is quasi-hereditary, and the uniqueness result of Theorem 3.2 completes the proof. ∎

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