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ON SOME FORMULA IN CONNECTED COCOMMUTATIVE HOPF ALGEBRAS OVER A FIELD OF CHARACTERISTIC 0

BY

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Abstract. Let H be a cocommutative connected Hopf algebra, where K is a field of characteristic zero. Let $H^+ = \operatorname{Ker}\varepsilon$ and $h^+ = h - \varepsilon(h)$ for $h \in H$. We prove that $d_h = \sum_{r=1}^{\infty}((-1)^{r+1}/r)\sum h_1^+ \ldots h_r^+$ is primitive, where $\sum h_1 \otimes \ldots \otimes h_r = \Delta_{r-1}(h)$.

1. Introduction. Let K be a field of characteristic 0. In [2] it is proved that if $D = (D_0, D_1, \ldots)$ is a higher derivation of a commutative algebra A, then the linear maps

$$d_n = \sum_{r=1}^{n} \frac{(-1)^{r+1}}{r} \sum_{\substack{i_1+\ldots+i_r=n \\ i_1,\ldots,i_r>0}} D_{i_1} \ldots D_{i_r}, \qquad n \geq 1,$$

are derivations of A.

Inspired by this result we prove the following:

THEOREM. *Let H be a connected, cocommutative Hopf algebra over K with comultiplication $\Delta : H \to H \otimes H$ and counity $\varepsilon : H \to K$, let $H^+ = \operatorname{Ker}\varepsilon$, and let $h^+ = h - \varepsilon(h)$ for $h \in H$. Then for any $h \in H^+$ the element*

$$d_h = \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r} \sum h_1^+ \ldots h_r^+$$

is primitive, where $\sum h_1 \otimes \ldots \otimes h_r = \Delta_{r-1}(h)$ (the infinite sum has only a finite number of non-zero summands).

As consequences of this theorem one gets:

1.1. COROLLARY. *Let H be as in the Theorem, and let A be an arbitrary (not necessarily commutative) H-module algebra. Then for any $h \in H^+$ the linear map $\widetilde{d}_h : A \to A$, $\widetilde{d}_h(a) = d_h a$, is a derivation of A.*

Corollary 1.1 gives us Saymeh's above-mentioned result for the connected and cocommutative Hopf algebra $H = K\langle x_0, x_1, \ldots\rangle$, $x_0 = 1$, $\Delta(x_n)$

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$=\sum_{i+j=n} x_i \otimes x_j$, $\varepsilon(x_i)=\delta_{i,0}$, where the antipode is given by the inductive formula: $S(x_0)=x_0$, $S(x_n)=-\sum_{i+j=n-1} x_{i+1}S(x_j)$ ($d_n=d_h$ for $h=x_n$, $n\ge 1$).

1.2. Corollary ([3, 13.0.1], [1, 5.6.5]). *Every connected, cocommutative Hopf algebra over a field of characteristic* 0 *is isomorphic to the universal enveloping algebra $U(L)$, where L is the Lie algebra of all primitive elements in H.*

Throughout the paper K is a fixed field of characteristic 0 and H denotes a connected Hopf algebra over K with comultiplication $\Delta : H \to H\otimes H$ and counity $\varepsilon : H\to K$. Connectedness of H means that $K1_H$ is the unique simple subcoalgebra of H ([1], [3]). The ideal $\operatorname{Ker}\varepsilon$ will be denoted by H^+. We define the maps $\Delta_n : H \to H^{\otimes n+1}$, $n\ge 0$, by induction: $\Delta_0=\mathrm{id}$, $\Delta_n=(\Delta\otimes\mathrm{id}\otimes\ldots\otimes\mathrm{id})\Delta_{n-1}$, $n>0$. Moreover, we write $\Delta_n(h)=\sum h_1\otimes\ldots\otimes h_{n+1}$. In particular, $\Delta(h)=\sum h_1\otimes h_2$.

As usual, $\mathbb{Z}$ stands for the set of rational integers.

2. Results. Let $H_0\subset H_1\subset\ldots$ be the coradical filtration of H [3, 9.1], and let $H_n^+=H_n\cap H^+$. For every $h\in H$ we have the unique decomposition $h=\varepsilon(h)+h^+$, where $\varepsilon(h)\in H_0$, $h^+\in H^+$.

If $h\in H^+$, then we know that $\Delta(h)=h\otimes 1+1\otimes h+f$, where $f\in H_{n-1}^+\otimes H_{n-1}^+$(this is a simple consequence of [3, Corollary 9.1.7]).

Let $D : H\to H\otimes H$ denote the linear map defined by $D(h)=1\otimes h+h\otimes 1$. Observe that D is not coassociative. Using D we define the map $\Delta^+ : H\to H\otimes H$ via $\Delta^+=\Delta-D$. Observe that $\Delta^+(h)=\sum h_1^+\otimes h_2^+$ for $h\in H^+$.

2.1. Lemma. *The map Δ^+ is coassociative, i.e.,*

$$(\Delta^+\otimes\mathrm{id})\Delta^+=(\mathrm{id}\otimes\Delta^+)\Delta^+.$$

Moreover, if Δ is cocommutative, then so is Δ^+.

Proof. For the first part, observe that since Δ is coassociative, it is enough to show that $L=R$, where

$$L=(\Delta^+\otimes\mathrm{id})\Delta^+-(\Delta\otimes\mathrm{id})\Delta=(\Delta\otimes\mathrm{id})D+(D\otimes\mathrm{id})\Delta-(D\otimes\mathrm{id})D,$$
$$R=(\mathrm{id}\otimes\Delta^+)\Delta^+-(\mathrm{id}\otimes\Delta)\Delta=(\mathrm{id}\otimes\Delta)D+(\mathrm{id}\otimes D)\Delta-(\mathrm{id}\otimes D)D.$$

We have

$$\begin{aligned}L(h)&=(\Delta\otimes\mathrm{id})(h\otimes 1+1\otimes h)\\&\quad+(D\otimes\mathrm{id})\Big(\sum h_1\otimes h_2\Big)-(D\otimes\mathrm{id})(h\otimes 1+1\otimes h)\\&=\sum h_1\otimes h_2\otimes 1+\sum h_1\otimes 1\otimes h_2\\&\quad+\sum 1\otimes h_1\otimes h_2-h\otimes 1\otimes 1-1\otimes h\otimes 1-1\otimes 1\otimes h\end{aligned}$$

$$
\begin{aligned}
&= \Big(h\otimes 1\otimes 1+\sum 1\otimes h_1\otimes h_2\Big)\\
&\quad + \Big(\sum h_1\otimes h_2\otimes 1+\sum h_1\otimes 1\otimes h_2\Big)\\
&\quad - (h\otimes 2\otimes 1+1\otimes h\otimes 1+1\otimes 1\otimes h)\\
&= (\mathrm{id}\otimes\Delta)(h\otimes 1+1\otimes h)\\
&\quad + (\mathrm{id}\otimes D)\Big(\sum h_1\otimes h_2\Big) - (\mathrm{id}\otimes D)(h\otimes 1+1\otimes h)\\
&= R(h).
\end{aligned}
$$

If Δ is cocommutative, then cocommutativity of Δ^+ is obtained directly from the definition. ∎

Now we define the linear maps $\Delta_n^+ : H \to H^{\otimes n+1}$ by the inductive formula

$$\Delta_0^+ = \mathrm{id}, \quad \Delta_n^+ = (\Delta^+\otimes \mathrm{id}\otimes\ldots\otimes\mathrm{id})\Delta_{n-1}^+, \quad n\ge 1.$$

It is easy to see that if $h\in H^+$, then $\Delta_n^+(h)=\sum h_1^+\otimes\ldots\otimes h_{n+1}^+$. Assume that $h\in H_n^+$. Then using the inclusions $\Delta(H_n)\subset\sum_{i+j=n}H_i\otimes H_j$ [3, 9.1.7] we have $\Delta_r(h)\in\sum_{i_1+\ldots+i_{r+1}=n}H_{i_1}\otimes\ldots\otimes H_{i_{r+1}}$ for every $r\ge 0$. Hence

$$\Delta_r^+(h)=\sum h_1^+\otimes\ldots\otimes h_{r+1}^+\in\sum_{i_1+\ldots+i_{r+1}=n}H_{i_1}^+\otimes\ldots\otimes H_{i_{r+1}}^+ \quad \text{for all } r\ge 0,$$

which implies that $\Delta_r^+(h)=0$ for all $r\ge n$, because $H_0^+=0$.

From now on, we assume that H is cocommutative.

Definition. Let t,e,s be integers. We define the non-negative integers $Q_{t,e,s}$ by

$$Q_{t,e,s}=\binom{t}{e}\binom{e}{t-s},$$

where $\binom{u}{v}=0$ for $u<0$ or $v<0$ or $u<v$. It is obvious that $Q_{t,e,s}\ne 0$ if and only if t,e,s satisfy the conditions: $t\ge 0$, $0\le e\le t$, $0\le s\le t$, $t\le e+s$.

2.2. Lemma. *Let t,e,s be integers.*

(1) *If $t>0$, then $Q_{t,e,s}=Q_{t-1,e-1,s}+Q_{t-1,e,s-1}+Q_{t-1,e-1,s-1}$.*
(2) *If $F:\mathbb{Z}^3\to\mathbb{Z}$ is a function satisfying the conditions:*
   (a) *$F(x,y,z)=0$ for integers x,y,z which do not satisfy one of the conditions: $t\ge 0$, $0\le e\le t$, $0\le s\le t$, $e+s\ge t$,*
   (b) *$F(0,0,0)=1$, $F(0,y,z)=0$, provided $y\ne 0$ or $z\ne 0$,*
   (c) *$F(x,y,z)=F(x-1,y-1,z+F(x-1,y,z-1))$*
   $\quad +F(x-1,y-1,z-1)$,

then $F(t,e,s)=Q_{t,e,s}$ for all $t,e,s\in\mathbb{Z}$.

Proof. (1) First we notice that $Q_{0,0,0}=1$. Now let $t>0$. If e,s do not satisfy one of the conditions: $0\le e\le t$, $0\le s\le t$, $t\le e+s$, then clearly $Q_{t,e,s}=Q_{t-1,e-1,s}=Q_{t-1,e,s-1}=Q_{t-1,e-1,s-1}=0$ and equality (1) is obvious. Now, assume that $0\le e\le t$, $0\le s\le t$, $t=e+s$. Then

$$Q_{t-1,e-1,s-1}=0,\qquad Q_{t,e,s}=\binom{t}{e},$$
$$Q_{t-1,e-1,s}=\binom{t-1}{e-1},\qquad Q_{t-1,e,s-1}=\binom{t-1}{e}$$

and the equality $Q_{t,e,s}=Q_{t-1,e-1,s}+Q_{t-1,e,s-1}+Q_{t-1,e-1,s-1}$ is the well known property of the Newton symbols.

The second case is $0\le e\le t$, $0\le s\le t$, $t<e+s$. In this situation

$$\begin{aligned}
&Q_{t-1,e-1,s}+Q_{t-1,e,s-1}+Q_{t-1,e-1,s-1}\\
&\quad=\binom{t-1}{e-1}\binom{e-1}{t-1-s}+\binom{t-1}{e}\binom{e}{t-s}+\binom{t-1}{e-1}\binom{e-1}{t-s}\\
&\quad=\frac{(t-1)!}{(t-e)!(t-1-s)!(e-t+s)!}\\
&\qquad+\frac{(t-1)!}{(t-1-e)!(t-s)!(e-t+s)!}+\frac{(t-1)!}{(t-e)!(t-s)!(e-1-t+s)!}\\
&\quad=\frac{(t-1)!((t-s)+(t-e)+(e-t+s))}{(t-e)!(t-s)!(e+s-t)!}=\frac{t!}{(t-e)!(t-s)!(e+s-t)!}\\
&\quad=\binom{t}{e}\binom{e}{t-s}=Q_{t,e,s}.
\end{aligned}$$

(2) If $x<0$, then $F(x,y,z)=0=Q_{x,y,z}$. If $x=0$ and $y\ne 0$ or $z\ne 0$, then $F(x,y,z)=0=Q_{x,y,z}$ and $F(0,0,0)=Q_{0,0,0}$. Now we show the equality $F(x,y,z)=Q_{x,y,z}$ for $x>0$. We proceed by induction on x. Assume that $F(x,y,z)=Q_{x,y,z}$ for a fixed $x\ge 0$ and all y,z. Then

$$\begin{aligned}
F(x+1,y,z)&=F(x,y-1,z)+F(x,y,z-1)+F(x,y-1,z-1)\\
&=Q_{x,y-1,z}+Q_{x,y,z-1}+Q_{x,y-1,z-1}=Q_{x+1,y,z},
\end{aligned}$$

by the inductive assumption and part (1) of the lemma. ∎

2.3. Lemma. *For all integers $e,s>0$,*

$$\sum_{p=0}^{s}(-1)^p\binom{e+p-1}{p}\binom{e}{s-p}=0.$$

Proof. This is equality (35) in [4, Chap. 2]. ∎

2.4. THEOREM. *If* $h \in H^+$, *then*

$$d = \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r} \sum h_1^+ \dots h_r^+$$

is a primitive element in H, *where* $\sum h_1^+ \otimes \dots \otimes h_r^+ = \Delta_{r-1}^+(h)$.

Proof. Obviously, $h \in H_n^+$ for some $n \geq 0$. We have to show that $\Delta(d) = 1 \otimes d + d \otimes 1$. We will use the following notation:

$$\begin{aligned} f_i &= \sum h_1^+ \dots h_i^+, \\ h_{k,l,m} &= \sum \Delta(h_1^+ \dots h_k^+)(h_{k+1}^+ \dots h_{k+l}^+ \otimes h_{k+l+1}^+ \dots h_{k+l+m}^+), \\ g_{i,j} &= \sum h_1^+ \dots h_i^+ \otimes h_{i+1}^+ \dots h_{i+j}^+. \end{aligned}$$

Clearly, $h_{k,0,0} = \Delta(f_k)$, $h_{0,l,m} = g_{l,m}$, and $d = \sum_{r=1}^{n}((-1)^{r+1}/r)f_r$, because $\Delta_r^+(h) = 0$ for $r \geq n$. Now we show the following equality:

$$(*) \qquad h_{k,l,m} = h_{k-1,l+1,m} + h_{k-1,l,m+1} + h_{k-1,l+1,m+1}.$$

One knows that $\Delta(h) = h \otimes 1 + 1 \otimes h + \sum h_1^+ \otimes h_2^+$ and that Δ^+ is cocommutative. Hence

$$\begin{aligned} &\sum \Delta(h_1^+ \dots h_k^+)(h_{k+1}^+ \dots h_{k+l}^+ \otimes h_{k+l+1}^+ \dots h_{k+l+m}^+) \\ &\quad = \sum \Delta(h_1^+ \dots h_{k-1}^+)(h_k^+ \dots h_{k+l}^+ \otimes h_{k+l+1}^+ \dots h_{k+l+m}^+) \\ &\qquad + \sum \Delta(h_1^+ \dots h_{k-1}^+)(h_{k+1}^+ \dots h_{k+l}^+ \otimes h_k^+ h_{k+l+1}^+ \dots h_{k+l+m}^+) \\ &\qquad + \sum \Delta(h_1^+ \dots h_{k-1}^+)(h_k^+ h_{k+2}^+ \dots h_{k+l+1}^+ \otimes h_{k+1}^+ h_{k+l+2}^+ \dots h_{k+l+m+1}^+) \\ &\quad = \sum \Delta(h_1^+ \dots h_{k-1}^+)(h_k^+ \dots h_{k+l}^+ \otimes h_{k+l+1}^+ \dots h_{k+l+m}^+) \\ &\qquad + \sum \Delta(h_1^+ \dots h_{k-1}^+)(h_k^+ \dots h_{k+l-1}^+ \otimes h_{k+l}^+ \dots h_{k+l+m}^+) \\ &\qquad + \sum \Delta(h_1^+ \dots h_{k-1}^+)(h_k^+ \dots h_{k+l}^+ \otimes h_{k+l+1}^+ \dots h_{k+l+m+1}^+), \end{aligned}$$

which proves $(*)$.

Next we apply $(*)$ to prove by induction on t that

$$(**) \qquad h_{k,l,m} = \sum_{\substack{0 \leq e,s \leq t \\ e+s \geq t}} Q_{t,e,s} h_{k-t,l+e,m+s} \qquad \text{for all } t \leq k.$$

If $t = 0$, then it is obvious. Assume that $(**)$ is true for some $t < k$. From $(*)$ it follows that

$$\begin{aligned}
&h_{k,l,m}\\
&= \sum_{\substack{0\le e,s\le t\\ e+s\ge t}} Q_{t,e,s}(h_{k-t-1,l+e+1,m+s}+h_{k-t-1,l+e,m+s+1}+h_{k-t-1,l+e+1,m+s+1})\\
&= \sum_{\substack{0\le e,s\le t\\ e+s\ge t}} Q_{t,e,s}h_{k-t-1,l+e+1,m+s}\\
&\quad + \sum_{\substack{0\le e,s\le t\\ e+s\ge t}} Q_{t,e,s}h_{k-t-1,l+e,m+s+1} + \sum_{\substack{0\le e,s\le t\\ e+s\ge t}} Q_{t,e,s}h_{k-t-1,l+e+1,m+s+1}\\
&= \sum_{\substack{0\le s\le t\\ 1\le e\le t+1\\ e+s\ge t+1}} Q_{t,e-1,s}h_{k-t-1,l+e,m+s}\\
&\quad + \sum_{\substack{0\le e\le t\\ 1\le s\le t+1\\ e+s\ge t+1}} Q_{t,e,s-1}h_{k-t-1,l+e,m+s} + \sum_{\substack{1\le e\\ s\le t+1\\ e+s\ge t+2}} Q_{t,e-1,s-1}h_{k-t-1,l+e,m+s}.
\end{aligned}$$

But

$$\sum_{\substack{0\le s\le t\\ 1\le e\le t+1\\ e+s\ge t+1}} Q_{t,e-1,s}h_{k-t-1,l+e,m+s} = \sum_{\substack{0\le e,s\le t+1\\ e+s\ge t+1}} Q_{t,e-1,s}h_{k-t-1,l+e,m+s},$$

because $Q_{t,-1,s} = Q_{t,e-1,t+1} = 0$. Further,

$$\sum_{\substack{0\le e\le t\\ 1\le s\le t+1\\ e+s\ge t+1}} Q_{t,e,s-1}h_{k-t-1,l+e,m+s} = \sum_{\substack{0\le e,s\le t+1\\ e+s\ge t+1}} Q_{t,e,s-1}h_{k-t-1,l+e,m+s},$$

because $Q_{t,e,-1} = Q_{t,t+1,s-1} = 0$, and

$$\sum_{\substack{1\le e,s\le t+1\\ e+s\ge t+2}} Q_{t,e-1,s-1}h_{k-t-1,l+e,m+s} = \sum_{\substack{0\le e,s\le t+1\\ e+s\ge t+1}} Q_{t,e-1,s-1}h_{k-t-1,l+e,m+s},$$

because $Q_{t,e-1,s-1} = 0$ if e,s satisfy one of the conditions $e = 0$, $s = 0$, $e+s = t+1$.

Hence

$$h_{k,l,m} = \sum_{\substack{0\le e,s\le t+1\\ e+s\ge t+1}} (Q_{t,e-1,s} + Q_{t,e,s-1} + Q_{t,e-1,s-1})h_{k-t-1,l+e,m+s}.$$

By Lemma 2.2, $Q_{t+1,e,s} = Q_{t,e-1,s} + Q_{t,e,s-1} + Q_{t,e-1,s-1}$, whence

$$h_{k,l,m} = \sum_{\substack{0\le e,s\le t+1\\ e+s\ge t+1}} Q_{t,e,s}h_{k-t-1,l+e,m+s},$$

which proves $(**)$.

Now using $(**)$ for $t = k$, $l = m = 0$ and the definition of $Q_{t,e,s}$, we have

$$h_{k,0,0} = \sum_{\substack{0\le e,s\le k\\ e+s\ge k}} \binom{k}{e}\binom{e}{k-s} h_{0,e,s},$$

whence

$$\Delta(f_k) = h_{k,0,0} = \sum_{\substack{0\le e,s\le k\\ e+s\ge k}} \binom{k}{e}\binom{e}{k-s} g_{e,s},$$

because $h_{0,e,s} = g_{e,s}$. It follows that

$$\Delta(d) = \sum_{r=1}^{n} \frac{(-1)^{r+1}}{r}\Delta(f_r) = \sum_{r=1}^{n} \frac{(-1)^{r+1}}{r} \sum_{\substack{0\le e,s\le r\\ e+s\ge r}} \binom{r}{e}\binom{e}{r-s} g_{e,s}.$$

Denote by $w_{e,s}$ the coefficient at $g_{e,s}$ in the above sum. If $e, s \ge 1$ and $e + s \le n$, then we have, for $p = r - e$,

$$\begin{aligned} w_{e,s} &= \sum_{p=0}^{s} \frac{(-1)^{e+p+1}}{e+p}\binom{e+p}{e}\binom{e}{e+p-s} \\ &= \sum_{p=0}^{s} \frac{(-1)^{e+p+1}}{e+p}\binom{e+p}{e}\binom{e}{s-p}. \end{aligned}$$

Since

$$\frac{1}{e+p}\binom{e+p}{p} = \frac{(e+p-1)!(e+p)}{(e+p)e(e-1)!p!} = \frac{1}{e}\binom{e+p-1}{p}$$

we get

$$w_{e,s} = \frac{(-1)^{e+1}}{e}\sum_{p=0}^{s}(-1)^p\binom{e+p-1}{p}\binom{e}{s-p} = 0,$$

by Lemma 2.3. Thus we have shown that $w_{e,s} = 0$ for $e, s \ge 1$, $e + s \le n$. If $e + s > n$, then clearly $g_{e,s} = 0$, as $\Delta_n^+(h) = 0$. The last case is $e = 0$ or $s = 0$, but then it is obvious that $w_{0,s} = (-1)^{s+1}/s$, $w_{e,0} = (-1)^{e+1}/e$. Consequently we have

$$\Delta(d) = \sum_{r=1}^{n} \frac{(-1)^{r+1}}{r}(g_{r,0} + g_{0,r}) = d \otimes 1 + 1 \otimes d. \ \blacksquare$$

2.5. Corollary. *If* $h \in H_n^+$, *then*

$$d = \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r}\sum h_1^+ \dots h_r^+ = \sum_{r=1}^{n} \frac{(-1)^{r+1}}{r}\sum h_1^+ \dots h_r^+. \ \blacksquare$$

2.6. Corollary. *The Hopf algebra H is generated, as an algebra, by the set $P(H)$ of all primitive elements in H.*

P r o o f. Let $A \subset H$ be the subalgebra of H generated by $P(H)$. We need only show that $H_n^+ \subset A$ for all $n \geq 1$. This will be done by induction on n. Clearly, $H_1^+ = P(H) \subset A$. Assume that $H_{n-1}^+ \subset A$ and take an $h \in H_n^+$. From the theorem above we know that

$$d = \sum_{r=1}^{n} \frac{(-1)^{r+1}}{r} \sum h_1^+ \ldots h_r^+ \in P(H) \subset A.$$

Hence by the induction assumption,

$$e = \sum_{r=2}^{n} \frac{(-1)^{r+1}}{r} \sum h_1^+ \ldots h_r^+ \in A,$$

because $\sum h_1^+ \otimes \ldots \otimes h_r^+ = \Delta_{r-1}^+(h) \in \sum_{i_1+\ldots+i_r=n} H_{i_1}^+ \otimes \ldots \otimes + H_{i_r}^+$, and $H_0^+ = 0$. This implies that $h = d - e \in A$, and consequently $A = H$. ∎

2.7. Corollary ([3, 13.0.1], [1, 5.6.5]). *The Hopf algebra H is isomorphic to the universal enveloping Hopf algebra $U(L)$, where L is the Lie algebra of all primitive elements in H with $[x,y] = xy - yx$.*

P r o o f. Let $f : U(L) \to H$ be the morphism of Hopf algebras induced by the inclusion $L \subset H$ ($f(y) = y$ for $y \in L$). Since, as we showed above in Corollary 2.6, H is generated by L, we see that f is surjective. Let $P(U(L))$ denote the set of all primitive elements in $U(L)$. From the P–B–W theorem it easily follows that the natural map $L \to U(L)$ induces an isomorphism $L \approx P(U(L))$. Hence, in view of [3, 11.0.1], f is injective. ∎

Example. Let H be the Hopf algebra defined as follows:

$$H = K\langle x_0, x_1, \ldots\rangle, \quad x_0 = 1 \qquad \text{(the free algebra on } x_1, x_2, \ldots),$$
$$\Delta(x_n) = \sum_{i+j=n} x_i \otimes x_j, \qquad \varepsilon(x_n) = \delta_{n,0}.$$

The antipode S is given by the inductive formula

$$S(x_0) = x_0 = 1, \qquad S(x_{n+1}) = -\sum_{i+j=n} x_{i+1} S(x_j), \quad n \geq 0.$$

It is not difficult to show, using [3, 11.0.2, 11.0.6, 9.0.1, (b), Exercise (4), p. 182], that H is connected.

Observe that an action of H on an algebra A is nothing else than a higher derivation $(D_0, D_1, \ldots)$ of A ($D_i(a) = x_i a$, $i \geq 0$). Let us apply Theorem 2.4 to $h = x_n$, $n \geq 1$. Since

$$\Delta_{r-1}^+(h) = \sum_{\substack{i_1+\ldots+i_r=n \\ i_1,\ldots,i_r>0}} x_{i_1} \otimes \ldots \otimes x_{i_r},$$

we see by Theorem 2.4 that the element

$$d = \sum_{r=1}^{n} \frac{(-1)^{r+1}}{r} \sum_{\substack{i_1+\ldots+i_r=n \\ i_1,\ldots,i_r>0}} x_{i_1}\ldots x_{i_r}$$

is primitive. Hence

$$\widetilde{d}_h = \sum_{r=1}^{n} \frac{(-1)^{r+1}}{r} \sum_{\substack{i_1+\ldots+i_r=n \\ i_1,\ldots,i_r>0}} D_{i_1}\ldots D_{i_r} : A \to A$$

is a derivation of A. This is just Saymeh's result [2, Prop. 1].

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