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"COUNTEREXAMPLES" TO THE HARMONIC LIOUVILLE THEOREM AND HARMONIC FUNCTIONS WITH ZERO NONTANGENTIAL LIMITS

BY

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Abstract. We prove that, if $\mu > 0$, then there exists a linear manifold M of harmonic functions in $\mathbb{R}^N$ which is dense in the space of all harmonic functions in $\mathbb{R}^N$ and

$$\lim_{\substack{\|x\|\to\infty \\ x\in S}} \|x\|^{\mu} D^{\alpha} v(x) = 0$$

for every $v \in M$ and multi-index α, where S denotes any hyperplane strip. Moreover, every nonnull function in M is universal. In particular, if $\mu \ge N+1$, then every function $v \in M$ satisfies $\int_H v\,d\lambda = 0$ for every $(N-1)$-dimensional hyperplane H, where λ denotes the $(N-1)$-dimensional Lebesgue measure. On the other hand, we prove that there exists a linear manifold M of harmonic functions in the unit ball $\mathbb{B}$ of $\mathbb{R}^N$, which is dense in the space of all harmonic functions and each function in M has zero nontangential limit at every point of the boundary of $\mathbb{B}$.

1. Introduction. Liouville's theorem states that a bounded holomorphic function on the complex plane $\mathbb{C}$ is constant. A similar result holds for harmonic functions on $\mathbb{R}^N$, that is, a bounded harmonic function on $\mathbb{R}^N$ is constant. Nevertheless, if the boundedness condition is slightly weakened, then nonconstant entire functions in $\mathbb{C}$ or harmonic functions in $\mathbb{R}^N$ can be obtained.

For example, Armitage and Goldstein [3] proved that, for each $\mu > 0$, there exists a nonconstant entire function f such that

$$\exp(r^{\mu}) f(re^{i\theta}) \to 0 \quad (r \to \infty)$$

for all $\theta \in [0, 2\pi)$. And, for each $\mu > 0$, there exists a nonconstant harmonic function v in $\mathbb{R}^n$ such that

$$\|x\|^{\mu} v(x) \to 0 \ (\|x\| \to \infty,\ x \in L)$$

for every semi-infinite line L in $\mathbb{R}^N$.

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Moreover, Armitage and Goldstein [5] showed that there exists a nonconstant harmonic function v on $\mathbb{R}^N$ such that

$$\int_H v\, d\lambda = 0 \tag{1}$$

for every $(N-1)$-dimensional hyperplane H, where λ denotes the $(N-1)$-dimensional Lebesgue measure. Also, Armitage and Gauthier [2] have proved that the space of harmonic functions on $\mathbb{R}^N$ for which (1) is true is dense in $h(\mathbb{R}^N)$, the space of all harmonic functions in $\mathbb{R}^N$.

A harmonic function f is *universal* if to each harmonic function g in $\mathbb{R}^n$ corresponds a sequence $\{a_n\}_{n\ge 1}$ depending on g and satisfying

$$\lim_{n\to\infty} f(x+a_n) = g(x)$$

uniformly on compact sets. In [2] it is proved that the set of universal harmonic functions is residual in $h(\mathbb{R}^N)$.

On the other hand, Bernal [9] proved that, given $\alpha \in (0,1/2)$, there exists a linear manifold M of entire functions which is dense in the space of all entire functions and, in addition,

$$\lim_{\substack{z\to\infty\\ z\in S}} \exp(|z|^\alpha) f^{(j)}(z) = 0$$

for every $f \in M$ and $j \in \mathbb{N}$, where S denotes any plane strip. See also [10] for a strengthening of this result.

In this paper, we prove that, if $\mu > 0$, then there exists a linear manifold M of harmonic functions in $\mathbb{R}^N$ which is dense in $h(\mathbb{R}^N)$ and

$$\lim_{\substack{\|x\|\to\infty\\ x\in S}} \|x\|^\mu D^\alpha v(x) = 0$$

for every $v \in M$ and multi-index α, where S denotes any hyperplane strip. Moreover, every nonnull function in M is universal.

In particular, we see that if $\mu \ge N+1$, then all functions in M satisfy (1).

In [7] Ash and Brown prove that there is a harmonic function, which is not identically zero, in the unit disc of the complex plane which has zero nontangential limit at every point of the boundary of the disc.

We prove that there exists a linear manifold M of harmonic functions in the unit ball of $\mathbb{R}^N$, which is dense in the space of all harmonic functions in the unit ball (with the compact-open topology) and each function in M has zero nontangential limit at every point of the boundary.

2. "Counterexamples" to the harmonic Liouville theorem. We need some notation. A *hyperplane strip* is the region lying between two parallel hyperplanes in $\mathbb{R}^N$. $h(\mathbb{R}^N)$ is endowed with the compact-open topology. $(\mathbb{R}^N)^*$ is the one-point compactification of $\mathbb{R}^N$. If F is a closed subset of $\mathbb{R}^N$, then $h(F)$ is the space of all harmonic functions on a neighbourhood of F.

THEOREM 1. *If $\mu > 0$, then there exists a linear manifold M of harmonic functions in $\mathbb{R}^N$ with the following properties*:

(a) *M is dense in $h(\mathbb{R}^N)$*,

(b) *$\lim_{\|x\|\to\infty,\, x\in S} \|x\|^\mu D^\alpha v(x) = 0$ on any hyperplane strip S, for every $v \in M$ and for every multi-index α*,

(c) *$D^\alpha v$ is bounded on any hyperplane strip*,

(d) *Every nonnull function in M is universal.*

Hence, *if $\mu \geq N+1$, then*

(e) *$D^\alpha v$ is integrable with respect to N-dimensional Lebesgue measure on any hyperplane strip*,

(f) *$D^\alpha v$ is integrable with respect to $(N-1)$-dimensional Lebesgue measure on any hyperplane H and $\int_H D^\alpha v = 0$.*

Proof. Let $\mu > 0$ and consider a denumerable family $\{p_n\}_{n=1}^\infty$ of harmonic polynomials dense in $h(\mathbb{R}^N)$. (Moreover, given $u \in h(\mathbb{R}^N)$ and a compact set $K \subset \mathbb{R}^N$, there exists a sequence $n_1 < n_2 < \ldots$ such that $p_{n_k} \to u$ uniformly on K.)

Let L denote the curve

$$L := \{(x_1,\ldots,x_N) : x_1 = t,\ x_2 = t^2,\ldots,x_N = t^N;\ t \geq 0\}$$

and T the tract

$$\{x \in \mathbb{R}^N : \operatorname{dist}(x,L) < (1+\|x\|)^{1/2}\}.$$

Then there exists a disjoint sequence $D_j^* = D(a_j, 1+2^j)$ of open discs such that $D_j^* \subset T \setminus L$ and $\|a_{j+1}\| > \|a_j\| + 2^{j+2}$ for $j = 1,2,\ldots$

Consider now $B_n := \{x : \|x\| \leq n\}$ and

$$E_n := \{x \in \mathbb{R}^N : \|x\| \geq n+1 \text{ and } \operatorname{dist}(x,L) \geq (1+\|x\|)^{1/2}\}$$

and define

$$F_n := B_n \cup E_n \cup \bigcup_{j>j_0} \overline{D}(a_j, 2^j)$$

where j_0 is the last index such that $D_{j_0}^* \cap B_{n+1} \neq \emptyset$. Then F_n is a closed subset of $\mathbb{R}^N$, and $(\mathbb{R}^N)^* \setminus F_n$ is connected and locally connected at $\{*\}$.

Now, let Ω_{B_n} and Ω_{E_n} be disjoint open subsets of $\mathbb{R}^N$ containing B_n and E_n respectively, which intersect no D_j^* for $j > j_0$. Dividing the sequence $\{a_j\}$ into infinitely many disjoint subsequences $\{a_{i(m,j)}\}$ by setting $i(m,j) = (j+m)(j+m+1)/2 + m$, we define the function $u_n : F_n \to \mathbb{R}$ by

$$u_n(z) = \begin{cases} p_n(x), & x \in \Omega_{B_n}, \\ 0, & x \in \Omega_{E_n}, \\ p_j(x - a_{i(n,j)}), & x \in D^*_{i(n,j)}, \\ 0, & x \in D^*_{i(m,j)},\ m \neq n. \end{cases}$$

Then $u_n \in h(F_n)$, and by Theorem 1.1 of [3], there exists $v_n \in h(\mathbb{R}^N)$ such that

$$|(v_n - u_n)(x)| < \frac{1}{n}(1 + \|x\|)^{-\mu-1}, \quad x \in F_n. \tag{2}$$

Hence

$$|(v_n - u_n)(x)| < 1/n, \quad x \in B_n.$$

Thus the sequence $\{v_n\}_{n=1}^{\infty}$ is dense in $h(\mathbb{R}^N)$. Now, we define M as the linear span of $\{v_n\}$. Evidently, M is a linear dense submanifold of $h(\mathbb{R}^N)$. In order to verify (b), it suffices to check it for v_n.

Define E_n^* as

$$E_n^* := \{x \in \mathbb{R}^N : \|x\| \ge n + 2 \text{ and } \operatorname{dist}(x, L) \ge (1 + \|x\|)^{1/2} + 1\}.$$

Then by using the Cauchy estimates for harmonic functions [8, p. 33], from (2), since $u_n = 0$ in E_n, we infer that

$$|D^\alpha v_n(x)| \le C_\alpha \max\{|v_n(y)| : \|y - x\| \le 1\} \le C_\alpha \frac{1}{\|x\|^{\mu+1}}$$

for all $x \in E_n^*$ and all multi-indices α.

Note that every fixed hyperplane strip S is wholly contained in E_n^* except for a bounded set. Therefore,

$$\lim_{\substack{\|x\| \to \infty \\ x \in S}} \|x\|^\mu D^\alpha v(x) = 0$$

for every $v \in M$ and multi-index α on any hyperplane strip S. Moreover, every function in M satisfies (c), and (e) if $\mu \ge N + 1$.

Also every nonnull function of M is universal. If $v \in M$, $v = \sum_{j \in I} \alpha_j v_j$, I finite, since every nonzero scalar multiple of a universal function is again universal, we may suppose that $\alpha_{j_1} = 1$ with $j_1 \in I$. In order to prove that $v = \sum_{j \in I} \alpha_j v_j$ is universal, it is enough to check that

$$\lim_{n \to \infty} \Big(\sum_{j \in I} \alpha_j v_j(x + a_{i(j_1, n)}) - p_n(x) \Big) = 0 \tag{3}$$

uniformly on compact subsets. This is readily seen by computing

$$\max_{\overline{D}(0, 2^n)} \Big| \sum_{j \in I} \alpha_j v_j(x + a_{i(j_1, n)}) - p_n(x) \Big|. \tag{4}$$

By the triangle inequality, (4) is less than

$$\max_{\overline{D}(0, 2^n)} |v_{j_1}(x + a_{i(j_1, n)}) - p_n(x)| + \max_{\overline{D}(0, 2^n)} \sum_{\substack{j \in I \\ j \ne j_1}} |\alpha_j| \cdot |v_j(x + a_{i(j_1, n)})|.$$

Now by (2), we conclude that v is universal.

For details of (f), see [5].

Remark 1. As in the holomorphic case (see [9]), it is possible to prove that any set $M \subset h(\mathbb{R}^N)$ satisfying (b) is of the first category in $h(\mathbb{R}^N)$.

3. Harmonic functions with zero nontangential limits. By $h(\mathbb{B})$ we denote the space of all harmonic functions in the unit ball of $\mathbb{R}^N$. $h(\mathbb{B})$ is endowed with the compact-open topology. $\mathbb{B}^*$ is the one-point compactification of $\mathbb{B}$. If F is a relatively closed subset of $\mathbb{B}$, then $h(F)$ is the space of all harmonic functions on a neighbourhood of F.

Theorem 2. *There exists a dense linear manifold M of harmonic functions in $\mathbb{B}$ which have zero nontangential limit at every point of $\partial\mathbb{B}$.*

Proof. Consider a denumerable family $\{p_n\}_{n=1}^{\infty}$ of harmonic polynomials dense in $h(\mathbb{B})$ with the compact-open topology.

Let $F_n = B_n \cup E_n$, where $B_n = \{x : \|x\| \le 1 - 1/n\}$ and

$$E_n = \mathbb{B} \setminus (B_{n+1} \cup \{x = (x_1, \ldots, x_N) \in \mathbb{B} : \\ x_N < 0,\ x \in B((1/2, 0, \ldots, 0), 1/2) \setminus B((3/4, 0, \ldots, 0), 1/4)\}).$$

Then F_n is a relatively closed subset of $\mathbb{B}$, and $\mathbb{B}^* \setminus F_n$ is connected and locally connected at $\{*\}$.

Let Ω_{B_n} and Ω_{E_n} be disjoint open subsets of $\mathbb{B}$ containing B_n and E_n respectively, and define the function $u_n : F_n \to \mathbb{R}$ by

$$u_n(z) = \begin{cases} p_n(x), & x \in \Omega_{B_n}, \\ 0, & x \in \Omega_{E_n}. \end{cases}$$

Then $u_n \in h(F_n)$, and by Theorem 1.1 and Remark 9.1 of [6], there exists $v_n \in h(\mathbb{B})$ such that

$$|(v_n - u_n)(x)| < \frac{1}{n} \operatorname{dist}(x, \partial\mathbb{B}), \qquad x \in F_n. \tag{5}$$

Hence

$$|(v_n - u_n)(x)| < 1/n, \qquad x \in B_n.$$

Thus, the sequence $\{v_n\}_{n=1}^{\infty}$ is dense in $h(\mathbb{B})$. Now, we define M as the linear span of $\{v_n\}$. Evidently, M is a linear dense manifold in $h(\mathbb{R}^n)$.

In order to verify that every $v \in M$ has zero nontangential limit at every point of $\partial\mathbb{B}$, it suffices to check it for every v_n. But this is a consequence of

$$|v_n(x)| < \operatorname{dist}(x, \partial\mathbb{B}), \qquad \forall x \in E_n.$$

Remark 2. The set of harmonic functions in $\mathbb{B}$ with zero nontangential limit at all points is of the first category. Indeed, given $x \in \partial\mathbb{B}$ and $n, m \in \mathbb{N}$, let

$$E_{n,m}(x) := \{u \in h(\mathbb{B}) : |u(rx)| \le 1/n,\ 1 - 1/m < r < 1\}$$

and

$$A := \{u \in h(\mathbb{B}) : \lim_{r \to 1} u(rx) = 0,\ \forall x \in \partial\mathbb{B}\}.$$

Then

$$A = \bigcap_{x\in\partial\mathbb{B}} \bigcap_{n} \bigcup_{m} E_{n,m}(x)$$

and $E_{n,m}(x)$ is closed and nowhere dense.

REMARK 3. If $N = 2$, then the nonzero functions u in M do not have the property that there exists a positive number α so that $m(r) = \sup_{|z|\le r} |u(z)| = o((1-r)^{-\alpha})$ as $r \to 1$ (see [7, Corollary 1]).

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