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# *COMPLEXITY OF THE CLASS OF PEANO FUNCTIONS*

BY

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**Abstract.** We evaluate the descriptive set theoretic complexity of the space of continuous surjections from $\mathbb{R}^m$ to $\mathbb{R}^n$.

The stimulus for our research comes from certain problems in topology concerning the homogeneity of locally compact separable metric spaces. Starting with a paper of Effros [2] several results appeared (see, for example, papers by Charatonik, Maćkowiak and Krupski [1], [5]) establishing that in certain situations homogeneity of $X$ with respect to a family $M$ of continuous selfmaps of $X$ implies strong forms of homogeneity with respect to $M$. (Typically one tries to prove that given $a, b \in X$, for some $f \in M$ with $f(a) = b$, almost all points, in the sense of category, which are close to $b$ are of the form $g(a)$ where $g \in M$ is close to $f$.) The possibility of proving such theorems without invoking strong axioms of set theory depends on the family $M$ being Borel or analytic. Many natural families $M$ were proved Borel by Krupski [5]. In this context, he asked (personal communication) whether the family of all continuous surjections from a locally compact, separable, metric space onto itself is Borel. We prove here that this is not the case for $\mathbb{R}^2$. More generally, we evaluate the descriptive set theoretic complexity of the class

$$\mathrm{Peano}(m, n) = \{f : \mathbb{R}^m \to \mathbb{R}^n \mid f \text{ is continuous and onto}\}.$$

This is a subset of $C(\mathbb{R}^m, \mathbb{R}^n)$, the space of all continuous function from $\mathbb{R}^m$ to $\mathbb{R}^n$, which is given the topology of uniform convergence on compact sets, also called the compact-open topology. This topology is Polish.

For a class $\Gamma$ of subsets of Polish spaces which is closed under taking continuous preimages, a set $A \subseteq X$, $X$ Polish, is called $\Gamma$ *hard* if for any zero-dimensional Polish space $Y$ and any $B \subseteq Y$, $B \in \Gamma$, there exists a continuous function $\phi : Y \to X$, called a *reduction*, such that $y \in B$ if, and only if, $\phi(y) \in A$ for $y \in Y$. Moreover, $A$ is said to be $\Gamma$ *complete* if $A \in \Gamma$

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and $A$ is $\Gamma$ hard. Note that to show that $A$ is $\Gamma$ complete it is enough to find a set $B$ which is known to be $\Gamma$ complete and a continuous reduction of $B$ to $A$. Note also that since there are co-analytic sets which are not analytic (or, equivalently, not Borel) a co-analytic complete set is necessarily not analytic and so not Borel. Similarly a $G_\delta$ complete set is not $F_\sigma$.

For a compact space $X$ we denote by $K(X)$ the compact space of all compact nonempty subsets of $X$ with the Vietoris topology (see [4, Section 4F]). By $\mathcal{C}$ we denote the Cantor set $2^{\mathbb{N}} = \{0,1\}^{\mathbb{N}}$. We identify it, via the usual construction, with a subset of the interval $[0,1]$. For a countable set $X$, by identifying $\mathcal{P}(X)$ with $2^X = \{0,1\}^X$ we can endow $\mathcal{P}(X)$ with the product topology. If $X$ is infinite, this space is homeomorphic to $\mathcal{C}$, hence Polish. We can thus also assign a topology to the space $\mathrm{Tr}(\mathbb{N})$ of all trees on $\mathbb{N}$ since $\mathrm{Tr}(\mathbb{N}) \subseteq 2^{\mathbb{N}^{<\mathbb{N}}}$. (A set $T \subseteq \mathbb{N}^{<\mathbb{N}}$ is a *tree* if for each $t \in T$ each initial segment of $t$ belongs to $T$.) Since $\mathrm{Tr}(\mathbb{N})$ is closed in $2^{\mathbb{N}^{<\mathbb{N}}}$, this topology is Polish. A tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ is *well-founded* if there is no $x \in \mathbb{N}^{\mathbb{N}}$ with $x|n \in T$ for each $n$. We will use the standard result that: *the set* WF *of all well-founded trees is co-analytic complete*. (For a review of Polish spaces and descriptive set theory, including Borel, analytic, and co-analytic sets, see [4].)

THEOREM 1. (i) $\mathrm{Peano}(m,1)$ *is* $G_\delta$ *complete.*
(ii) *For* $n \geq 2$, $\mathrm{Peano}(m,n)$ *is co-analytic complete.*

Proof. By definition,

$$\mathrm{Peano}(m,n) = \{f \in C(\mathbb{R}^m,\mathbb{R}^n) \mid \forall y \in \mathbb{R}^n\ \exists x \in \mathbb{R}^m \text{ such that } f(x) = y\}.$$

Thus $\mathrm{Peano}(m,n)$ is co-analytic being the co-projection of the countable union of closed sets $S_k$, $k \in \mathbb{N}$, where

$$S_k = \{(f,y) \in C(\mathbb{R}^m,\mathbb{R}^n) \times \mathbb{R}^n \mid \exists x \in \mathbb{R}^m \text{ such that } |x| \leq k \text{ and } f(x)=y\}.$$

Now we show that $\mathrm{Peano}(m,1)$ is $G_\delta$. For a continuous function $f : \mathbb{R}^m \to \mathbb{R}$,

$$f \in \mathrm{Peano}(m,1) \Leftrightarrow \forall k \in \mathbb{N}(\exists x \in \mathbb{R}^m\ f(x) > k \text{ and } \exists x \in \mathbb{R}^m\ f(x) < -k).$$

The condition $f(x) > k$ defines an open set in $C(\mathbb{R}^m,\mathbb{R}) \times \mathbb{R}^m$. Since projections preserve open sets, $\exists x \in \mathbb{R}^m\ f(x) > k$ defines an open set in $C(\mathbb{R}^m,\mathbb{R})$ as does $\exists x \in \mathbb{R}^m\ f(x) < -k$. So $\mathrm{Peano}(m,1)$ is $G_\delta$.

Peano functions were first studied with a compact domain and range, like the closed unit interval and square. The proof above shows that in such a setting, the class of all Peano functions is closed.

To complete the proof of (i), we will continuously reduce to $\mathrm{Peano}(m,1)$ the set

$$N_2 = \{x \in \mathcal{C} \mid x(n) = 1 \text{ for infinitely many } n\},$$

which is known to be $G_\delta$ complete [4, p. 179]. We will carry this out only for Peano$(1,1)$, as the extension to higher dimensions is straightforward. To do so, we define $\phi : \mathcal{C} \to C(\mathbb{R}, \mathbb{R})$ which sends any $x \in \mathcal{C}$ to the unique function $f$ such that

1. for all negative integers $z$, $f(z) = z$,
2. for all nonnegative integers $n$, $f(n) = n \cdot x(n)$,
3. $f$ is linear between successive integers.

Clearly, $x \in N_2 \Leftrightarrow \phi(x) \in$ Peano$(1,1)$. The map $\phi$ is continuous, since if $(x_n) \to x$ in $\mathcal{C}$, then for any compact interval $I$ in $\mathbb{R}$, $\phi(x_n)|I = \phi(x)|I$ for large enough $n$.

To finish the proof we only need to show that for all $n \geq 2$, Peano$(m,n)$ is co-analytic hard. To simplify notation we will carry it out only for Peano$(1,2)$. We will present two arguments for it. In the first one we will reduce WF to Peano$(1,2)$. The reduction we will produce will not be continuous but merely Borel. However, Kechris [3] has recently shown that if a set is co-analytic complete with respect to Borel reductions, then it is co-analytic complete with respect to continuous reductions. Moreover, one can easily see directly that a co-analytic set complete with respect to Borel reductions is not Borel. The second argument will continuously reduce the so-called Hurewicz set.

Argument 1

Lemma 1. *The set* $\mathcal{A} = \{(K_n) \in K(\mathcal{C})^{\mathbb{N}} \mid \bigcup_n K_n = \mathcal{C}\}$ *is co-analytic hard.*

Proof. This seems to be a folklore result in descriptive set theory, yet it is so directly relevant to our work that we sketch a proof.

For $X$ a set of finite sequences of 0's and 1's, i.e., $X \in 2^{2^{<\mathbb{N}}}$, let

$$G(X) = \{c \in \mathcal{C} \mid (c(0), \ldots, c(n-1)) \in X \text{ for infinitely many } n\}.$$

First we need to see that the set $\mathcal{B} = \{X \in 2^{2^{<\mathbb{N}}} \mid G(X) = \emptyset\}$ is co-analytic hard. Let $0^n$ be the sequence consisting of $n$ zeros. For any finite sequence $a \in \mathbb{N}^{<\mathbb{N}}$ of length $n$ let $\phi(a) = 0^{a(0)}10^{a(1)}1\ldots0^{a(n-1)}1$ (in particular, $\phi(\emptyset) = \emptyset$). The function $\widetilde{\phi} : \mathrm{Tr}(\mathbb{N}) \to 2^{2^{<\mathbb{N}}}$ which continuously reduces WF to $\mathcal{B}$ is now defined by $\widetilde{\phi}(T) = \{\phi(t) : t \in T\}$.

Now we will find a Borel reduction of $\mathcal{B}$ to $\mathcal{A}$. For all $X \subset 2^{<\mathbb{N}}$ and $k \in \mathbb{N}$, we define

$$K_X^k = \{c \in \mathcal{C} \mid (c(0), \ldots, c(n-1)) \in X \text{ for at most } k \text{ } n\text{'s}\}.$$

It is easy to see that each $K_X^k$ is compact. We now define $\psi : 2^{2^{<\mathbb{N}}} \to K(\mathcal{C})^{\mathbb{N}}$ by

$$\psi(X) = (K_X^k).$$

It is straightforward to check that $\psi$ reduces $\mathcal{B}$ to $\mathcal{A}$. Showing that $\psi$ is Borel is also not difficult and we leave it to the reader.

Our intention now is to construct a continuous reduction

$$\phi : K(\mathcal{C})^{\mathbb{N}} \to C(\mathbb{R}, \mathbb{R}^2)$$

of $\mathcal{A}$ to Peano$(1,2)$. This will finish the proof by Lemma 1. Let $g$ be a continuous surjection from $(-\infty,-1]$ onto $\mathbb{R}^2 \setminus (\mathcal{C} \times \{0\})$ with $g(-1) = (-1,1)$. For $K \in K(\mathcal{C})$ let $f_K : [-1,2] \to \mathbb{R}^2$ be given by

$$f_K(x) = (x, \min\{|x-a| : a \in K\}) \quad \text{for } \min K \le x \le \max K$$

and let it be linear on $[-1, \min K]$ and $[\max K, 2]$ and such that $f_K(-1) = (-1,1)$ and $f_K(2) = (2,1)$.

Let $\phi((K_n)) = f_{(K_n)} : \mathbb{R} \to \mathbb{R}^2$ be given by

$$f_{(K_n)}(x) = \begin{cases} g(x) & \text{for } x \in (-\infty,-1], \\ f_{K_n}(x-6n) & \text{for } x \in [6n-1, 6n+2],\ n = 0,1,2,\ldots, \\ (6n+4-x, 1) & \text{for } x \in [6n+2, 6n+5],\ n = 0,1,2,\ldots \end{cases}$$

Observe that $(\mathcal{C} \times \{0\}) \cap f_{(K_n)}[\mathbb{R}] = (\bigcup_n K_n) \times \{0\}$, so $f_{(K_n)}$ is a surjection iff $(K_n) \in \mathcal{A}$, hence $\phi$ is a reduction. It is not hard to verify that it is continuous.

Argument 2. This proof of co-analytic hardness of Peano$(1,2)$ uses co-analytic hardness of *Hurewicz's set*

$$\mathcal{H} = \{K \in K([0,1]) : K \subseteq \mathbb{Q}\}.$$

For a proof of this fact see [4, 27.4].

It is enough to construct a continuous map

$$\phi : K([0,1]) \to C(\mathbb{R}, \mathbb{R}^2)$$

such that $\phi(K)$ is in Peano$(1,2)$ iff $K \in \mathcal{H}$.

First let us fix some auxiliary maps:

The set $W = (\mathbb{R}^2 \setminus [0,1] \times \{0\}) \cup (\mathbb{Q} \times \{0\})$ is a countable union of locally connected continua and it is arcwise connected hence there exists a continuous surjection $g : (-\infty,-1] \to W$ with $g(-1) = (-1,0)$. Let $h : [0,3] \to \mathbb{R}^2$ be defined by the formula $h(x) = (2-x, |3/2-x| - 3/2)$.

For each $n \in \mathbb{N}$ and $K \in K([0,1])$ define $f_{K,n} : [-1,2] \to \mathbb{R}^2$ by

$$f_{K,n}(x) = (x, \max\{0, 1 - 2^n \min\{|x-a| : a \in K\}\}).$$

Roughly speaking: for fixed $K$ and for large $n$ the image of $f_{K,n}$ looks like an approximation of the graph of the characteristic function of $K$.

Finally, we define $\phi(K) = f_K : \mathbb{R} \to \mathbb{R}^2$ as follows:

$$f_K(x) = \begin{cases} g(x) & \text{for } x \in (-\infty,-1], \\ f_{K,n}(x-6n) & \text{for } x \in [6n-1, 6n+2],\ n = 0,1,2,\ldots, \\ h(x-6n-2) & \text{for } x \in [6n+2, 6n+5],\ n = 0,1,2,\ldots \end{cases}$$

Now, it is easy to check that

$$([0,1] \times \{0\}) \cap f_K(\mathbb{R}) = ((\mathbb{Q} \cap [0,1]) \cup ([0,1] \setminus K)) \times \{0\},$$

so $f_K$ is a surjection iff $K \subset \mathbb{Q}$. Note that the family $\{f_K\}_{K \in K([0,1])}$ is equicontinuous. It is now not hard to verify that $\phi$ has all the desired properties.

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