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SEPARATION PROPERTIES IN CONGRUENCE LATTICES OF LATTICES

BY

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Abstract. We investigate the congruence lattices of lattices in the varieties $\mathcal{M}_n$. Our approach is to represent congruences by open sets of suitable topological spaces. We introduce some special separation properties and show that for different n the lattices in $\mathcal{M}_n$ have different congruence lattices.

1. Introduction. It is well known that a lattice is algebraic if and only if it is the congruence lattice of some algebra. Much less is known if we require this algebra to be of a special type. For instance, there is a longstanding problem of whether every distributive algebraic lattice is isomorphic to the congruence lattice of some lattice. We refer to this problem as CLP.

The only class of lattices for which the congruence lattices are well understood is the variety of distributive lattices. (We recall these results in Section 4.) Recent investigations show that the congruence lattices of large nondistributive lattices have much more complicated structure. Especially, some refinement properties come into play. (See [10], [14].)

In this paper we investigate the congruence lattices of lattices belonging to the varieties $\mathcal{M}_n$ generated by the lattices M_n, $n \geq 3$. Here M_n denotes the lattice of length 2 with n middle elements. As our main achievement we consider the discovery of special conditions, called here n-separability. These conditions show that lattices in $\mathcal{M}_n$ for different n have, in general, different congruence lattices. We believe that further investigation of conditions of this kind can contribute significantly to the solution of CLP. We also hope that the topological method used in this paper can be successfully applied to other varieties of lattices and other algebras.

Our reference books are [8] for universal algebra, [4] for lattice theory and [6] for topology. All notions unexplained in our paper can be found there. We adopt the following notations. If f is a function, then $\mathrm{dom}(f)$ and $\mathrm{rng}(f)$ stand for its domain and range, respectively. By $\ker(f)$ (the kernel) we denote the binary relation on $\mathrm{dom}(f)$ given by $(x, y) \in \ker(f)$ iff $f(x) = f(y)$. The composition of functions is written in such a way that

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$(fg)(x) = f(g(x))$. By $f\restriction X$ we mean the restriction of f to X. If P is an ordered set and $x \in P$ then $\uparrow x = \{y \in P \mid y \geq x\}$ and $\downarrow x = \{y \in P \mid y \leq x\}$.

2. Topological representation. An element a of a lattice L is called *strictly meet irreducible* iff $a = \bigwedge X$ implies that $a \in X$ for every subset X of L. The greatest element of L is not strictly meet irreducible. Let $M(L)$ denote the set of all strictly meet irreducible elements of L. We shall call a set $Z \subseteq M(L)$ *closed* if $Z = M(L) \cap \uparrow x$ for some $x \in L$. A set is called *open* if its complement is closed.

2.1. LEMMA. *Let L be a complete lattice. The sets $\emptyset$, $M(L)$ are closed. The intersection of any family of closed sets is closed. If L is distributive then the union of two closed sets is closed.*

Proof. It is easy to see that $\emptyset = M(L) \cap \uparrow 1$, $M(L) = M(L) \cap \uparrow 0$, $\bigcap_{i\in I}(M(L) \cap \uparrow x_i) = M(L) \cap \uparrow(\bigvee_{i\in I} x_i)$ for any $x_i \in L$. Further, if L is distributive then $(M(L) \cap \uparrow x_1) \cup (M(L) \cap \uparrow x_2) = M(L) \cap \uparrow(x_1 \wedge x_2)$ for any $x_1, x_2 \in L$. ∎

Thus, if L is complete and distributive, then $M(L)$ is a topological space. For any topological space T let $\mathcal{O}(T)$ denote the family of all open subsets, ordered by set inclusion.

2.2. THEOREM. *If L is a distributive algebraic lattice, then L is isomorphic to $\mathcal{O}(M(L))$.*

Proof. The assignment $x \mapsto M(L) \setminus \uparrow x$ is clearly surjective and order preserving. The rest follows from the fact that if L is algebraic, then $x = \bigwedge(M(L) \cap \uparrow x)$ for every $x \in L$. (See 2.19 in [8] or I.4.23 in [3].) ∎

Let us note that (under the assumptions of 2.2) the assignment $x \mapsto \uparrow x$ is a bijection between elements of L and closed subsets of $M(L)$. However, it reverses order ($x \leq y$ iff $\uparrow y \subseteq \uparrow x$), so we choose the representation by open sets.

Topological representations for distributive (algebraic) lattices appear quite often in the literature, starting with M. H. Stone ([12]). Let us just mention that the representations in [3] (Section V.4) and [4] (Section II.5) are very close (but, in general, not equivalent) to our construction.

We are interested in the case when $L = \operatorname{Con} A$ is the congruence lattice of an algebra A. Then $M(\operatorname{Con} A)$ is the set of all $\varrho \in \operatorname{Con} A$ such that the factor algebra A/ϱ is subdirectly irreducible. In the sequel we investigate connections between subdirect decomposition of an algebra A and topological properties of the space $M(\operatorname{Con} A)$. We wish to mention that our original source of motivation here is the natural duality theory of B. A. Davey and H. Werner ([2], [1]). Although we make no explicit use of this theory, the reader might observe some of its characteristic features, especially the correspondence between congruences of an algebra and open subsets of its dual space.

For the rest of this section we make the following assumptions. Let the algebra A belong to a congruence distributive, locally finite variety $\mathcal{V}$. Let $\mathrm{SI}(\mathcal{V})$ be the class of all subdirectly irreducible algebras in $\mathcal{V}$. Choose $\mathcal{S} \subseteq \mathrm{SI}(\mathcal{V})$ such that every $T \in \mathrm{SI}(\mathcal{V})$ is isomorphic to some $S \in \mathcal{S}$ and no two members of $\mathcal{S}$ are isomorphic. Note that $\mathcal{S}$ may be a proper class.

We are going to describe $M(\mathrm{Con}\, A)$ by means of homomorphisms. For $S \in \mathcal{S}$ let $H_S(A)$ denote the set of all surjective homomorphisms $A \to S$ and $H(A) = \bigcup\{H_S(A) \mid S \in \mathcal{S}\}$. Clearly, $\ker(f) \in M(\mathrm{Con}\, A)$ for every $f \in H(A)$. Let φ denote the surjective mapping $H(A) \to M(\mathrm{Con}\, A)$ defined by $\varphi(f) = \ker(f)$. We say that a set $G \subseteq H(A)$ is closed iff $G = \varphi^{-1}(Z)$ for some closed subset Z of $M(\mathrm{Con}\, A)$. Equivalently G is closed iff $G = \{f \in H(A) \mid \ker(f) \supseteq \alpha\}$ for some $\alpha \in \mathrm{Con}\, A$. This obviously defines a topology on $H(A)$ and φ becomes continuous.

2.3. THEOREM. *The lattices $\mathcal{O}(H(A))$ and $\mathrm{Con}\, A$ are isomorphic.*

Proof. Let $\psi : \mathcal{O}(M(\mathrm{Con}\, A)) \to \mathcal{O}(H(A))$ be defined by $\psi(Y) = \varphi^{-1}(Y)$. It is easy to see that ψ is a bijective lattice homomorphism. By 2.2, $\mathcal{O}(M(\mathrm{Con}\, A))$ is isomorphic to $\mathrm{Con}\, A$. ∎

The map φ need not be injective. It is easy to see that $\ker(f) = \ker(g)$ for $f \in H_S(A)$, $g \in H_T(A)$ iff there exists an isomorphism $h : S \to T$ such that $hf = g$. Since we assume that all $S \in \mathcal{S}$ are mutually nonisomorphic, this can only happen if $S = T$ and h is an automorphism of S. In such a case f and g are topologically indistinguishable in $H(A)$, i.e. for every closed (or open) set G we have $f \in G$ iff $g \in G$. By identifying the indistinguishable points we obtain a topological space $\overline{H(A)}$. More precisely, let $\sim$ be the equivalence relation on $H(A)$ given by $f \sim g$ iff $\ker(f) = \ker(g)$. Let $\overline{H(A)}$ be the factor set $H(A)/{\sim}$. The $\sim$-class containing $g \in H(A)$ will be denoted by $\overline{g}$. The set $\overline{G} \subseteq \overline{H(A)}$ is defined to be closed if $\overline{G} = \{\overline{g} \mid g \in G\}$ for some closed set $G \subseteq H(A)$. Equivalently, $\overline{G}$ is closed if $\overline{G} = \{\overline{g} \mid \ker(g) \supseteq \alpha\}$ for some $\alpha \in \mathrm{Con}\, A$. Now the assignment $\overline{g} \mapsto \ker(g)$ defines a homeomorphism $\overline{H(A)} \to M(\mathrm{Con}\, A)$ and we have the following assertion.

2.4. THEOREM. *The spaces $\overline{H(A)}$ and $M(\mathrm{Con}\, A)$ are homeomorphic. Consequently, the lattices $\mathcal{O}(\overline{H(A)})$ and $\mathrm{Con}\, A$ are isomorphic.*

For a subset $B \subseteq A$ let $\langle B\rangle$ denote the subalgebra of A generated by B. Let $X \subseteq A$ generate the whole A. (We are mainly interested in the case when X is a free generating set, but this is not essential here.) Let $X_0 \subseteq X$ be finite, $S \in \mathcal{S}$ and let $r : X_0 \to S$ be a map that can be extended to a surjective homomorphism $r^* : \langle X_0\rangle \to S$. For any such r we define

$$G_r = \{f \in H(A) \mid \ker(f{\restriction}\langle X_0\rangle) \subseteq \ker(r^*)\}.$$

Equivalently, $f \in G_r$ iff there exists a homomorphism $h : f(\langle X_0\rangle) \to S$ with $hf{\restriction}\langle X_0\rangle = r^*$.

2.5. THEOREM. *Every set of the form G_r is open and compact in $H(A)$. Every open set in $H(A)$ is a union of sets of the form G_r. Consequently, these sets form a basis of the topology on $H(A)$.*

Proof. Let $\alpha = \bigcap\{\ker(f) \mid f \in H(A) \setminus G_r\}$. Then clearly $H(A) \setminus G_r \subseteq \{f \in H(A) \mid \ker(f) \supseteq \alpha\}$. To show that G_r is open, it suffices to prove the inverse inclusion. Let $g \in G_r$. We need to show that $\ker(g) \not\supseteq \alpha$.

Define an equivalence relation $\approx$ on $H(A)\setminus G_r$ by $f_1 \approx f_2$ iff $\ker(f_1{\restriction}\langle X_0\rangle) = \ker(f_2{\restriction}\langle X_0\rangle)$. Because of the local finiteness of $\mathcal{V}$, $\langle X_0\rangle$ is finite and we have only a finite number of equivalence classes $C_1, \ldots, C_m$. Let $\alpha_i = \bigcap\{\ker(f) \mid f \in C_i\}$. Clearly, $\alpha = \alpha_1 \wedge \ldots \wedge \alpha_m$. Since $\ker(g)$ is meet irreducible it suffices to show that $\ker(g) \not\supseteq \alpha_i$ for every i. Choose $f \in C_i$ arbitrarily. Since $f \notin G_r$, we have $\ker(f{\restriction}\langle X_0\rangle) \not\subseteq \ker(r^*)$. Hence, there are $u, v \in \langle X_0\rangle$ with $(u, v) \in \ker(f)$ and $(u, v) \notin \ker(r^*)$. Then $(u, v) \in \ker(f')$ for every $f' \approx f$, hence $(u, v) \in \alpha_i$. On the other hand, $\ker(g{\restriction}\langle X_0\rangle) \subseteq \ker(r^*)$, hence $(u, v) \notin \ker(g)$.

To prove the compactness, we first show that α is compact. The relations $\alpha'_i = \alpha_i \cap \langle X_0\rangle^2$ are congruences of $\langle X_0\rangle$. We already know that $\alpha'_i \not\subseteq \ker(r^*)$. Since $r^*(\langle X_0\rangle) \in \mathcal{S}$, $\ker(r^*)$ is strictly meet irreducible in $\mathrm{Con}\langle X_0\rangle$ and we deduce that $\alpha \cap \langle X_0\rangle^2 = \alpha'_1 \wedge \ldots \wedge \alpha'_m \not\subseteq \ker(r^*)$. Let $\beta \in \mathrm{Con}\, A$ be generated by $\alpha \cap \langle X_0\rangle^2$. We claim that $\alpha = \beta$. Clearly $\beta \subseteq \alpha$. Assume now that $(u, v) \notin \beta$. There is a surjective homomorphism $f : A \to T \in \mathcal{V}$ with $\ker(f) = \beta$. The algebra T need not be subdirectly irreducible. However, by the subdirect representation theorem, there exists $S \in \mathcal{S}$ and a surjective homomorphism $h : T \to S$ such that $(f(u), f(v)) \notin \ker(h)$. Then $hf \in H(A)$ and $(u, v) \notin \ker(hf)$. Further, $\alpha \cap \langle X_0\rangle^2 \subseteq \ker(f{\restriction}\langle X_0\rangle) \subseteq \ker(hf{\restriction}\langle X_0\rangle)$, hence $\ker(hf{\restriction}\langle X_0\rangle) \not\subseteq \ker(r^*)$, $hf \notin G_r$ and $\alpha \subseteq \ker(hf)$. Thus, $(u, v) \notin \alpha$, which shows that $\alpha = \beta$. Since $\langle X_0\rangle$ is finite, β is finitely generated and hence compact.

By 2.3 the lattices $\mathcal{O}(H(A))$ and $\mathrm{Con}\, A$ are isomorphic and G_r in this isomorphism corresponds to α (since $G_r = \{f \in H(A) \mid \ker(f) \not\supseteq \alpha\}$). Hence, G_r is a compact element of the lattice $\mathcal{O}(H(A))$, which clearly means that G_r is a compact subspace of $H(A)$.

For the last assertion, let $G \subseteq H(A)$ be open and $g \in G$. We need to find G_r such that $g \in G_r \subseteq G$. Since G is open, we have $G = \{f \in H(A) \mid \ker(f) \not\supseteq \theta\}$ for some $\theta \in \mathrm{Con}\, A$. Hence, there are $u, v \in A$ with $(u, v) \in \theta \setminus \ker(g)$. There is a finite set $X_0 \subseteq X$ such that $u, v \in \langle X_0\rangle$.

There exist $T \in \mathcal{S}$ and a surjective homomorphism $h : g(\langle X_0 \rangle) \to T$ such that $(g(u), g(v)) \notin \ker(h)$. Set $r = hg\restriction X_0$. Then clearly $r^* = hg\restriction\langle X_0\rangle$, hence $\ker(g\restriction\langle X_0\rangle) \subseteq \ker(r^*)$ and $g \in G_r$. Further, for every $f \in G_r$ we have $(u, v) \notin \ker(r^*) \supseteq \ker(f\restriction\langle X_0\rangle)$, hence $\theta \nsubseteq \ker(f)$ and therefore $f \in G$. ∎

2.6. Corollary. *An open subset of $H(A)$ is compact iff it is a finite union of sets of the form G_r.*

Another consequence is that $H(A)$ has a basis of compact open sets, but this also follows easily from the fact that $\mathcal{O}(H(A))$ is isomorphic to the algebraic lattice $\operatorname{Con} A$.

Further, we define $\overline{G}_r = \{\overline{f} \in \overline{H(A)} \mid f \in G_r\}$ and obtain the following consequence.

2.7. Corollary. *All sets $\overline{G}_r$ are compact and form a basis of the topology on $\overline{H(A)}$. Compact open sets in $\overline{H(A)}$ are exactly the finite unions of sets of the form $\overline{G}_r$.*

Since the whole space $H(A)$ is an open set, it is also a union of sets of the form G_r.

2.8. Corollary. *The spaces $M(\operatorname{Con} A)$, $H(A)$ and $\overline{H(A)}$ are locally compact.*

The spaces $M(\operatorname{Con} A)$, $H(A)$ and $\overline{H(A)}$ need not be compact. Also, note that the topology on $H(A)$ is often close to the usual product topology. This is especially true if $\mathcal{S}$ consists of a single algebra S with no proper automorphisms. In this case $H(A)$ inherits its topology from the product topology of S^A (with S discrete). We discuss such an example in Section 4.

3. Free algebras. Let $\mathcal{V}$ and $\mathcal{S}$ be as in the previous section. Let $F(X)$ denote the free algebra in $\mathcal{V}$ with X as free generating set.

3.1. Lemma. *Let L be a distributive lattice, $x \in L$ and $Z = M(L) \cap \uparrow x$. Then the subspace Z of $M(L)$ is homeomorphic to $M(\uparrow x)$.*

Proof. It is easy to see that $q \in M(\uparrow x)$ iff $q \in M(L)$ and $q \geq x$. Hence the sets Z and $M(\uparrow x)$ are equal and it is easy to check that their topologies coincide. ∎

3.2. Theorem. *For a distributive lattice L the following conditions are equivalent.*

(1) $L \cong \operatorname{Con} A$ *for some* $A \in \mathcal{V}$;

(2) $M(L)$ *is homeomorphic to a closed subspace of* $M(\operatorname{Con} F(X))$ *for some set* X;

(3) $L \cong \mathcal{O}(Z)$ *for some X and some closed subspace Z of* $M(\operatorname{Con} F(X))$;

(4) $M(L)$ *is homeomorphic to a closed subspace of* $\overline{H(F(X))}$ *for some* X;

(5) $L \cong \mathcal{O}(Z)$ *for some X and a closed subspace Z of* $\overline{H(F(X))}$.

P r o o f. (1)⇒(2). Let $L \cong \operatorname{Con} A$. There is a set X and $\theta \in \operatorname{Con} F(X)$ such that $A \cong F(X)/\theta$. Set $Z = M(\operatorname{Con} F(X)) \cap \uparrow\theta$. Then $\operatorname{Con} A \cong \uparrow\theta$ and therefore $M(L)$ is homeomorphic to $M(\uparrow\theta)$, which by 3.1 is homeomorphic to Z.

(2)⇒(3). If $M(L)$ is homeomorphic to Z then, by 2.2, $L \cong \mathcal{O}(M(L)) \cong \mathcal{O}(Z)$.

(3)⇒(1). Let $L \cong O(Z)$ and $Z = M(\operatorname{Con} F(X)) \cap \uparrow\theta$. We set $A = F(X)/\theta$. By 3.1, Z is homeomorphic to $M(\uparrow\theta)$, hence $L \cong \mathcal{O}(M(\uparrow\theta)) \cong \mathcal{O}(M(\operatorname{Con} A)) \cong \operatorname{Con} A$.

(2)⇔(4) and (3)⇔(5) follow from 2.4. ∎

Thus, our strategy is to describe the spaces $\overline{H(F(X))}$ and their closed subspaces. (Note that, in accordance with 2.2, we use the lattices of *open* sets in such closed subspaces for the representation of congruence lattices.) The existence of a free generating set allows some simplification in the description of the spaces $H(A)$ and $\overline{H(A)}$. Every map $f : X \to S \in \mathcal{S}$ extends uniquely to a homomorphism $f^* : F(X) \to S$. Therefore, we can identify the set $H(F(X))$ with the set of all maps $f : X \to S \in \mathcal{S}$ such that $f(X)$ generates S. The sets G_r are defined for all $r : X_0 \to S \in \mathcal{S}$ such that $r(X_0)$ generates S.

4. Distributive lattices. Before we turn to the varieties $\mathcal{M}_n$, let us briefly discuss the well known case of distributive lattices. The variety $\mathcal{D}$ of distributive lattices contains only one subdirectly irreducible member, namely the two-element lattice $D = \{0, 1\}$. In this case $H(F(X)) = \overline{H(F(X))}$ and this space is isomorphic to the space $D^X \setminus \{o, i\}$ with the usual product topology. (Here o and i denote the two constant functions $X \to \{0, 1\}$.)

4.1. THEOREM. *For a distributive lattice L, the following conditions are equivalent.*

(1) $L \cong \operatorname{Con} M$ *for some* $M \in \mathcal{D}$;

(2) $L \cong \mathcal{O}(Z \setminus \{z\})$ *for some Boolean space* Z *and some* $z \in Z$;

(3) $L \cong \mathcal{O}(Z \setminus \{z_1, z_2\})$ *for some Boolean space* Z *and some* $z_1, z_2 \in Z$;

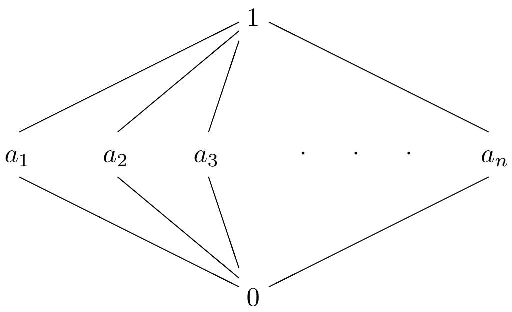
(4) $L \cong \mathcal{O}(Z')$ *for some locally compact, Hausdorff, zero-dimensional space* Z'.

P r o o f. It is a topological exercise to show that Z' satisfies the conditions in (4) iff $Z' = Z \setminus \{z\}$ iff $Z' = Z \setminus \{z_1, z_2\}$ for some Boolean (compact, Hausdorff, zero-dimensional) space Z. We omit this proof. The equivalence of (1) and (3) follows from 3.2 and is also known from Priestley or Stone duality. (See e.g. [4] or [11].) ∎

Boolean spaces are rather well investigated and some of their properties are easy to deduce. In a Hausdorff space, every compact set is closed and therefore every compact open set is clopen (closed and open). Furthermore, the intersection of two compact sets is compact ([6], exercise B in Chapter 5). Thus, the compact elements in the congruence lattice of a distributive lattice are closed under intersection. In fact, $L \cong \operatorname{Con} M$ for some $M \in \mathcal{D}$ iff the compact elements of L form a generalized Boolean algebra. (See [4], II.3 and II.4.) This nontopological characterization can also be deduced from (4).

The situation becomes even simpler if we consider the class $\mathcal{D}_{01}$ of bounded distributive lattices. Then $L \cong \operatorname{Con} M$ for some $M \in \mathcal{D}_{01}$ iff $L \cong \mathcal{O}(Z)$ for some Boolean space Z iff the compact elements of L form a Boolean algebra.

5. The varieties $\mathcal{M}_n$. For $n \geq 3$ let $M_n = \{0, 1, a_1, \ldots, a_n\}$ denote the lattice having the smallest element 0, the greatest element 1 and n mutually incomparable "middle" elements $a_1, \ldots, a_n$.



Let $\mathcal{M}_n$ denote the variety generated by the lattice M_n. It is well known that subdirectly irreducible members of $\mathcal{M}_n$ are (up to isomorphism) the lattice $D = \{0, 1\}$ and the lattices $M_3, M_4, \ldots, M_n$. (This follows easily from Jónsson's lemma.)

For any set X let $F_n(X)$ denote the free algebra in $\mathcal{M}_n$ with X as the set of free generators. According to Section 3, the points of $H(F_n(X))$ are all maps $f : X \to M_n$ such that either $f(X) = \{0, 1\}$ or $\{a_1, \ldots, a_k\} \subseteq f(X) \subseteq \{0, 1, a_1, \ldots, a_k\}$ for some $k \geq 3$. (Notice that all subdirectly irreducible algebras in $\mathcal{M}_n$ are subalgebras of M_n, so we only consider maps $X \to M_n$.) For every $f \in H(F_n(X))$ let f^* denote its unique extension to a homomorphism $F_n(X) \to M_n$.

The sets G_r are defined for all finite $X_0 \subseteq X$ and all $r : X_0 \to M_n$ such that $r(X_0) = \{0, 1\}$ or $\{a_1, \ldots, a_k\} \subseteq r(X_0) \subseteq \{0, 1, a_1, \ldots, a_k\}$ for some $k \geq 3$. Now we shall look closer at these sets. Let A_n be the set of all automorphisms of M_n, i.e. all permutations $\pi : M_n \to M_n$ with $\pi(0) = 0$ and $\pi(1) = 1$. Thus, $f, g \in H(F_n(X))$ are indistinguishable (i.e. $\ker(f^*) = \ker(g^*)$) iff $\pi f = g$ for some $\pi \in A_n$. In such a case we write $f \sim g$.

5.1. LEMMA. *If* $r(X_0) = \{0, 1\}$ *then*

$$G_r = \{f \in H(F_n(X)) \mid r^{-1}(0) \subseteq (\pi f)^{-1}(\{0, a_1\}) \text{ and } r^{-1}(1) \subseteq (\pi f)^{-1}(\{1, a_2\}) \text{ for some } \pi \in A_n\}.$$

If $\{a_1, \ldots, a_k\} \subseteq r(X_0) \subseteq \{0, 1, a_1, \ldots, a_k\}$ *for some* $k \geq 3$ *then*

$$G_r = \{f \in H(F_n(X)) \mid r = \pi f \restriction X_0 \text{ for some } \pi \in A_n\}$$

and the set G_r *is clopen.*

Proof. Consider the lattice homomorphism $s : \{0, 1, a_1, a_2\} \to \{0, 1\}$ defined by $s(0) = s(a_1) = 0$ and $s(1) = s(a_2) = 1$. If $r^{-1}(0) \subseteq (\pi f)^{-1}(\{0, a_1\})$ and $r^{-1}(1) \subseteq (\pi f)^{-1}(\{1, a_2\})$ then $r = s\pi f \restriction X_0$, which implies that $\ker(r^*) \supseteq \ker(\pi f^* \restriction \langle X_0 \rangle) = \ker(f^* \restriction \langle X_0 \rangle)$, hence $f \in G_r$.

Conversely, let $f \in G_r$, i.e. $\ker(f^* \restriction \langle X_0 \rangle) \subseteq \ker(r^*)$. Since $r^*(X_0) = \{0, 1\}$, there is a lattice homomorphism $g : f^*(\langle X_0 \rangle) \to \{0, 1\}$ with $r^* = gf^* \restriction \langle X_0 \rangle$. For $k \geq 3$ there is no lattice homomorphism from M_k onto $\{0, 1\}$. Thus, $f^*(\langle X_0 \rangle)$ is a sublattice of M_n not isomorphic to M_k and we can assume that $f^*(\langle X_0 \rangle) \subseteq \{0, 1, a_j, a_k\}$ for some $j \neq k$. The equality $g(a_j) = g(a_k)$ is impossible, because it implies that g is constant while r^* is not. Therefore we can assume that $g^{-1}(0) \subseteq \{0, a_j\}$ and $g^{-1}(1) \subseteq \{1, a_k\}$. Choose $\pi \in A_n$ such that $\pi(a_j) = a_1$ and $\pi(a_k) = a_2$. Then $r^{-1}(0) \subseteq (\pi f)^{-1}(\{0, a_1\})$ and $r^{-1}(1) \subseteq (\pi f)^{-1}(\{1, a_2\})$.

Now the second formula. If $\pi f \restriction X_0 = r$ then obviously $\ker(r^*) = \ker(f^* \restriction \langle X_0 \rangle)$, hence $f \in G_r$. Conversely, let $f \in G_r$. Now we have $r^*(\langle X_0 \rangle) = M_k$ and there is a surjective homomorphism $g : f^*(\langle X_0 \rangle) \to M_k$ with $r^* = gf^* \restriction \langle X_0 \rangle$. The only sublattices of M_n that can be homomorphically mapped onto M_k are those isomorphic to M_k. Thus, we can assume that $f^*(\langle X_0 \rangle) = \{0, 1, a_{i_1}, \ldots, a_{i_k}\}$, $g(a_{i_1}) = a_1, \ldots, g(a_{i_k}) = a_k$. Choose $\pi \in A_n$ such that $\pi(a_{i_1}) = a_1, \ldots, \pi(a_{i_k}) = a_k$. Then clearly $r = \pi f \restriction X_0$.

It remains to show that G_r is closed. Let $f \in H(F_n(X)) \setminus G_r$. Since $f(X)$ is finite, it is possible to choose a finite set X_1 such that $X_0 \subseteq X_1 \subseteq X$ and $f(X_1) = f(X)$. Let $s = f \restriction X_1$. Then G_s is defined, $f \in G_s$ and we claim that $G_r \cap G_s = \emptyset$. Let $g \in G_r$. Then $g^*(\langle X_0 \rangle)$ is isomorphic to $r^*(\langle X_0 \rangle)$ and hence to M_k. Thus, g cannot belong to G_s if $s(X_1) = \{0, 1\}$. Finally, if $s(X_1) \neq \{0, 1\}$ then $g \in G_s$ would imply $f \restriction X_1 = s = \pi g \restriction X_1$ for some $\pi \in A_n$ and then also $f \restriction X_0 = \pi g \restriction X_0$, hence $f \in G_r$, which is not true. ■

Recall that the space $\overline{H(F_n(X))}$ arises from $H(F_n(X))$ by identifying the indistinguishable points. By 3.2, the congruence lattices of algebras in $\mathcal{M}_n$ are exactly the lattices of the form $\mathcal{O}(Z)$, where Z is a closed subspace of $\overline{H(F_n(X))}$. Thus, we would like to describe the closed subspaces of $\overline{H(F_n(X))}$. Before we try to do so, let us present an illustrative example.

Let $n = 3$ and let ω denote the set of all nonnegative integers. The points of $\overline{H(F_3(\omega))}$ will be denoted as sequences of elements $0, 1, a_1, a_2, a_3$.

Let W be the subspace of $\overline{H(F_3(\omega))}$ whose points are $x = (0,1,0,0,0\ldots)$, $y = (1,0,0,0,0\ldots)$, $z_1 = (a_1,a_2,a_3,0,0\ldots)$, $z_2 = (a_1,a_2,0,a_3,0\ldots)$, $z_3 = (a_1,a_2,0,0,a_3,0\ldots),\ldots$ (More precisely, the points of W are the $\sim$-equivalence classes containing these sequences.) It is not difficult to check that W is indeed a closed subspace. Topologically, W consists of a discrete sequence $\{z_1,z_2,z_3,\ldots\}$ converging to its two limit points x, y. Hence $\mathcal{O}(W)$ is the family of all sets $G \subseteq W$ satisfying the condition

$$\text{if } x \in G \text{ or } y \in G \text{ then } W \setminus G \text{ is finite.}$$

Such a set G is compact if it is finite or contains x or y. Notice that W is not Hausdorff and the intersection of two compact sets need not be compact. (Indeed, if $\{x,y\} \cap G_1 = \{x\}$, $\{x,y\} \cap G_2 = \{y\}$, then $G_1 \cap G_2$ is not compact.) This is probably the simplest example of a congruence lattice representable in $\mathcal{M}_3$ but not in the variety $\mathcal{D}$ of distributive lattices. Note that we can explicitly indicate the lattice $L \in \mathcal{M}_3$ whose congruence lattice is $\mathcal{O}(W)$. Namely, L is the sublattice of $(M_3)^\omega$ generated by the elements $(0,1,a_1,a_1,a_1\ldots)$, $(1,0,a_2,a_2,a_2\ldots)$, $(0,0,a_3,0,0\ldots)$, $(0,0,0,a_3,0\ldots)$, $(0,0,0,0,a_3,0\ldots)$, etc.

Now we turn to the question of what can be said about the closed subspaces of $\overline{H(F_n(X))}$ in general. Some properties are easy to observe. Let $Z \subseteq \overline{H(F_n(X))}$ be closed. It is natural to consider Z as the union of two sets $Z = Z_0 \cup Z_n$, where $Z_0 = \{\overline{f} \in Z \mid f(X) = \{0,1\}\}$ and $Z_n = Z \setminus Z_0$.

Note that if Z corresponds (in the sense of 3.2) to $L \in \mathcal{M}_n$ then Z_0 corresponds to the largest distributive quotient of L. However, if Z is an abstract space, then its partition into Z_0 and Z_n is not determined uniquely. It is easy to find closed subspaces Y and Z of $\overline{H(F_n(X))}$ such that Y and Z are homeomorphic, while Y_0 and Z_0 are not. This ambiguity is natural, since nonisomorphic lattices may have isomorphic congruence lattices.

5.2. Lemma. *Let $W \subseteq Z$ be closed and $\overline{f} \in Z_n \setminus W$. Then $\overline{f} \in \overline{G}_r$ and $\overline{G}_r \cap W = \emptyset$ for some $r : X_0 \to M_n$ with $\{a_1,\ldots,a_k\} \subseteq r(X_0) \subseteq \{0,1,a_1,\ldots,a_k\}$ for some $k \geq 3$.*

Proof. Since W is closed, there is some set $\overline{G}_s$ with $\overline{f} \in \overline{G}_s$ and $\overline{G}_s \cap W = \emptyset$. The only difficulty is that s may be a function $X_0 \to \{0,1\}$. Since $\overline{f} \in Z_n$, there is a finite set $Y \subseteq X$ with $X_0 \subseteq Y$ and $\{a_1,\ldots,a_k\} \subseteq f(Y) \subseteq \{0,1,a_1,\ldots,a_k\}$. Set $r = f{\restriction}Y$. Then clearly $\overline{G}_r \subseteq \overline{G}_s$ and $\overline{f} \in \overline{G}_r$. ∎

5.3. Lemma.

(1) *Z is a T_1-space* (*i.e. all singletons are closed sets*);
(2) *Z has a basis of compact open sets*;
(3) *Z_0 is a closed subspace of Z*;
(4) *both Z_0 and Z_n are locally compact*, *Hausdorff*, *zero-dimensional.*

P r o o f. (1) is a direct consequence of the fact that all subdirectly irreducible algebras in $\mathcal{M}_n$ are simple. Indeed, for every $\alpha \in M(\operatorname{Con} F_n(X))$ the closed set $M(\operatorname{Con} F_n(X)) \cap \uparrow\alpha$ is equal to α. Thus, $\overline{H(F_n(X))}$ is homeomorphic to the T_1-space $M(\operatorname{Con} F_n(X))$ and Z is its subspace.

(2) is satisfied because $\mathcal{O}(Z)$ is isomorphic to $\operatorname{Con} A$ for some $A \in \mathcal{M}_n$, which is an algebraic lattice.

For every $\overline{f} \in Z_n$ there is a finite set $X_0 \subseteq X$ such that $f(X_0) = \{a_1, a_2, a_3\}$. For $r = f{\restriction}X_0$ we have $\overline{f} \in \overline{G}_r$ and $\overline{G}_r \cap Z_0 = \emptyset$. Hence, every point of Z_n has a neighbourbood disjoint from Z_0, which means that Z_n is open and Z_0 closed. Thus, (3) holds.

The basis of the topology on Z_0 consists of all sets of the form $\overline{G}_r \cap Z_0 = \{\overline{f} \in Z_0 \mid f{\restriction}X_0 = r{\restriction}X_0\}$, where $X_0 \subseteq X$ is finite and $r : X_0 \to \{0,1\}$ is surjective. It is easy to see that all these sets are clopen (in Z_0) and they separate the points of Z_0, hence Z_0 is Hausdorff and zero-dimensional. Further, any $\overline{f} \in Z_0$ belongs to some $\overline{G}_r$, which is compact by 2.7. Since Z_0 is closed, $\overline{G}_r \cap Z_0$ is compact, showing that Z_0 is locally compact.

By 5.2, every open subset of Z_n is a union of clopen sets. Hence, Z_n is zero-dimensional. Since one-element sets are closed, 5.2 implies that any two points can be separated by a clopen set, hence Z_n is Hausdorff. Further, any $\overline{f} \in Z_n$ belongs to some $\overline{G}_r$ with $r : X_0 \to M_n$ such that $\{a_1, \dots, a_k\} \subseteq r(X_0) \subseteq \{0, 1, a_1, \dots, a_k\}$ for some $k \geq 3$. Since Z is closed in $\overline{H(F_n(X))}$, the set $Z_n \cap \overline{G}_r = Z \cap \overline{G}_r$ is compact and hence Z_n is locally compact. ∎

For later use, let us explicitly state the following consequence of 5.2 and 5.3(1).

5.4. Corollary. *If $\overline{f} \in Z_n$ and $\overline{g}_1, \dots, \overline{g}_m \in Z$ are mutually different then there is a clopen set $C \subseteq Z$ such that $\overline{f} \in C$ and $\overline{g}_1, \dots, \overline{g}_m \notin C$.*

By 5.3, the spaces Z_0 and Z_n look very much the same. By 4.1 they arise from Boolean spaces by deleting one point. In other words, they are dual spaces of some distributive lattices. The difficult point is how Z_0 and Z_n are glued together. We have seen in our example that while Z_0 and Z_n are Hausdorff, Z need not be Hausdorff. Namely, Z_0 may contain points that do not have disjoint neighbourhoods in Z. Equivalently a sequence (or, more generally, a net) in Z_n may have two limit points in Z_0. The next lemma actually says that such a net cannot have three limit points.

5.5. Lemma. *If $\overline{f}, \overline{g}, \overline{h} \in Z$ are mutually different then there are open sets $A, B, C \subseteq Z$ such that $\overline{f} \in A$, $\overline{g} \in B$, $\overline{h} \in C$ and $A \cap B \cap C = \emptyset$.*

P r o o f. If one of $\overline{f}, \overline{g}, \overline{h}$ belongs to Z_n (say $\overline{f} \in Z_n$) then by 5.4 there is a clopen set $\overline{G}_r$ with $\overline{f} \in \overline{G}_r$ and $\overline{g}, \overline{h} \notin \overline{G}_r$. We can set $A = Z \cap \overline{G}_r$ and $B = C = Z \setminus \overline{G}_r$.

Assume now that $\overline{f}, \overline{g}, \overline{h} \in Z_0$, hence f, g, h are functions $X \to \{0,1\}$. Since they are different, we have $x, y, z \in X$ such that $f(x) \neq g(x)$, $f(y) \neq h(y)$, $g(z) \neq h(z)$. Let X_0 be a finite subset of X such that $x, y, z \in X_0$ and $f(X_0) = g(X_0) = h(X_0) = \{0,1\}$. Set $r = f\restriction X_0$, $s = g\restriction X_0$, $t = h\restriction X_0$, $A = \overline{G}_r$, $B = \overline{G}_s$, $C = \overline{G}_t$. Then clearly $\overline{f} \in A$, $\overline{g} \in B$, $\overline{h} \in C$. It remains to show that $A \cap B \cap C = \emptyset$. For contradiction, suppose that $k \in G_r \cap G_s \cap G_t$. Since $\{r(x), s(x)\} = \{0,1\}$, $k(x)$ must be in $\{a_1, \ldots, a_n\}$ by 5.1. Similarly, $k(y), k(z) \in \{a_1, \ldots, a_n\}$. Now we claim that $k(x), k(y), k(z)$ are different. Suppose $k(x) = k(y)$. Then, by 5.1, $r(x) = r(y)$, $s(x) = s(y)$, $t(x) = t(y)$. Since $r(x) \neq s(x)$ and $r(y) \neq t(y)$, necessarily $s(x) = t(x) = t(y) = s(y)$. Since $s(z) \neq t(z)$, we have without loss of generality $s(y) = s(z)$, $t(y) \neq t(z)$. Since $k \in G_s \cap G_t$, the equality $s(y) = s(z)$ implies $k(y) = k(z)$, while $t(y) \neq t(z)$ implies $k(y) \neq k(z)$, which is impossible. Hence, $k(x) \neq k(y)$ and similarly $k(y) \neq k(z) \neq k(x)$. By 5.1, such a k cannot belong to G_r. ∎

6. Uniform separation. In the previous sections we have seen that the spaces $\overline{H(F_n(X))}$ for different n look very similar. In this section we introduce topological properties that distinguish these spaces.

A subset Q of a topological space T is called *discrete* if for every $q \in Q$ there is an open set C with $C \cap Q = \{q\}$. The space T will be called *n-uniformly separable* ($n \geq 3$) if for every discrete set $Q \subseteq T$ there is a family $\{B_{pq} \mid p, q \in Q,\ p \neq q\}$ of open sets such that $p \in B_{pq}$ for every $p, q \in Q$ and $\bigcap\{B_{pq} \mid p, q \in Q_0,\ p \neq q\} = \emptyset$ for every n-element set $Q_0 \subseteq Q$.

By 5.5, any three points have disjoint neighbourhoods. However, n-uniform separability requires that disjoint neighbourhoods can be chosen simultaneously for large families of points.

6.1. Lemma. *Let $X_1, \ldots, X_k$ be finite sets, $k \geq 3$, and $Y = X_1 \cup \ldots \cup X_k$. Let $f_1, \ldots, f_k$ be functions $Y \to \{0,1\}$ such that for all $i \neq j$ there exists $x \in X_i$ with $f_i(x) \neq f_j(x)$. Let h be a function such that* $\operatorname{dom}(h) = Y$ *and* $\ker(f_i\restriction(X_i \cup X_j)) = \ker(h\restriction(X_i \cup X_j))$ *for all i, j. Then h takes at least k values* (*i.e.*, $\operatorname{rng}(h)$ *is at least k-element*).

Proof. Our assumptions imply that the sets X_i are nonempty and $\ker(h) \subseteq \ker(f_i)$ for every i. Indeed, suppose that $(x, y) \notin \ker(f_i)$, $x \in X_j$, $y \in X_m$, and choose $z \in X_i$ arbitrarily. Then either $(z, x) \in \ker(f_i)$ or $(y, z) \in \ker(f_i)$; we can assume that $(z, x) \in \ker(f_i)$. Then $x, z \in X_i \cup X_j$, hence $(x, z) \in \ker(h)$. Similarly, $y, z \in X_i \cup X_m$ implies that $(y, z) \notin \ker(h)$. Then clearly $(x, y) \notin \ker(h)$.

Our next claim is that $h\restriction(X_1 \cup X_2)$ is not constant. Suppose it is. Then every f_i must be constant on $X_1 \cup X_2$. Without loss of generality, $f_1(X_1 \cup X_2) = \{0\}$. For every $i \neq 1$ the functions f_i and f_1 differ on X_1, hence $f_i(X_1 \cup X_2) = \{1\}$. But then there is no $x \in X_2$ with $f_2(x) \neq f_3(x)$, a contradiction.

Hence, h is not constant on $X_1 \cup X_2$. On the other hand, $\ker(h\restriction(X_1 \cup X_2)) = \ker(f_1\restriction(X_1 \cup X_2))$, so h cannot take three values on $X_1 \cup X_2$. We conclude that h takes exactly two values on $X_1 \cup X_2$. (And similarly for any $X_i \cup X_j$.)

Now we show that h takes at least three values on $X_1 \cup X_2 \cup X_3$. Suppose it takes only two values on $X_1 \cup X_2 \cup X_3$. We claim that $\ker(f_1\restriction(X_1 \cup X_2 \cup X_3)) = \ker(h\restriction(X_1 \cup X_2 \cup X_3))$. The inclusion $\ker(h) \subseteq \ker(f_1)$ is already proved. Let $x, y \in X_1 \cup X_2 \cup X_3$, $h(x) \neq h(y)$. We need to show that $f_1(x) \neq f_1(y)$. This is clear if $x, y \in X_1 \cup X_2$ or $(x, y) \in X_1 \cup X_3$. The remaining case is $x \in X_2$, $y \in X_3$. Choose $z \in X_1$ arbitrarily. Since h takes only two values on $X_1 \cup X_2 \cup X_3$, we have either $h(x) = h(z)$ or $h(y) = h(z)$. Without loss of generality, $h(x) = h(z) \neq h(y)$. Since $x, z \in X_1 \cup X_2$ and $y, z \in X_1 \cup X_3$, we conclude that $f_1(x) = f_1(z) \neq f_1(y)$.

Similar arguments hold for f_2 and f_3. Hence, for every $i = 1, 2, 3$ we have $\ker(f_i\restriction(X_1 \cup X_2 \cup X_3)) = \ker(h\restriction(X_1 \cup X_2 \cup X_3))$. There are only two functions $X_1 \cup X_2 \cup X_3 \to \{0, 1\}$ with this property, hence two of f_1, f_2, f_3 must coincide on $X_1 \cup X_2 \cup X_3$, which contradicts our assumptions.

We have proved that h takes two values on $X_1 \cup X_2$, two values on $X_1 \cup X_3$ but at least three values on $X_1 \cup X_2 \cup X_3$. This is only possible if h is constant on X_1. For similar reasons, h is constant on each X_i. Since h is not constant on $X_i \cup X_j$, all the sets X_i must be disjoint and h takes a different value on each X_i. ■

6.2. THEOREM. *For any set X and every $n \geq 3$, the spaces $H(F_n(X))$ and $\overline{H(F_n(X))}$ are $(n+1)$-uniformly separable.*

Proof. We present the proof for $H(F_n(X))$. (Transition to $\overline{H(F_n(X))}$ is obvious.) As in the previous section, let $H_0 = \{f \in H(F_n(X)) \mid f(X) = \{0, 1\}\}$ and $H_n = H(F_n(X)) \setminus H_0$. Let $Q \subseteq H(F_n(X))$ be discrete. If $p \in Q \cap H_n$ then, by 5.4, for every $q \in Q$, $q \neq p$, there are open sets B_{pq} and B_{qp} such that $p \in B_{pq}$, $q \in B_{qp}$ and $B_{pq} \cap B_{qp} = \emptyset$. Suppose now that $p, q \in Q \cap H_0$. By our assumption, for every $t \in Q$ there exists an open set C_t with $C_t \cap Q = \{t\}$. We can assume that $C_t = G_{r_t}$ for a suitable function r_t. If $t \in H_0$ then clearly $r_t = t\restriction X_t$ for some finite set X_t. We set $r_{pq} = p\restriction(X_p \cup X_q)$ and $B_{pq} = G_{r_{pq}}$. It remains to verify that the family $\{B_{pq} \mid p, q \in Q,\ p \neq q\}$ has the required property.

It is obvious that $p \in B_{pq}$ for every p, q. Now let $q_1, \ldots, q_{n+1}$ be different elements of Q. Write B_{ij} instead of $B_{q_i q_j}$. If any of q_i belongs to H_n then clearly $\bigcap\{B_{ij} \mid i \neq j\} = \emptyset$. So assume that $q_1, \ldots, q_{n+1} \in H_0$. For contradiction suppose that $p \in \bigcap\{B_{ij} \mid i \neq j\}$. Set $X'_i = \{x \in X_{q_i} \mid q_j(x) \neq q_i(x)$ for some $j \in \{1, \ldots, n+1\}\}$ and $Y = X'_1 \cup \ldots \cup X'_{n+1}$. For any $y \in Y$, $p(y)$ cannot be 0, because if $y \in X'_i$ then $q_i(y) = 1$ or $q_j(y) = 1$ for some j and we suppose that $p \in B_{ij} \cap B_{ji}$. (See 5.1.) For similar reasons, $p(y) \neq 1$ and therefore $p(y) \in \{a_1, \ldots, a_n\}$.

By 5.1 we have $\ker(p\restriction(X_{q_i}\cup X_{q_j}))=\ker(r_{q_iq_j}) = \ker(q_i\restriction(X_{q_i}\cup X_{q_j}))$. Since $X'_i \cup X'_j \subseteq X_{q_i} \cup X_{q_j}$, we also have $\ker(p\restriction(X'_i \cup X'_j)) = \ker(q_i\restriction(X'_i \cup X'_j))$. Further, if $i \neq j$ then $q_j \notin C_{q_i}$, hence $q_j(x) \neq r_{q_i}(x) = q_i(x)$ for some $x \in X_i$, which shows that $x \in X'_i$. Thus, we can apply 6.1 to the functions $f_i = q_i\restriction Y$ and $h = p\restriction Y$. We deduce that h takes at least $n+1$ values, which is impossible because $p(Y) \subseteq \{a_1, \ldots, a_n\}$. ■

It is easy to see that if a space is n-uniformly separable then all its subspaces are n-uniformly separable. By 3.2 we have the following consequence.

6.3. Theorem. *If $L \cong \operatorname{Con} K$ for some $K \in \mathcal{M}_n$ then $M(L)$ is $(n+1)$-uniformly separable.*

Now we show that $H(F_n(X))$ is not n-uniformly separable if the cardinality of X is at least $\aleph_2$. We need the following combinatorial principle of Hajnal and Máté. For a set X let $[X]^2$ denote the set of all two-element subsets of X. Analogously, let $[X]^{<\omega}$ be the set of all finite subsets of X.

6.4. Lemma (see [5], Section 3.2). *Let $|X| \geq \aleph_2$ and let f be a function $[X]^2 \to [X]^{<\omega}$. Then for every natural number n there are $x_1, \ldots, x_n \in X$ such that $x_i \notin f(\{x_j, x_k\})$ whenever $i \neq j \neq k \neq i$.*

The special case $n = 3$ was proved by Kuratowski [7]. The importance of this principle for congruence lattices was discovered by Wehrung in [13] and [14]. The principle was subsequently used in [10] and [9].

6.5. Theorem. *If $|X| \geq \aleph_2$ and $n \geq 3$, then the spaces $H(F_n(X))$ and $\overline{H(F_n(X))}$ are not n-uniformly separable.*

Proof. For every $x \in X$ we define $f_x : X \to \{0,1\}$ by $f_x(x) = 1$ and $f_x(y) = 0$ for all $y \neq x$. Let $Q = \{f_x \mid x \in X\}$. Clearly, Q is discrete. (Indeed, if $\{x\} \subsetneq X_0 \subseteq X$ and $r = f_x\restriction X_0$, then G_r is defined, $f_x \in G_r$ and $G_r \cap Q = \{f_x\}$.) Suppose that $\{B_{xy} \mid x, y \in X,\ x \neq y\}$ is a family of open sets with $f_x \in B_{xy}$. We can assume that $B_{xy} = G_{r_{xy}}$ for some $r_{xy} : X_{xy} \to \{0,1\}$. (Necessarily, r_{xy} is the restriction of f_x to some finite set.) Let $f : [X]^2 \to [X]^{<\omega}$ be defined by $f(\{x,y\}) = \operatorname{dom}(r_{xy}) \cup \operatorname{dom}(r_{yx})$. Using 6.4 we find $x_1, \ldots, x_n \in X$ such that $x_i \notin \operatorname{dom}(r_{x_jx_k})$ whenever $i \neq j \neq k \neq i$. Define $g : X \to M_n$ by $g(x_i) = a_i$ and $g(y) = 0$ otherwise.

We claim that $g \in \bigcap\{B_{x_ix_j} \mid i, j \in \{1, \ldots, n\},\ i \neq j\}$. Set $r = r_{x_ix_j}$. Let $\pi \in A_n$ be such that $\pi(a_i) = a_2$, $\pi(a_j) = a_1$. If $x \in r^{-1}(\{1\})$ then $x = x_i$, $g(x) = a_i$ and $\pi g(x) = a_2$. If $x \in r^{-1}(\{0\})$ then either $x = x_j$ and $\pi g(x) = a_1$ or $x \notin \{x_1, \ldots, x_n\}$ and $g(x) = \pi g(x) = 0$. Hence, $r^{-1}(\{1\}) \subseteq (\pi g)^{-1}(\{1, a_2\})$ and $r^{-1}(\{0\}) \subseteq (\pi g)^{-1}(\{0, a_1\})$, which by 5.1 means that $g \in B_{x_ix_j}$. ■

If $L = \operatorname{Con} F_{n+1}(X)$ then $M(L)$ is homeomorphic to $\overline{H(F_{n+1}(X))}$ and hence not $(n+1)$-uniformly separable. By 6.3 we have the following result.

6.6. THEOREM. *Let* $n \geq 3$, $|X| \geq \aleph_2$ *and* $L = \operatorname{Con} F_{n+1}(X)$. *Then there is no* $K \in \mathcal{M}_n$ *such that* $\operatorname{Con} K \cong L$.

Hence, the lattices in different $\mathcal{M}_n$ have different congruence lattices. On the other hand, it is an open question if for $|X| \leq \aleph_1$ the lattice $\operatorname{Con} F_{n+1}(X)$ can be represented in $\mathcal{M}_n$. (Equivalently, if $\overline{H(F_{n+1}(X))}$ is homeomorphic to a closed subspace of $\overline{H(F_n(X))}$.)

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