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HERMITIAN AND QUADRATIC FORMS OVER LOCAL CLASSICAL CROSSED PRODUCT ORDERS

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Abstract. Let R be a complete discrete valuation ring with quotient field K, L/K be a Galois extension with Galois group G and S be the integral closure of R in L. If a is a factor set of G with values in the group of units of S, then $(L/K,a)$ (resp. $\Lambda=(S/R,a)$) denotes the crossed product K-algebra (resp. crossed product R-order in A). In this paper hermitian and quadratic forms on Λ-lattices are studied and the existence of at most two irreducible non-singular quadratic Λ-lattices is proved (Theorem 3.5). Further the orthogonal decomposition of an arbitrary non-singular quadratic Λ-lattice is given.

1. Introduction. Let R be a complete discrete valuation ring with quotient field K and finite residue class field k, L be a finite Galois extension of K of degree n with valuation ring S, and G be the Galois group of L/K. We denote by $A=(L/K,a)$ the crossed product algebra of G over L with factor set $a:G\times G\to S^{\cdot}$, where $S^{\cdot}$ is the group of units of S. We assume that the factor set α is normalized. The K-algebra A is central simple with L-basis u_σ, $\sigma\in G$, and multiplication given by the relations

$$u_\sigma l=\sigma(l)u_\sigma,\qquad u_\sigma u_\tau=a(\sigma,\tau)u_{\sigma\tau},\qquad l\in L,\ \sigma,\tau\in G$$

([Re], §29). Moreover we denote by Λ the ring $(S/R,a)=\bigoplus_{\sigma\in G}Su_\sigma$, which is an R-order in A.

In this paper we study the hermitian and non-singular quadratic forms on Λ-lattices. In [Ri] C. Riehm considered hermitian forms over a hereditary order in a central simple K-algebra. We remark that the crossed product order Λ is a hereditary order if and only if the extension L/K is tamely ramified ([C-R], §28). In the main result of this paper (Theorem 3.5) we prove the existence of at most two (up to isometry) irreducible non-singular quadratic Λ-lattices.

In the second section we prove the validity of the reduction theorem ([Q-S-S], Theorem 2.2) for the categories latt(Λ) of left Λ-lattices and latt$(\Lambda)/\mathcal{I}_I$ for a suitable ideal $\mathcal{I}_I$ of latt(Λ).

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In the third section, besides Theorem 3.5 mentioned above, we state the Krull–Schmidt–Azumaya theorem for the category of non-singular quadratic Λ-lattices. Also the orthogonal decomposition of an arbitrary non-singular quadratic Λ-lattice is given.

We refer to [C-R] and [Re] for the theory of orders and lattices over orders, to [Kn] and [Sch] for the theory of quadratic and hermitian forms and to [Re] for the classical crossed products.

2. Reduction theorem for the category latt(Λ)**.** We keep the notation of the first section and let V be an irreducible left A-module. By the Wedderburn–Artin theory the K-algebra $D = \mathrm{End}_A(V)$ is a division ring which is Brauer equivalent to A. Moreover, V carries a natural right D-vector space structure and

$$A \cong \mathrm{End}_D(V) \cong M_r(D).$$

Let m be the index of D, so $(D : K) = m^2$. An *involution* on A is an anti-automorphism of degree 2. An involution on A is said to be of the *first kind* if its restriction to K is the identity. We assume that the crossed product A admits an involution of the first kind. Then by ([A], X Theorem 19) the index m is equal to 1 or 2. From ([H-Th], Theorem 3.1) there exists an involution $^-$ on A such that $\overline{\Lambda} = \Lambda$ and the restriction $^-|L$ is the identity on L.

Let $X^* = \mathrm{Hom}_R(X, R)$ for a left Λ-lattice X. Then $X^{**} \cong X$ as left R-lattices ([C-R], §10), and thus we get the following proposition.

Proposition 2.1. *The pair* $(\mathrm{latt}(\Lambda),^*)$ *is an additive Krull–Schmidt category with duality.*

The extension L/K is separable, thus the reduced trace $\mathrm{tr}_{S/R}$ induces a symmetric associative non-degenerate R-bilinear form on S and so the image is given by $\mathrm{tr}_{S/R}(S) = R\vartheta$ for some non-zero $\vartheta \in R$. For every element $x = \sum_{\sigma\in G} s_\sigma(x)u_\sigma$, $s_\sigma(x) \in S$, of Λ, let

$$\phi(x) = \frac{1}{\vartheta}\,\mathrm{tr}_{S/R}(s_1(x)),$$

where 1 is the unit element of G. The map $\Phi : \Lambda \to \Lambda^*$ given by $\Phi(x)(\lambda) = \phi(\lambda x)$, where $x, \lambda \in \Lambda$, is a $(\Lambda\text{-}\Lambda)$-bimodule isomorphism. In particular, Λ is a symmetric R-order ([Th-W], Theorem 1). Moreover Φ is a hermitian form, since

$$\Phi(x)(\lambda) = \overline{\Phi(\lambda)(x)}$$

for $x, \lambda \in \Lambda$, because $\phi(x\lambda) = \phi(\lambda x)$ (see the proof of [Th-W], Theorem 1). Thus we get the following proposition.

PROPOSITION 2.2. *The pair* (Λ, Φ) *is a non-singular hermitian lattice.*

Let Δ be the unique maximal order in D, π_0 (resp. π) a prime element of R (resp. S) and π_D a prime element of Δ such that $\pi_D^m = \pi_0$ ([Re], §14), where m is the index of A. If M is an irreducible left Λ-lattice and right Δ-module, then $\mathrm{End}_\Delta(\pi^i M)$, $0 \le i \le e/m - 1$, are the maximal R-orders in A containing Λ and their intersection Γ is a hereditary R-order in A containing Λ of type e/m and with invariants $(f, f, \ldots, f)$, where e (resp. f) is the ramification index (resp. degree) of $\pi_0 R$ in the extension L/K ([Ch-Th], Theorem 2.2). Set

$$I = \pi^k \Gamma,$$

where $k = d - (e-1)$ and d is the different of L/K. Then I is the maximal two-sided Γ-ideal contained in Λ ([Th-W]).

Following Kelly [Kel], by the Jacobson radical Rad_Λ of the category $\mathrm{latt}(\Lambda)$ we mean the intersection of all maximal two-sided ideals of $\mathrm{latt}(\Lambda)$. Since $\mathrm{latt}(\Lambda)$ is a Krull–Schmidt category, Rad_Λ is generated by all non-invertible morphisms between indecomposable objects in $\mathrm{latt}(\Lambda)$, that is, $\mathrm{Rad}_\Lambda(X, Y)$ consists of all non-isomorphisms $f : X \to Y$ for any pair X, Y of indecomposables in $\mathrm{latt}(\Lambda)$ ([A-R-S] and [Sim]). In particular $\mathrm{Rad}_\Lambda(X, X)$ is the Jacobson radical of the endomorphism ring $\mathrm{End}(X, X)$.

We consider the two-sided ideal $\mathcal{I}_I \subseteq \mathrm{latt}(\Lambda)$ consisting of all homomorphisms $f : X \to IY$, where X and Y are in $\mathrm{latt}(\Lambda)$.

PROPOSITION 2.3. *The ideal* $\mathcal{I}_I$ *is contained in the radical ideal* $\mathrm{Rad}_\Lambda(X, Y)$ *of the category* $\mathrm{latt}(\Lambda)$ *for all* $X, Y \in \mathrm{latt}(\Lambda)$.

Proof. It is sufficient to prove that

$$\mathcal{I}_I \subset R(X, X) \tag{2.1}$$

for all $X \in \mathrm{latt}(\Lambda)$ ([Kn], II 4.1.1). If X is an indecomposable left Λ-lattice, then (2.1) holds, since then $\mathrm{End}_\Lambda(X)$ is a local ring ([C-R], Proposition 6.10) and so $\mathrm{Rad}_\Lambda(X, X)$ is its unique maximal ideal. The Krull–Schmidt–Azumaya theorem holds in $\mathrm{latt}(\Lambda)$ ([C-R], Theorem 6.12); so if $X = \bigoplus_{i=1}^s X_i$ is the decomposition of $X \in \mathrm{latt}(\Lambda)$ into indecomposable Λ-lattices, then

$$\mathrm{Hom}_\Lambda(X, \pi^k \Gamma X) \cong \bigoplus_{i,j=1}^{s} \mathrm{Hom}_\Lambda(M_i, \pi^k \Gamma M_j) \tag{2.2}$$

and

$$\mathrm{Rad}_\Lambda(X, X) = \bigoplus_{i,j=1}^{s} \mathrm{Rad}_\Lambda(X_i, X_j). \tag{2.3}$$

We remark that $\operatorname{rad} X_i = (\operatorname{rad} \Lambda)X_i$, $1 \le i \le s$, and $\operatorname{Rad}_\Lambda(X_i, X_j)$, $0 \le i, j \le s$, does not contain invertible elements, hence

$$\operatorname{Hom}_\Lambda(X_i, \pi^k \Gamma X_j) \subset \operatorname{Hom}_\Lambda(X_i, \operatorname{rad} X_j) \subset \operatorname{Rad}_\Lambda(X_i, X_j),$$

$0 \le i, j \le s$. The above relation together with (2.2) and (2.3) completes the proof of the proposition. ∎

Lemma 2.4. *Let $X \in \operatorname{latt}(\Lambda)$ and $I = \pi^k \Gamma$, where $\Gamma \supseteq \Lambda$ is as above. Then there exists a functorial isomorphism $\operatorname{Hom}_\Lambda(I, X) = IX$ of Λ-modules.*

Proof. From ([Th], Proposition 8) we can write $I = \sum_{i=1}^{e} \Lambda\omega_i$ for $\omega_i \in I$. Moreover the elements of IX are finite sums $\sum_{i=1}^{e} a_i x_i$ for $a_i \in I$, $x_i \in X$. The map

$$\vartheta : IX \to \operatorname{Hom}_\Lambda(I, X)$$

sending $\omega_i m$ to f_i such that $f_i(\omega_i) = x$ for $x \in X$ and extended Λ-linearly is an R-isomorphism. In fact, it is easy to see that ϑ is surjective. Let $f \in \operatorname{Hom}_\Lambda(I, X)$ and $x = \sum_{i=1}^{e} \lambda_i\omega_i$ be an element of I for $\lambda_i \in \Lambda$, $1 \le i \le e$. Then $f(x) = \sum_{i=1}^{e} \lambda_i f(\omega_i)$. We put $f_i(\omega_i) = x_i$ and $\vartheta(\omega_i x_i) = g_i$ with $g_i(\omega_i) = x_i$, $1 \le i \le e$. If $y = \sum_{i=1}^{e} \omega_i x_i$,then $\vartheta(y)(x) = f(x)$ for all $x \in X$, hence ϑ is surjective. The formulas

$$(\lambda f)(a) = f(x) \quad \text{for all } \lambda \in \Lambda,\ f \in \operatorname{Hom}_\Lambda(I, X),\ a \in I,$$

and

$$\lambda x = x\overline{\lambda} \quad \text{for } x \in X,\ \lambda_i \in \Lambda$$

give a left Λ-structure on $\operatorname{Hom}_\Lambda(I, X)$ and a right Λ-structure on X respectively. Thus, if $\lambda \in \Lambda$ and $x = \omega_i x_i \in IX$, then

$$\vartheta(\lambda x)(\omega_i) = f_{\lambda x}(\omega_i) = x_i\overline{\lambda},$$

hence ϑ is Λ-linear. Finally, ϑ is a natural isomorphism, since the diagram

$$\begin{array}{ccc} IX & \xrightarrow{\vartheta_X} & \operatorname{Hom}_\Lambda(I, X) \\ {\scriptstyle \phi|IX}\downarrow & & \downarrow{\scriptstyle \phi_*} \\ IY & \xrightarrow{\vartheta_Y} & \operatorname{Hom}_\Lambda(I, Y) \end{array}$$

where $X, Y \in \operatorname{latt}(\Lambda)$ and $\phi \in \operatorname{Hom}_\Lambda(X, Y)$, is commutative. Therefore we may identify $\operatorname{Hom}_\Lambda(I, X)$ with IX. ∎

Proposition 2.5. *The ideal $\mathcal{I}_I$ of $\operatorname{latt}(\Lambda)$ is *-invariant, i.e.*

$$\mathcal{I}_I(X^*, Y^*) = \mathcal{I}_I(X, Y)^* = \{f^* : f \in I(X, Y)\}.$$

P r o o f. From Lemma 2.4 and the adjointness theorem ([C-R], Theorem 3.19) it follows that

$$\begin{aligned}\mathcal{I}_I(X,Y) &= \mathrm{Hom}_\Lambda(X, \mathrm{Hom}_\Lambda(I,Y))\\ &= \mathrm{Hom}_\Lambda(I\otimes_\Lambda X, Y) = \mathrm{Hom}_\Lambda(IX,Y).\end{aligned}$$

Now the assertion follows because $\pi\Gamma = \Gamma\pi$ ([Th], Proposition 3), $\Gamma = \overline{\Gamma}$ ([H-Th], Proposition 3.2) and so $I = \bar{I}$, where $^-$ means the involution on A. ∎

We consider the category $\mathrm{latt}(\Lambda)/\mathcal{I}_I$ with objects the objects of $\mathrm{latt}(\Lambda)$ and morphisms the Λ/I-homomorphisms $X/IX \to Y/IY$ for $X, Y \in \mathrm{latt}(\Lambda)$ and $I = \pi^k\Gamma$. We define the following residue class functor:

$$F_\Lambda : \mathrm{latt}(\Lambda) \to \mathrm{latt}(\Lambda)/\mathcal{I}_I$$

given by $F_\Lambda(X) = X$ for $X \in \mathrm{latt}(\Lambda)$ and $F_\Lambda(f) : X/IX \to Y/IY$ with $F_\Lambda(f)(x + IX) = f(x) + IY$ for $x \in X$.

Moreover, we can define a duality in $\mathrm{latt}(\Lambda)/\mathcal{I}_I$ as follows:

$$X^* = \mathrm{Hom}_{R/I_0}(X/IX, R/I_0),$$

where $I_0 = \pi_0 R$.

The following lemma is obvious and useful to state the reduction theorem.

Lemma 2.6. *The residue class functor F_Λ is duality preserving.*

We recall the definition of a quadratic module ([Kn], II 2.4). Let $\varepsilon = \pm 1$ and Ω be a family of groups $(X_M)_{M\in\mathrm{latt}(\Lambda)}$ such that X_M is a subgroup of $\mathrm{Hom}(M, M^*)$ and

$$\{f - \varepsilon f^* \mid f \in \mathrm{Hom}(M, M^*)\} \subset X_M \subset \{f \in \mathrm{Hom}(M, M^*) \mid f + \varepsilon f^* = 0\}$$

and

$$f^* X_N f \subset X_M \quad \text{ for all } f \in \mathrm{Hom}(M, N).$$

The pair (ε, Ω) is a *form parameter* in $(\mathrm{latt}(\Lambda),^*)$. An (ε, Ω)-*quadratic module* is the pair $(M, [g])$, where $M \in \mathrm{latt}(\Lambda)$ and $[g] = g + X_M$. For $M_1, M_2 \in \mathrm{latt}(\Lambda)$, a morphism $\sigma : (M_1, [g_1]) \to (M_2, [g_2])$ is an element of $\mathrm{Hom}_\Lambda(M_1, M_2)$ such that $[\sigma^* g_2 \sigma] = [g_1]$. For every $[g]$, $g \in \mathrm{Hom}(M, M^*)$, the even hermitian form $h = g + \varepsilon g^*$ is well defined. If $g + \varepsilon g^*$ is non-singular then $[g]$ is called *non-singular*. In case $[g]$ is non-singular, the pair $(M, [g])$ is called a *non-singular* quadratic module.

We define, in a canonical way, a form parameter (ε, Ω') in the category $(\mathrm{latt}(\Lambda)/\mathcal{I}_I,^*)$, where

$$\Omega'_X = \frac{\Omega_X}{\Omega_X \cap \mathcal{I}_I(X, X^*)}.$$

Let $\mathfrak{D}^{(\varepsilon,\Omega)}(\mathrm{latt}(\Lambda))$ and $\mathfrak{D}^{(\varepsilon,\Omega')}(\mathrm{latt}(\Lambda)/\mathcal{I}_I)$ be the categories of non-singular quadratic modules over $\mathrm{latt}(\Lambda)$ and $\mathrm{latt}(\Lambda)/\mathcal{I}_I$ respectively. Since $\mathrm{End}_\Lambda(X)$

is $\mathcal{I}_I(X,X)$-complete for all $X \in \mathrm{latt}(\Lambda)$, we are now ready to state the reduction theorem ([Q-S-S], Theorem 2.2) for the categories $\mathfrak{D}^{(\varepsilon,\Omega)}(\mathrm{latt}(\Lambda))$ and $\mathfrak{D}^{(\varepsilon,\Omega')}(\mathrm{latt}(\Lambda)/\mathcal{I}_I)$.

THEOREM 2.7 (Reduction Theorem). *The reduction functor*

$$\widetilde{F}_\Lambda : \mathfrak{D}^{(\varepsilon,\Omega)}(\mathrm{latt}(\Lambda)) \to \mathfrak{D}^{(\varepsilon,\Omega')}(\mathrm{latt}(\Lambda)/\mathcal{I}_I) \tag{2.4}$$

defined by $\widetilde{F}_\Lambda(X,[g]) = (X,[F_\Lambda(g)])$ *has the following properties*:

(i) *Every non-singular quadratic module over* $\mathrm{latt}(\Lambda)/\mathcal{I}_I$ *is the image of a non-singular quadratic module over* $\mathrm{latt}(\Lambda)$, *or equivalently*, $\widetilde{F}_\Lambda$ *is surjective on objects and on orthogonal sums.*

(ii) *The functor* $\widetilde{F}_\Lambda$ *is surjective on isometries.*

Similarly, we define the residue class functor

$$F_\Gamma : \mathrm{latt}(\Gamma) \to \mathrm{latt}(\Gamma)/\mathcal{I}_I$$

by setting $F_\Gamma(X) = X$, and $F_\Gamma(f) : X/IX \to Y/IY$, $F_\Gamma(f)(x+IX) = f(x)+IY$, for $X,Y \in \mathrm{latt}(\Gamma)$ and $f \in \mathrm{Hom}_\Gamma(X,Y)$. Then the reduction functor

$$\widetilde{F}_\Gamma : \mathfrak{D}^{(\varepsilon,\Omega)}(\mathrm{latt}(\Gamma)) \to \mathfrak{D}^{(\varepsilon,\Omega')}(\mathrm{latt}(\Gamma)/\mathcal{I}_I) \tag{2.5}$$

is defined by

$$\widetilde{F}_\Gamma(X,[g]) = (X,[F_\Gamma(g)]) \quad \text{for } X \in \mathrm{latt}(\Gamma)$$

where (ε,Ω) (resp. (ε,Ω')) is a form parameter in $(\mathrm{latt}(\Gamma),^*)$ (resp. $\mathrm{latt}(\Gamma)/\mathcal{I}_I$). Moreover we can state the reduction theorem for the categories $\mathrm{latt}(\Gamma)$ and $\mathrm{latt}(\Gamma)/\mathcal{I}_I$ and the functor $\widetilde{F}_\Gamma$ in a manner analogous to Theorem 2.7.

3. The orthogonal decomposition in $\mathfrak{D}^{(\varepsilon,\Omega)}(\mathrm{latt}(\Lambda))$. We keep the notation of the previous sections. $\Lambda = (S/R,a)$ denotes the crossed product order in A throughout this section. We remark first that the Krull–Schmidt–Azumaya theorem holds in $\mathfrak{D}^{(\varepsilon,\Omega)}(\mathrm{latt}(\Lambda))$, as it holds in $\mathrm{latt}(\Lambda)$ ([Q-S-S], Theorems 3.2 and 3.3). Let $M = \bigoplus_{i=1}^s N_i$ be the decomposition of $M \in \mathrm{latt}(\Lambda)$ into the idecomposables $N_i \in \mathrm{latt}(\Lambda)$, $1 \le i \le s$. We recall that for a fixed family Σ of indecomposable lattices in $\mathrm{latt}(\Lambda)$, M is of *type* Σ if each N_i is isomorphic to some element in Σ. In particular, we say that $\{N_1,\dots,N_s\}$, where $N_i \not\cong N_j$ for $i \ne j$, is the *type of* M.

By $H^{(\varepsilon,\Omega)}(X)$ we denote a hyperbolic quadratic (ε,Ω)-module, for any $X \in \mathrm{latt}(\Lambda)$. In particular we have $H^{(\varepsilon,\Omega)}(X) = (X \oplus X^*,[g])$, where $g \in \mathrm{Hom}_\Lambda(X\oplus X^*, X\oplus X^*)$ is given by the matrix $\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)$, and the hermitian form $h = g+\varepsilon g^*$ is non-singular on $X \oplus X^*$.

Thus we get the following theorem:

THEOREM 3.1. (i) *Every non-singular quadratic module* $(M,[g])$ *in the category* $\mathfrak{D}^{(\varepsilon,\Omega)}(\mathrm{latt}(\Lambda))$ *has an orthogonal decomposition*

$$(M,[g]) \cong \mathop{\perp}_{i=1}^{s} (M_i,[g_i])$$

where M_i *is of type* $\{N_i, N_i^*\}$ *and* N_i, $1 \le i \le s$, *are indecomposable left* Λ*-lattices such that* $N_i \oplus N_i^* \ncong N_j \oplus N_j^*$ *for* $i \ne j$, $1 \le i, j \le s$.

(ii) *If* N *is an indecomposable left* Λ*-lattice with* $N \ncong N^*$, *then every non-singular quadratic module* $(M,[g])$ *of type* $\{N, N^*\}$ *is the hyperbolic quadratic* (ε,Ω)*-module* $H^{(\varepsilon,\Omega)}(N^{(r)})$, *where* $N^{(r)}$ *is* r *copies of* N, *for some* r.

Proof. (i) Isometries in the category $\mathfrak{D}^{(\varepsilon,\Omega)}(\mathrm{latt}(\Lambda)/\mathcal{I}_I)$ lift to isometries in the category $\mathfrak{D}^{(\varepsilon,\Omega)}(\mathrm{latt}(\Lambda))$ by Theorem 2.7. Now the result follows from the fact that $\mathrm{End}_\Lambda(X)$ is $\mathcal{I}_I(X,X)$-complete for $X \in \mathrm{latt}(\Lambda)$ and by induction on S, analogously to ([Kn], II 6.3.1).

(ii) The proof is analogous to ([Kn], II 6.4.1). ∎

Because of the above theorem we are interested in determining those orthogonal summands $(N,[g])$ of a quadratic module $(M,[f])$ of the category $\mathfrak{D}^{(\varepsilon,\Omega)}(\mathrm{latt}(\Lambda))$ for which $N \cong N^*$. For this aim we use the following functor which is due to Green and Reiner [G-R] and Ringel and Roggenkamp [R-R]:

For the R-orders $\Lambda \subset \Gamma$ in the K-algebra $A = (L/K, a)$ and the two-sided Γ-ideal I in Λ we get the diagram

$$\begin{array}{ccc} \Lambda & \hookrightarrow & \Gamma \\ \downarrow & & \downarrow \\ \Lambda/I & \hookrightarrow & \Gamma/I \end{array}$$

Given a left Λ-lattice X, consider ΓX computed inside KX. Thus ΓX is a Γ-lattice. We remark that the algebras Λ/I and Γ/I are artinian, X/IX is a finitely generated left Λ/I-module and $\Gamma X/I\Gamma X$ is a finitely generated projective left Γ/I-module, since Γ is a hereditary R-order ([Re], 10.7). Moreover the inclusion $X \subset \Gamma X$ induces an inclusion $X/IX \xrightarrow{\sigma} \Gamma X/IX$ such that $(\Gamma/I)(\mathrm{Im}\,\sigma) = \Gamma X/IX$. This construction induces a functor F from the category $\mathrm{latt}(\Lambda)$ to the category $\mathcal{C}$ with objects the pairs $Y \xrightarrow{\sigma} Z$, where Y is a finitely generated left Λ/I-module, Z is a finitely generated projective left Γ/I-module and σ is a Λ/I-monomorphism such that $(\Gamma/I)\sigma(Y) = Z$. Morphisms in $\mathcal{C}$ are commutative diagrams

$$\begin{array}{ccc} Y & \xrightarrow{\sigma} & Z \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \phi} \\ Y' & \xrightarrow{\sigma'} & Z' \end{array}$$

where f is a Λ/I-homomorphism and ϕ is a Γ/I-homomorphism. The functor F is given by

$$F : \mathrm{latt}(\Lambda) \to \mathcal{C}, \quad X \to X/IX \overset{\sigma}{\to} \Gamma X/IX. \tag{3.1}$$

THEOREM 3.2. *The functor F is a representation equivalence of categories.*

Proof. E. L. Green and I. Reiner ([G-R], Section 2) and C. M. Ringel and K. W. Roggenkamp ([R-R], Theorem A) proved that F is a representation equivalence. ■

LEMMA 3.3. *The functor F is duality preserving.*

Proof. For X in $\mathrm{latt}(\Lambda)$, $\Gamma \otimes_R X \cong \Gamma X$ as left Γ-lattices, since X is a projective R-module. Therefore we get the natural isomorphism

$$\phi_X : F(X^*) = \Gamma \otimes_R X^* \cong F(X)^*$$

([Re], Theorem 2.3.8), and this proves the lemma. ■

Let now (ε, Ω) be a form parameter in $\mathrm{latt}(\Lambda)$. With the functor F (3.1) we can associate a form parameter $(\varepsilon, F(\Omega))$ in $\mathcal{C}$. Therefore we can define the reduction functor

$$\widetilde{F} : \mathfrak{D}^{(\varepsilon,\Omega)}(\mathrm{latt}(\Lambda)) \to \mathfrak{D}^{(\varepsilon,F(\Omega))}(\mathcal{C}) \tag{3.2}$$

where $\widetilde{F}(X, [g]) = (F(X), [F(g)])$.

We now come back to the involution of the first kind on $A = (L/K, a)$, mentioned in Section 2, such that $\overline{\Lambda} = \Lambda$. From ([H-Th], Proposition 3.2) $\overline{\Gamma} = \Gamma$ and from ([Ch-Th], Theorem 2.2) $\pi^i M$, $0 \le i \le e/m - 1$, are the non-isomorphic indecomposable Γ-lattices, where M is a left Λ- and right Δ-lattice full in an irreducible left A-module V such that $A \cong \mathrm{End}_D(V)$. Moreover we recall that m is equal to 1 or 2, because of the involution on A.

Let now (ε, Ω) be a form parameter in $\mathrm{latt}(\Gamma)$, $g \in \mathrm{Hom}_\Gamma(V, V^*)$, and let $h = g + \varepsilon g^*$ be the corresponding even hermitian form, for $\varepsilon = \pm 1$. We remark that the hermitian module (V, h) is uniquely determined by the quadratic (ε, Ω)-module $(V, [g])$. The map h is an A-homomorphism because of the involution on A. Thus a map $\varphi : Z \to Z$ is defined by the relation

$$h(\pi^i M) = \mathrm{Hom}_\Gamma(\pi^{\varphi(i)} M, \Gamma), \quad 0 \le i \le e/m - 1,$$

where the $\mathrm{Hom}_\Gamma(\pi^{\varphi(i)} M, \Gamma)$ are also the non-isomorphic indecomposable left Γ-lattices for $0 \le i \le e/m - 1$. In addition φ satisfies the relation

$$\varphi(i) = \varphi(0) - i$$

([Ri], §2). We remark that $(\pi^i M, [g_i])$, for $i \in \{0, \ldots, e/m - 1\}$ and g_i the restriction of g on $\pi^i M$, is a non-singular quadratic indecomposable module over $\mathrm{latt}(\Gamma)$ if and only if

$$h(\pi^i M) \cong \mathrm{Hom}_\Gamma(\pi^i M, \Gamma)$$

hence if and only if

$$2i \equiv \varphi(0) \bmod (e/m).$$

This equivalence has exactly one solution if e/m is odd, two solutions if both e/m and $\varphi(0)$ are even and no solutions if e/m is even and $\varphi(0)$ is odd. Thus we have proved the following proposition, using the same notation.

PROPOSITION 3.4. *Let Λ be the crossed product order $(S/R, a)$ in the crossed product algebra $A = (L/K, a) \cong \mathrm{End}_D(V)$ and $h = g + \varepsilon g^*$ be a non-singular hermitian form on V. Then if e/m is odd there exists exactly one* (*up to isometry*) *indecomposable non-singular quadratic module in* $\mathrm{latt}(\Gamma)$; *if both e/m and $\varphi(0)$ are even then there exist exactly two such modules*; *and if e/m is even and $\varphi(0)$ is odd there is no such module in* $\mathrm{latt}(\Gamma)$.

THEOREM 3.5. *Let Λ be the crossed product order $(S/R, a)$ in the crossed product algebra $A = (L/K, a) \cong \mathrm{End}_D(V)$ and $h = g + \varepsilon g^*$ be a non-singular hermitian form on V. Then if e/m is odd there exists exactly one* (*up to isometry*) *irreducible non-singular quadratic module in* $\mathrm{latt}(\Lambda)$; *if both e/m and $\varphi(0)$ are even then there exist exactly two such modules in* $\mathrm{latt}(\Lambda)$; *and there is no such module in* $\mathrm{latt}(\Lambda)$ *if e/m is even and $\varphi(0)$ is odd.*

Proof. Let $h = g + \varepsilon g^*$ be a non-singular hermitian form on V and (ε, Ω) be a form parameter in $\mathrm{latt}(\Gamma)$. Let also $(N, [g_N])$ be an indecomposable non-singular quadratic module in $\mathrm{latt}(\Gamma)$ corresponding to h according to Proposition 3.4, where g_N is the restriction of g to N. Then N is Γ-isomorphic to $\pi^i M$ for some $i \in \{0, \ldots, e/m - 1\}$. If $(N, [F_\Gamma(g_N)])$ is the image of $(N, [g_N])$ via the functor $\widetilde{F}_\Gamma$, then there exists a unique, up to isometry, non-singular quadratic (ε, Ω')-module $(N, [g'])$ in $\mathrm{latt}(\Lambda)/\mathcal{I}_I$ such that $\widetilde{F}(N, [g']) = (N, [F_\Gamma(g_N)])$, where $F(\Omega') = F_\Gamma(\Omega)$. The functor $\widetilde{F}_\Lambda$ is onto, so from Theorem 2.7 there exists at least one non-singular quadratic (ε, Ω_1)-module $(N, [g_1])$ in $\mathrm{latt}(\Lambda)$ such that $\widetilde{F}_\Lambda(N, [g_1]) = (N, [g'])$, where $F_\Lambda(\Omega_1) = \Omega'$. Furthermore from Theorem 2.7 it follows that g_1 is an isomorphism, and so we get the existence of an irreducible non-singular quadratic (ε, Ω_1)-module in $\mathrm{latt}(\Lambda)$, and moreover $(N, [g_1])$ is unique. ∎

All irreducible left Λ-lattices are described in ([Ch-Th], Theorem 3.1). We shall follow this description to get the irreducible left Λ-lattices in case $m = 1$ or 2. Suppose that $m = 2$. If N is an irreducible left Λ-lattice, then there is an L-basis υ_1, υ_2 of V such that

$$N = \pi^j \{S\upsilon_1 + \pi^{a_2} S\upsilon_2\} \tag{3.3}$$

for $0 \le j \le ea/2 - 1$, $0 \le a_2 \le (12\sigma)$, where a is a natural number depending on N and (12σ) is the valuation of the coefficient of υ_2 in the expression of $u_\sigma \upsilon_1$ as a linear combination of υ_1, υ_2 with coefficients from S, where $\sigma \in G$. We remark that then $M = S\upsilon_1 + S\upsilon_2$ is a left Λ- and right Δ-lattice. In case $m = 1$, $\pi^i S$, $0 \le i \le e/m - 1$, are the non-isomorphic irreducible left Λ-lattices.

PROPOSITION 3.6. *Let N be an irreducible left Λ-lattice, as in the formula* (3.3), *with $a_2 > 0$. Then $N \not\cong N^*$.*

Proof. If $g : N \cong N^*$, then $h = g_0 + \varepsilon g_0^*$ is a non-singular hermitian form on $V = K \otimes_R N$, where g_0 is the extension of g on V. Further $(N, [g])$ is a non-singular quadratic module in latt(Λ) for some form parameter (ε, Ω) in latt(Λ). Therefore

$$\widetilde{F}\widetilde{F}_\Lambda(N, [g]) = (\Gamma N, [\widetilde{g}]),$$

where $\widetilde{F}$ is the functor (3.2). However, ΓN is isomorphic to $\pi^i M$ for some $i \in \{0, \ldots, e/m - 1\}$, where $M = Sv_1 + Sv_2$. Thus, because of Theorem 2.7 and 3.2 and the functor $\widetilde{F}$, we get an isometry

$$(N, [g]) \cong (\pi^i M, [g_0|\pi^i M])$$

for some $i \in \{0, \ldots, e/m - 1\}$, which is impossible whenever $a_2 > 0$. ∎

COROLLARY 3.7. *Every non-singular quadratic module $(X, [g])$ over* latt(Λ) *of type $\{N, N^*\}$, where N is as in* (3.3) *with $a_2 > 0$, is a hyperbolic module of the form $H^{(\varepsilon,\Omega)}(N)$ for a form parameter (ε, Ω) in* latt(Λ).

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