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NEW OSCILLATION CRITERIA FOR FIRST ORDER NONLINEAR DELAY DIFFERENTIAL EQUATIONS

BY

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Abstract. New oscillation criteria are obtained for all solutions of a class of first order nonlinear delay differential equations. Our results extend and improve the results recently obtained by Li and Kuang [7]. Some examples are given to demonstrate the advantage of our results over those in [7].

1. Introduction and preliminaries. Oscillation theory of delay differential equations has drawn much attention in recent years. This is evidenced by extensive references in the recent books of Ladde *et al.* [4], Győri and Ladas [3] and Erbe *et al.* [2]. In a recent paper [7], Li and Kuang obtained a sharp sufficient condition for the oscillation of a nonlinear delay differential equation of the form

$$x'(t) + p(t)f(x(\tau(t))) = 0, \qquad t \ge t_0 > 0, \tag{1.1}$$

where

$$p, \tau \in C([t_0,\infty),[0,\infty)), \qquad \tau(t) < t, \qquad \lim_{t\to\infty} \tau(t) = \infty, \tag{1.2}$$

$$f \in C(\mathbb{R},\mathbb{R}) \quad \text{and} \quad uf(u) > 0 \quad \text{for } u \ne 0. \tag{1.3}$$

The main result in [7] is the following:

THEOREM A. *Assume that* (1.2) *and* (1.3) *hold and that for some* $\varepsilon > 0$, $M \ge 0$ *and* $r > 0$,

$$|f(u) - u| \le M|u|^{1+r} \qquad \textit{for } |u| < \varepsilon. \tag{1.4}$$

Suppose that

$$\int_{\delta(t)}^{t} p(s)\,ds \ge e^{-1}, \qquad t \ge t_0, \tag{1.5}$$

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and

$$\int_{t_0}^{\infty} p(t)\Big[\exp\Big(\int_{\delta(t)}^{t} p(s)\,ds - e^{-1}\Big) - 1\Big]dt = \infty, \tag{1.6}$$

where $\delta(t) = \max\{\tau(s) : t_0 \le s \le t\}$. *Then every solution of* (1.1) *oscillates.*

Theorem A is an extension of a result in [5] for linear delay differential equations to the nonlinear delay differential equation (1.1). For further research on the oscillation of linear delay differential equations, see the recent papers by Li [6], Tang and Shen [8], and Elbert and Stavroulakis [1].

In this paper, we establish an improvement of Theorem A in the following sense: (a) the nonlinear restriction (1.4) can be relaxed; (b) the oscillation criteria (1.5) or (1.6) can be weakened. The methods employed allow us to consider a more general first order nonlinear delay differential equation of the form

$$x'(t) + f(t, x(\tau(t))) = 0, \qquad t \ge t_0 > 0, \tag{1.7}$$

where

$$\tau \in C([t_0,\infty),[0,\infty)), \qquad \tau(t) < t, \qquad \lim_{t\to\infty} \tau(t) = \infty, \tag{1.8}$$

$$f \in C([t_0,\infty)\times\mathbb{R},\mathbb{R}), \qquad uf(t,u) \ge 0, \qquad t \ge t_0. \tag{1.9}$$

In connection with the nonlinear function $f(t,u)$ in (1.7) we suppose that the following assumption (H) holds:

(H) There are a piecewise continuous function $p : [t_0,\infty) \to \mathbb{R}^+ = [0,\infty)$, a function $g \in C(\mathbb{R},\mathbb{R}^+)$ and a number $\varepsilon_0 > 0$ such that

- (i) g is nondecreasing on $\mathbb{R}^+$;
- (ii) $g(-u) = g(u)$, $\lim_{u\to 0} g(u) = 0$;
- (iii) $\int_0^\infty g(e^{-u})\,du < \infty$;
- (iv) $|f(t,u) - p(t)u|/|u| \le p(t)g(u)$ for $t \ge t_0$ and $0 < |u| < \varepsilon_0$;
- (v) for each $\psi \in C([t_0,\infty),\mathbb{R})$ with $\lim_{t\to\infty} \psi(t) > 0$,

$$\int_{t_0}^{\infty} f(t,\psi(\tau(t)))\,dt = \infty, \qquad \int_{t_0}^{\infty} f(t,-\psi(\tau(t)))\,dt = -\infty.$$

Remark 1.1. It is easily seen that when $f(t,u) = p(t)f(u)$ condition (1.4) implies conditions (i)–(iv) for $g(u) = |u|^r$, $r > 0$. Also, noting that $\lim_{t\to\infty} \psi(t) > 0$ in condition (v), we see that conditions (1.3) and (1.5) imply (v). On the other hand, for $f(t,u) = p(t)f(u)$, where

$$f(u) = \begin{cases} u[1 + (1 + \ln^2|u|)^{-1}], & u \ne 0, \\ 0, & u = 0, \end{cases} \tag{1.10}$$

and

$$g(u) = \begin{cases} 1, & |u| > 1, \\ (1+\ln^2|u|)^{-1}, & 0 < |u| \le 1, \\ 0, & u = 0, \end{cases} \tag{1.11}$$

and $p \in C([t_0,\infty),\mathbb{R}^+)$ with $\int_{t_0}^{\infty} p(t)\,dt = \infty$, it is easily seen that condition (H) holds, but (1.4) does not hold. In Section 4, we will apply our main results to (1.7) when $f(t,x) = p(t)f(u)$ and $f(u)$ and $g(u)$ are given by (1.10) and (1.11) respectively.

REMARK 1.2. Here, we also remark that in some sense condition (v) is necessary for the oscillation of all solutions of (1.7). To see this, we give the following

PROPOSITION 1.1. *Assume that* (1.8) *holds*, $f \in C([t_0,\infty)\times\mathbb{R},\mathbb{R})$ *with* $f(t,u) \ge 0$ *for* $t \ge t_0$ *and* $u \ge 0$, *and* f *is nondecreasing in* u. *Suppose that there exists a constant* $\alpha > 0$ *such that*

$$\int_{t_0}^{\infty} f(s,\alpha)\,ds < \infty.$$

Then (1.7) *has an eventually positive solution.*

Proof. Let $t_1 > t_0$ be such that

$$\int_{t_1}^{\infty} f(s,\alpha)\,ds \le \alpha/2.$$

Define a function $y(t)$ as follows:

$$y(t) = \begin{cases} \dfrac{\alpha}{2} + \displaystyle\int_t^{\infty} f(s,\alpha)\,ds, & t \ge t_1, \\ \dfrac{\alpha}{2} + \dfrac{t-t_0}{t_1-t_0}\displaystyle\int_{t_1}^{\infty} f(s,\alpha)\,ds, & t_0 \le t < t_1. \end{cases}$$

Clearly, $y(t)$ is continuous on $[t_0,\infty)$ and

$$\alpha/2 \le y(t) \le \alpha \quad \text{for } t \ge t_0.$$

Let $T \ge t_1$ be such that $\tau(t) \ge t_0$ for $t \ge T$. Then for $t \ge T$,

$$y(t) = \frac{\alpha}{2} + \int_t^{\infty} f(s,\alpha)\,ds \ge \frac{\alpha}{2} + \int_t^{\infty} f(s,y(\tau(s)))\,ds.$$

Set $b = T - \min_{t\ge T}\{\tau(t)\}$. Define a sequence $\{x_n\}$ of functions as follows:

$$x_0(t) = \begin{cases} y(t), & t \ge T, \\ y(T), & T-b \le t < T, \end{cases}$$

$$x_n(t) = \begin{cases} \dfrac{\alpha}{2} + \displaystyle\int_t^\infty f(s, x_{n-1}(\tau(s)))\, ds, & t \ge T, \\ \dfrac{\alpha}{2} + \dfrac{t - T + b}{b}\Big[x_n(T) - \dfrac{\alpha}{2}\Big], & T - b \le t < T. \end{cases}$$

By induction, we have

$$\alpha/2 \le x_n(t) \le x_{n-1}(t) \le \alpha, \qquad t \ge T - b,\ n = 1, 2, \ldots$$

Then for $t \ge T - b$, $x(t) = \lim_{n\to\infty} x_n(t)$ exists, and

$$x(t) = \frac{\alpha}{2} + \int_t^\infty f(s, x(\tau(s)))\, ds, \qquad t \ge T.$$

It is easy to see that $x(t)$ is an eventually positive solution of (1.7). The proof is complete.

We note that the function $f(t, u) = p(t)u^r$, where $p(t) \ge 0$, $r \ge 1$, and

$$\int_{t_0}^\infty p(t)\, dt < \infty,$$

satisfies all the conditions in Proposition 1.1.

Let $\delta(t) = \max\{\tau(s) : t_0 \le s \le t\}$ and $\delta^{-1}(t) = \min\{s \ge t_0 : \delta(s) = t\}$. Clearly, δ and δ^{-1} are nondecreasing and satisfy:

(A) $\delta(t) < t$ and $\delta^{-1}(t) > t$;
(B) $\delta(\delta^{-1}(t)) = t$ and $\delta^{-1}(\delta(t)) \le t$.

Let $\delta^{-k}(t)$ be defined on $[t_0, \infty)$ by

$$\delta^{-(k+1)}(t) = \delta^{-1}(\delta^{-k}(t)), \qquad k = 1, 2, \ldots \tag{1.12}$$

Throughout this paper, we use the sequence $\{p_k\}$ of functions defined as follows:

$$p_1(t) = \int_t^{\delta^{-1}(t)} p(s)\, ds, \qquad t \ge t_0,$$

$$p_{k+1}(t) = \int_t^{\delta^{-1}(t)} p(s) p_k(s)\, ds, \qquad t \ge t_0,\ k = 1, 2, \ldots$$

Our main results are the following:

Theorem 1.1. *Assume that* (1.8), (1.9) *and* (H) *hold, and that*

$$\int_{\tau(t)}^t p(s)\, ds \ge e^{-1}, \qquad t \ge t_0, \tag{1.13}$$

$$\int_{t_0}^{\infty} p(t)\Big[\exp\Big(\int_{\tau(t)}^{t} p(s)\,ds - e^{-1}\Big) - 1\Big]\,dt = \infty. \tag{1.14}$$

Then every solution of (1.7) *oscillates.*

REMARK 1.3. It is clear that Theorem 1.1 extends and improves Theorem A. The following results yield a further improvement on the oscillation criteria (1.13) and (1.14).

THEOREM 1.2. *Assume that* (1.8), (1.9) *and* (H) *hold*, *and that*

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\,ds > 0. \tag{1.15}$$

Suppose that there exists a positive integer n such that

$$\int_{t_0}^{\infty} p(t)\ln(e^n p_n(t))\,dt = \infty. \tag{1.16}$$

Then every solution of (1.7) *oscillates.*

COROLLARY 1.1. *Assume that* (1.8), (1.9), (1.15) *and* (H) *hold*, *and that*

$$\int_{t_0}^{\infty} p(t)\ln\Big(e\int_{t}^{\delta^{-1}(t)} p(s)\,ds\Big)\,dt = \infty. \tag{1.17}$$

Then every solution of (1.7) *oscillates.*

We note that if

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)\,ds > 2,$$

then by Lemma 2.3 in Section 2 every solution of (1.7) oscillates. Thus, we will assume throughout this paper that

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)\,ds \le 2.$$

This implies that for some $\varepsilon > 0$,

$$\int_{\tau(t)}^{t} p(s)\,ds \le 2+\varepsilon \quad \text{for large } t.$$

Thus we have

$$\liminf_{t\to\infty} p_k(t) \le (2+\varepsilon)^{k-1}\liminf_{t\to\infty}\int_{t}^{\delta^{-1}(t)} p(s)\,ds \le (2+\varepsilon)^{k-1}\liminf_{t\to\infty}\int_{\tau(t)}^{t} p(s)\,ds.$$

As a result, by Theorem 1.2 we have

COROLLARY 1.2. *Assume that* (1.8), (1.9) *and* (H) *hold*, *and that there exists a positive integer* n *such that*

$$\liminf_{t\to\infty} p_n(t) > 1/e^n. \tag{1.18}$$

Then every solution of (1.7) *oscillates.*

The proofs of the above theorems and also some lemmas to be used in these proofs will be given in the next two sections. Some examples which illustrate the above remarks and the advantage of our results over those in [7] will be given in Section 4.

As is customary, a solution is called *oscillatory* if it has arbitrarily large zeros. Otherwise it is called *nonoscillatory*.

2. Some lemmas

LEMMA 2.1. *Assume that* (1.8), (1.9) *and* (H) *hold. Then every nonoscillatory solution of* (1.7) *converges to zero monotonically as* $t \to \infty$.

Proof. Suppose that $x(t)$ is a nonoscillatory solution of (1.7). We shall assume that $x(t)$ is eventually positive. The case where $x(t)$ is eventually negative is similar and is omitted. Choose a $t_1 \geq t_0$ such that $x(t) > 0$ for $t \geq t_1$. From (1.7)–(1.9), it follows that there exists $t_2 > t_1$ such that $\tau(t) \geq t_1$ and $x'(t) \leq 0$ for $t \geq t_2$. Hence $\lim_{t\to\infty} x(t) = \alpha \geq 0$ exists. If $\alpha > 0$, then by (1.7) we have

$$x(t) - x(t_0) = -\int_{t_0}^{t} f(s, x(\tau(s)))\, ds.$$

It follows from assumption (H)(v) that $\lim_{t\to\infty} x(t) = -\infty$, which contradicts $x(t)$ being eventually positive. The proof is complete.

LEMMA 2.2. *Assume that* (1.8), (1.9) *and* (H) *hold. If* $x(t)$ *is a nonoscillatory solution of* (1.7), *then there exist* $A > 0$ *and* $T > t_0$ *such that*

$$|x(t)| \leq A \exp\left(-\frac{1}{2}\int_{T}^{t} p(s)\, ds\right), \qquad t \geq T. \tag{2.1}$$

LEMMA 2.3. *Assume that* (1.8), (1.9) *and* (H) *hold. If* (1.7) *has a nonoscillatory solution*, *then eventually*

$$\int_{\tau(t)}^{t} p(s)\, ds \leq 2 \quad \textit{and} \quad p_k(t) \leq 2^k, \quad k = 1, 2, \ldots \tag{2.2}$$

LEMMA 2.4. *Assume that* (1.8), (1.9), (1.15) *and* (H) *hold. If* $x(t)$ *is a nonoscillatory solution of* (1.7), *then* $x(\tau(t))/x(t)$, *which is well defined for large* t, *is bounded.*

Proofs of Lemmas 2.2–2.4. Suppose that $x(t)$ is a nonoscillatory solution of (1.7) which will be assumed to be eventually positive (if $x(t)$ is eventually negative the proof is similar). By Lemma 2.1, there exists $t_1 \geq t_0$ such that

$$\varepsilon_0 > x(\tau(t)) \geq x(t) > 0, \qquad t \geq t_1, \tag{2.3}$$

and $\lim_{t\to\infty} x(t) = 0$. By assumption (H), there exists $t_2 > t_1$ such that

$$f(t, x(\tau(t))) \geq \tfrac{1}{2}p(t)x(\tau(t)), \qquad t \geq t_2, \tag{2.4}$$

and it follows from (1,7) that

$$x'(t) + \tfrac{1}{2}p(t)x(\tau(t)) \leq 0, \qquad t \geq t_2. \tag{2.5}$$

The rest of the proof is similar to that of Lemmas 2–4 in [7] respectively, and thus is omitted.

3. Proofs of theorems

Proof of Theorem 1.1. Assume that (1.7) has a nonoscillatory solution $x(t)$ which will be assumed to be eventually positive (if $x(t)$ is eventually negative the proof is similar). By Lemma 2.1, there exists $t_1 \geq t_0$ such that

$$0 < x(t) \leq x(\delta(t)) \leq x(\tau(t)) < \varepsilon_0, \qquad t \geq t_1, \tag{3.1}$$

where ε_0 is given by assumption (H). From (3.1) and (H), we have

$$f(t, x(\tau(t))) \geq p(t)[1 - g(x(\tau(t)))]x(\tau(t)), \qquad t \geq t_1. \tag{3.2}$$

Set

$$\omega(t) = \frac{x(\tau(t))}{x(t)} \qquad \text{for } t \geq t_1.$$

Then $\omega(t) \geq 1$ for $t \geq t_1$. From (1.7) and (3.2), we have

$$\frac{x'(t)}{x(t)} + p(t)\omega(t)[1 - g(x(\tau(t)))] \leq 0, \qquad t \geq t_1. \tag{3.3}$$

Let $t_2 > t_1$ be such that $\tau(t) \geq t_1$ for $t \geq t_2$. Integrating both sides of (3.3) from $\tau(t)$ to t, we obtain

$$\omega(t) \geq \exp\Big(\int_{\tau(t)}^{t} p(s)\omega(s)[1 - g(x(\tau(s)))]\,ds\Big), \qquad t \geq t_2. \tag{3.4}$$

By (1.13), for $t \geq t_2$ we have

$$\int_{\delta(t)}^{t} p(s)\,ds = \int_{\tau(t^*)}^{t} p(s)\,ds \geq \int_{\tau(t^*)}^{t^*} p(s)\,ds \geq e^{-1}, \tag{3.5}$$

where $t^* \in [t_0, t]$ with $\tau(t^*) = \delta(t)$. From (1.13) and (3.4), we find that for $t \geq t_2$,

$$\begin{aligned}\omega(t) &\geq \exp \int_{\tau(t)}^{t} p(s)\omega(s)[1 - g(x(\tau(s)))]\,ds \\ &= \exp\Big(\int_{\tau(t)}^{t} p(s)[\omega(s) - 1]\,ds + e^{-1}\Big) \exp\Big(\int_{\tau(t)}^{t} p(s)\,ds - e^{-1}\Big) \\ &\quad \times \exp\Big(- \int_{\tau(t)}^{t} p(s)\omega(s)g(x(\tau(s)))\,ds\Big) \\ &\geq \Big(e \int_{\delta(t)}^{t} p(s)[\omega(s) - 1]\,ds + 1\Big) \exp\Big(\int_{\tau(t)}^{t} p(s)\,ds - e^{-1}\Big) \\ &\quad \times \exp\Big(- \int_{\tau(t)}^{t} p(s)\omega(s)g(x(\tau(s)))\,ds\Big).\end{aligned}$$

Let $\nu(t) = \omega(t) - 1$ for $t \geq t_1$. Then $\nu(t) \geq 0$ for $t \geq t_1$, and so for $t \geq t_2$,

$$\begin{aligned}\nu(t) - e \int_{\delta(t)}^{t} p(s)\nu(s)\,ds &\geq \Big(e \int_{\delta(t)}^{t} p(s)\nu(s)\,ds + 1\Big) \\ &\quad \times \Big[\exp\Big(\int_{\tau(t)}^{t} p(s)\,ds - e^{-1}\Big) \exp\Big(- \int_{\tau(t)}^{t} p(s)\omega(s)g(x(\tau(s)))\,ds\Big) - 1\Big],\end{aligned}$$

that is, for $t \geq t_2$,

$$\begin{aligned}&p(t)\nu(t) - ep(t) \int_{\delta(t)}^{t} p(s)\nu(s)\,ds \\ &\geq p(t)\Big(e \int_{\delta(t)}^{t} p(s)\nu(s)\,ds + 1\Big) \\ &\quad \times \Big[\exp\Big(\int_{\tau(t)}^{t} p(s)\,ds - e^{-1}\Big) \exp\Big(- \int_{\tau(t)}^{t} p(s)\omega(s)g(x(\tau(s)))\,ds\Big) - 1\Big].\end{aligned} \tag{3.6}$$

By Lemmas 2.2–2.4, there exist $T > t_2, A > 0$ and $M > 0$ such that for $t \geq T$,

$$x(\tau(t)) \leq A \exp\Big(-\frac{1}{2} \int_{T}^{\tau(t)} p(s)\,ds\Big), \tag{3.7}$$

$$\int_{\tau(t)}^{t} p(s)\,ds \le 2, \tag{3.8}$$

$$\omega(t) \le M. \tag{3.9}$$

Let

$$\alpha(t) = \frac{1}{2}\int_{T}^{t} p(s)\,ds, \qquad t \ge T.$$

Clearly, (1.13) implies that $\alpha(t) \to \infty$ as $t \to \infty$. For $t \ge t_2$, set

$$D(t) = p(t)\Big(e\int_{\delta(t)}^{t} p(s)\nu(s)\,ds + 1\Big)\exp\Big(\int_{\tau(t)}^{t} p(s)\,ds - e^{-1}\Big) \tag{3.10}$$
$$\times\Big[1 - \exp\Big(-\int_{\tau(t)}^{t} p(s)\omega(s)g(x(\tau(s)))\,ds\Big)\Big].$$

One can easily see that

$$0 \le 1 - e^{-c} \le c \qquad \text{for } c \ge 0. \tag{3.11}$$

It follows from (3.10) that for $t \ge t_2$,

$$D(t) \le p(t)\Big(e\int_{\delta(t)}^{t} p(s)\nu(s)\,ds + 1\Big)\exp\Big(e\int_{\tau(t)}^{t} p(s)\,ds - e^{-1}\Big) \tag{3.12}$$
$$\times\int_{\tau(t)}^{t} p(s)\omega(s)g(x(\tau(s)))\,ds.$$

Let $T^* > T$ be such that $\tau(\tau(t)) \ge T$ for $t \ge T^*$ and $\alpha(T^*) > 2 + \ln A$. Set $M_1 = e^{2e}M[2e(M-1)+1]$ and $A_1 = eA$. Noting that

$$e\int_{\delta(t)}^{t} p(s)\nu(s)\,ds + 1 \le 2e(M-1)+1 \qquad \text{for } t \ge T,$$

from (3.7)–(3.9), (3.12) and assumption (H), we obtain for $N > T^*$,

$$\int_{T^*}^{N} D(t)\,dt \le M_1\int_{T^*}^{N} p(t)\int_{\tau(t)}^{t} p(s)g\Big(A\exp\Big(-\frac{1}{2}\int_{T}^{\tau(s)} p(\mu)d\mu\Big)\Big)\,ds\,dt$$
$$= M_1\int_{T^*}^{N} p(t)\int_{\tau(t)}^{t} p(s)$$
$$\times g\Big(A\exp\Big(-\frac{1}{2}\int_{T}^{s} p(\mu)d\mu + \frac{1}{2}\int_{\tau(s)}^{s} p(\mu)d\mu\Big)\Big)\,ds\,dt$$

$$
\begin{aligned}
&\le M_1 \int_{T^*}^{N} p(t) \int_{\tau(t)}^{t} p(s) g(A_1 e^{-\alpha(s)})\, ds\, dt \\
&= 2M_1 \int_{T^*}^{N} p(t) \int_{\alpha(\tau(t))}^{\alpha(t)} g(A_1 e^{-u})\, du\, dt \\
&= 2M_1 \int_{T^*}^{N} p(t) \int_{\alpha(t)-\beta(t)}^{\alpha(t)} g(A_1 e^{-u})\, du\, dt \qquad \Big(\beta(t) = \frac{1}{2}\int_{\tau(t)}^{t} p(s)\, ds\Big) \\
&\le 4M_1 \int_{\alpha(T^*)}^{\alpha(N)} \int_{v-1}^{v} g(A_1 e^{-u})\, du\, dv \\
&\le 4M_1 \int_{\alpha(T^*)-1}^{\alpha(N)} g(A_1 e^{-u})\, du = 4M_1 \int_{\alpha(T^*)-\ln(eA_1)}^{\alpha(N)-\ln A_1} g(e^{-u})\, du \\
&\le 4M_1 \int_{0}^{\alpha(N)} g(e^{-u})\, du \le 4M_1 \int_{0}^{\infty} g(e^{-u})\, du < \infty.
\end{aligned}
$$

Thus

$$
\int_{T^*}^{\infty} D(t)\, dt < \infty. \tag{3.13}
$$

Substituting (3.10) into (3.6), for $t \ge t_2$ we obtain

$$
\begin{aligned}
&p(t)\nu(t) - ep(t) \int_{\delta(t)}^{t} p(s)\nu(s)\, ds \\
&\ge p(t)\Big(e \int_{\delta(t)}^{t} p(s)\nu(s)\, ds + 1\Big)\Big[\exp\Big(\int_{\tau(t)}^{t} p(s)\, ds - e^{-1}\Big) - 1\Big] - D(t) \\
&\ge p(t)\Big[\exp\Big(\int_{\tau(t)}^{t} p(s)\, ds - e^{-1}\Big) - 1\Big] - D(t).
\end{aligned} \tag{3.14}
$$

Integrating both sides of (3.14) from T^* to $N > \delta^{-1}(T^*)$, we have

$$
\begin{aligned}
&\int_{T^*}^{N} p(t)\nu(t)\, dt - e \int_{T^*}^{N} p(t) \int_{\delta(t)}^{t} p(s)\nu(s)\, ds\, dt \\
&\qquad \ge \int_{T^*}^{N} p(t)\Big[\exp\Big(\int_{\tau(t)}^{t} p(s)\, ds - e^{-1}\Big) - 1\Big] dt - \int_{T^*}^{N} D(t)\, dt.
\end{aligned} \tag{3.15}
$$

By interchanging the order of integrations and by (3.5), we have

$$(3.16)\qquad e\int_{T^*}^{N} p(t)\int_{\delta(t)}^{t} p(s)\nu(s)\,ds\,dt \ge e\int_{T^*}^{\delta(N)} p(t)\nu(t)\int_{t}^{\delta^{-1}(t)} p(s)\,ds\,dt \ge \int_{T^*}^{\delta(N)} p(t)\nu(t)\,dt.$$

From this and (3.15), it follows that

$$(3.17)\qquad \int_{\delta(N)}^{N} p(t)\nu(t)\,dt \ge \int_{T^*}^{N} p(t)\Big[\exp\Big(\int_{\tau(t)}^{t} p(s)\,ds - e^{-1}\Big) - 1\Big]\,dt - \int_{T^*}^{N} D(t)\,dt.$$

By (3.8) and (3.9),

$$\int_{\delta(N)}^{N} p(t)\nu(t)\,dt \le (M-1)\int_{\delta(N)}^{N} p(t)\,dt \le (M-1)\int_{\tau(N)}^{N} p(t)\,dt \le 2(M-1),$$

and so by (3.17),

$$2(M-1) \ge \int_{T^*}^{N} p(t)\Big[\exp\Big(\int_{\tau(t)}^{t} p(s)\,ds - e^{-1}\Big) - 1\Big]\,dt - \int_{T^*}^{N} D(t)\,dt.$$

This implies that

$$2(M-1) \ge \int_{T^*}^{\infty} p(t)\Big[\exp\Big(\int_{\tau(t)}^{t} p(s)\,ds - e^{-1}\Big) - 1\Big]\,dt - \int_{T^*}^{\infty} D(t)\,dt,$$

which together with (3.13) yields

$$(3.18)\qquad \int_{T^*}^{\infty} p(t)\Big[\exp\Big(\int_{\tau(t)}^{t} p(s)\,ds - e^{-1}\Big) - 1\Big]\,dt < \infty.$$

This contradicts (1.14) and so the proof is complete.

Proof of Theorem 1.2. Assume that (1.7) has a nonoscillatory solution $x(t)$ which will be assumed to be eventually positive (if $x(t)$ is eventually negative the proof is similar). By Lemma 2.1 and assumption (H), there exists $t_0^* \ge t_0$ such that

$$(3.19)\qquad 0 < x(t) \le x(\delta(t)) \le x(\tau(t)) \le \varepsilon_0, \quad g(x(\tau(t))) < 1, \quad t \ge t_0^*,$$

where ε_0 is given by assumption (H). (3.19) and (H) yield that for $t \geq t_0^*$,

$$f(x(\tau(t))) \geq p(t)[1-g(x(\tau(t)))]x(\tau(t)) \geq p(t)[1-g(x(\tau(t)))]x(\delta(t)), \tag{3.20}$$

and it follows from (1.7) that

$$\frac{x'(t)}{x(t)} + p(t)\frac{x(\delta(t))}{x(t)}[1-g(x(\tau(t)))] \leq 0, \qquad t \geq t_0^*. \tag{3.21}$$

By Lemmas 2.2–2.4, there exist $T > t_0^*$, $A > 0$ and $M > 0$ such that for $t \geq T$,

$$x(\tau(t)) \leq A \exp\Big(-\frac{1}{2}\int_T^{\tau(t)} p(s)\,ds\Big), \tag{3.22}$$

$$\int_{\delta(t)}^t p(s)\,ds \leq \int_{\tau(t)}^t p(s)\,ds \leq 2, \qquad p_k(t) \leq 2^k, \qquad k = 1, 2, \dots, \tag{3.23}$$

$$\frac{x(\delta(t))}{x(t)} \leq \frac{x(\tau(t))}{x(t)} \leq M. \tag{3.24}$$

Let $t_k = \delta^{-k}(T)$, $k = 1, 2, \dots$ Clearly $t_k \to \infty$ as $k \to \infty$. Set $\lambda(t) = -x'(t)/x(t)$, $t \geq T$. Then $x(\delta(t))/x(t) = \exp\int_{\delta(t)}^t \lambda(s)\,ds$, $t \geq t_1$, and from (3.21) we have

$$\lambda(t) \geq p(t)\exp\int_{\delta(t)}^t \lambda(s)\,ds - p(t)g(x(\tau(t)))\frac{x(\delta(t))}{x(t)}, \qquad t \geq t_1. \tag{3.25}$$

It follows from (3.22)–(3.25) that for $t \geq t_1$,

$$\begin{aligned} \lambda(t) &\geq p(t)\exp\int_{\delta(t)}^t \lambda(s)\,ds - Mp(t)g\Big(A\exp\Big(-\frac{1}{2}\int_T^{\tau(t)} p(s)\,ds\Big)\Big) \\ &\geq p(t)\exp\int_{\delta(t)}^t \lambda(s)\,ds - Mp(t)g\Big(A_1\exp\Big(-\frac{1}{2}\int_T^t p(s)\,ds\Big)\Big) \end{aligned} \tag{3.26}$$

where $A_1 = eA$. By the inequality $e^c \geq ec$ for $c \geq 0$, we have

$$\begin{aligned} \lambda(t) \geq{}& ep(t)\int_{\delta(t)}^t \lambda(s)\,ds \\ &- Mp(t)g\Big(A_1\exp\Big(-\frac{1}{2}\int_T^t p(s)\,ds\Big)\Big), \qquad t \geq t_1. \end{aligned} \tag{3.27}$$

Set

$$\alpha(t) = \frac{1}{2}\int_T^t p(s)\,ds, \qquad t \geq T; \tag{3.28}$$

and

$$\begin{cases} \lambda_0(t) = \lambda(t), \quad t \geq T, \\ \lambda_k(t) = p(t) \int\limits_{\delta(t)}^{t} \lambda_{k-1}(s)\,ds, \quad t \geq t_k,\ k = 1, \dots, n; \end{cases} \tag{3.29}$$

and

$$\begin{cases} G_0(t) = 0, \quad t \geq T, \\ G_k(t) = ep(t) \int\limits_{\delta(t)}^{t} G_{k-1}(s)\,ds \\ \qquad\qquad + Mp(t)g(A_1 \exp(-\alpha(t))), \quad t \geq t_k,\ k = 1, \dots, n. \end{cases} \tag{3.30}$$

Clearly, (1.15) implies that $\alpha(t)$ is nondecreasing on $[T, \infty)$ and $\alpha(t) \to \infty$ as $t \to \infty$. By iteration we deduce from (3.27) that

$$\lambda(t) \geq e^k \lambda_k(t) - G_k(t), \qquad t \geq t_k,\ k = 1, \dots, n-1, \tag{3.31}$$

and so by (3.26),

$$\begin{aligned} \lambda(t) \geq p(t) \exp\Big(e^{n-1} \int\limits_{\delta(t)}^{t} \lambda_{n-1}(s)\,ds\Big) \\ \times \exp\Big(-\int\limits_{\delta(t)}^{t} G_{n-1}(s)\,ds\Big) - G_1(t), \qquad t \geq t_n. \end{aligned} \tag{3.32}$$

From (3.30), one can easily obtain

$$\begin{aligned} G_{k+1}(t) - G_k(t) \\ = ep(t) \int\limits_{\delta(t)}^{t} [G_k(s) - G_{k-1}(s)]\,ds, \qquad t \geq t_{k+1},\ k = 1, \dots, n-1. \end{aligned} \tag{3.33}$$

By (3.23), (3.28) and (3.30), for $t \geq t_2$ we have

$$\begin{aligned} \int\limits_{\delta(t)}^{t} G_1(s)\,ds = M \int\limits_{\delta(t)}^{t} p(s)g(A_1 e^{-\alpha(s)})\,ds \\ = 2M \int\limits_{\alpha(\delta(t))}^{\alpha(t)} g(A_1 e^{-u})\,du \leq 2M \int\limits_{\alpha(t)-1}^{\alpha(t)} g(A_1 e^{-u})\,du. \end{aligned} \tag{3.34}$$

Thus, from (3.33), we get

$$\begin{aligned} G_2(t) - G_1(t) &= ep(t) \int_{\delta(t)}^{t} G_1(s)\, ds \\ &\le 2eMp(t) \int_{\alpha(t)-1}^{\alpha(t)} g(A_1 e^{-u})\, du, \qquad t \ge t_2, \\ G_3(t) - G_2(t) &= ep(t) \int_{\delta(t)}^{t} [G_2(s) - G_1(s)]\, ds \\ &\le 2e^2 Mp(t) \int_{\delta(t)}^{t} p(s) \int_{\alpha(s)-1}^{\alpha(s)} g(A_1 e^{-u})\, du\, ds \\ &= 4e^2 Mp(t) \int_{\alpha(\delta(t))}^{\alpha(t)} \int_{v-1}^{v} g(A_1 e^{-u})\, du\, dv \\ &\le 4e^2 Mp(t) \int_{\alpha(t)-1}^{\alpha(t)} \int_{v-1}^{v} g(A_1 e^{-u})\, du\, dv \\ &\le 4e^2 Mp(t) \int_{\alpha(t)-2}^{\alpha(t)} g(A_1 e^{-u})\, du, \quad t \ge t_3. \end{aligned}$$

By induction, one can prove in general that for $k = 2, \dots, n-1$,

$$\begin{aligned} &G_k(t) - G_{k-1}(t) \\ &\qquad \le (2e)^{k-1}(k-2)!Mp(t) \int_{\alpha(t)-(k-1)}^{\alpha(t)} g(A_1 e^{-u})\, du, \qquad t \ge t_k, \end{aligned}$$

and so

$$\begin{aligned} G_{n-1}(t) &= \sum_{k=1}^{n-1} [G_k(t) - G_{k-1}(t)] \\ &\le G_1(t) + Mp(t) \sum_{k=2}^{n-1} (2e)^{k-1} \\ &\quad \times (k-2)! \int_{\alpha(t)-(k-1)}^{\alpha(t)} g(A_1 e^{-u})\, du, \qquad t \ge t_{n-1}. \end{aligned} \tag{3.35}$$

By (3.23), (3.24) and (3.29), we obtain

$$
(3.36)\qquad \begin{cases}
\lambda_1(t) = p(t) \int\limits_{\delta(t)}^{t} \lambda(s)\,ds = p(t)\ln \dfrac{x(\delta(t))}{x(t)} \\
\qquad \le p(t)\ln M, \qquad t \ge t_1, \\
\lambda_2(t) = p(t) \int\limits_{\delta(t)}^{t} \lambda_1(s)\,ds \le (\ln M)p(t) \int\limits_{\delta(t)}^{t} p(s)\,ds \\
\qquad \le 2(\ln M)p(t), \qquad t \ge t_2, \\
\dots \\
\lambda_{n-1}(t) \le 2^{n-2}(\ln M)p(t), \qquad t \ge t_{n-1}.
\end{cases}
$$

Set

$$
\begin{aligned}
D(t) = {} & p(t)\exp\Big(e^{n-1} \int\limits_{\delta(t)}^{t} \lambda_{n-1}(s)\,ds\Big) \\
& \times \Big[1 - \exp\Big(- \int\limits_{\delta(t)}^{t} G_{n-1}(s)\,ds\Big)\Big] + G_1(t), \qquad t \ge t_n.
\end{aligned}
$$

From (3.11), (3.23), (3.34), (3.35) and (3.36) we have

$$
\begin{aligned}
(3.37)\qquad D(t) & \le p(t)\exp\Big(e^{n-1} \int\limits_{\delta(t)}^{t} \lambda_{n-1}(s)\,ds\Big) \int\limits_{\delta(t)}^{t} G_{n-1}(s)\,ds + G_1(t) \\
& \le G_1(t) + p(t)\exp\Big(2^{n-2}e^{n-1}\ln M \int\limits_{\delta(t)}^{t} p(s)\,ds\Big) \\
& \quad \times \int\limits_{\delta(t)}^{t} \Big(G_1(s) + Mp(s) \\
& \quad \times \sum_{k=2}^{n-1} (2e)^{k-1}(k-2)! \int\limits_{\alpha(s)-(k-1)}^{\alpha(s)} g(A_1e^{-u})\,du\Big)\,ds \\
& \le G_1(t) + 2Mp(t)\exp[(2e)^{n-1}\ln M] \int\limits_{\alpha(t)-1}^{\alpha(t)} g(A_1e^{-u})\,du \\
& \quad + Mp(t)\exp[(2e)^{n-1}\ln M] \\
& \quad \times \sum_{k=2}^{n-1} (2e)^{k-1}(k-2)! \int\limits_{\delta(t)}^{t} p(s) \int\limits_{\alpha(s)-(k-1)}^{\alpha(s)} g(A_1e^{-u})\,du\,ds
\end{aligned}
$$

$$\leq G_1(t) + M_1 p(t) \sum_{k=1}^{n-1} (2e)^{k-1}(k-1)! \int\limits_{\alpha(t)-k}^{\alpha(t)} g(A_1 e^{-u})\,du, \qquad t \geq t_n,$$

where $M_1 = 2M \exp[(2e)^{n-1} \ln M]$.

Let $T^* > t_n$ be such that $\alpha(T^*) > n + \ln A_1$. It follows from (3.37) and (H) that

$$\begin{aligned}
(3.38)\quad \int\limits_{T^*}^{\infty} D(t)\,dt &\leq \int\limits_{T^*}^{\infty} G_1(t)\,dt \\
&\quad + M_1 \sum_{k=1}^{n-1} (2e)^{k-1}(k-1)! \int\limits_{T^*}^{\infty} p(t) \int\limits_{\alpha(t)-k}^{\alpha(t)} g(A_1 e^{-u})\,du\,dt \\
&\leq 2M \int\limits_{\alpha(T^*)}^{\infty} g(A_1 e^{-u})\,du \\
&\quad + 2M_1 \sum_{k=1}^{n-1} (2e)^{k-1}(k-1)! \int\limits_{\alpha(T^*)}^{\infty} \int\limits_{v-k}^{v} g(A_1 e^{-u})\,du\,dv \\
&\leq 2M \int\limits_{\alpha(T^*)-\ln A_1}^{\infty} g(e^{-u})\,du \\
&\quad + 2M_1 \sum_{k=1}^{n-1} (2e)^{k-1} k! \int\limits_{\alpha(T^*)-(k+1)}^{\infty} g(A_1 e^{-u})\,du \\
&\leq 2M \int\limits_{0}^{\infty} g(e^{-u})\,du + 2M_1 \sum_{k=1}^{n-1} (2e)^{k-1} k! \int\limits_{0}^{\infty} g(e^{-u})\,du < \infty.
\end{aligned}$$

Since

$$\begin{aligned}
p(t) \exp\Big(e^{n-1} \int\limits_{\delta(t)}^{t} \lambda_{n-1}(s)\,ds\Big) \exp\Big(-\int\limits_{\delta(t)}^{t} G_{n-1}(s)\,ds\Big) - G_1(t) \\
= p(t) \exp\Big(e^{n-1} \int\limits_{\delta(t)}^{t} \lambda_{n-1}(s)\,ds\Big) - D(t), \qquad t \geq t_n,
\end{aligned}$$

it follows from (3.32) that

$$(3.39)\qquad \lambda(t) \geq p(t) \exp\Big(e^{n-1} \int\limits_{\delta(t)}^{t} \lambda_{n-1}(s)\,ds\Big) - D(t), \qquad t \geq t_n.$$

One can easily show that $\gamma e^x \geq x + \ln(e\gamma)$ for $\gamma > 0$, and so for $t \geq t_n$,

$$p_n(t)\lambda(t) \geq p(t)e^{1-n}[e^{n-1}p_n(t)]\exp\Big(e^{n-1}\int_{\delta(t)}^{t}\lambda_{n-1}(s)\,ds\Big) - p_n(t)D(t)$$
$$\geq p(t)\int_{\delta(t)}^{t}\lambda_{n-1}(s)\,ds + e^{1-n}p(t)\ln(e^n p_n(t)) - p_n(t)D(t),$$

that is, for $t \geq t_n$,

$$p_n(t)\lambda(t) - p(t)\int_{\delta(t)}^{t}\lambda_{n-1}(s)\,ds \geq e^{1-n}p(t)\ln(e^n p_n(t)) - p_n(t)D(t). \tag{3.40}$$

For $N > \delta^{-n}(T^*)$, we have

$$\int_{T^*}^{N}\lambda(t)p_n(t)\,dt - \int_{T^*}^{N}p(t)\int_{\delta(t)}^{t}\lambda_{n-1}(s)\,ds\,dt \geq e^{1-n}\int_{T^*}^{N}p(t)\ln(e^n p_n(t))\,dt - \int_{T^*}^{N}p_n(t)D(t)\,dt. \tag{3.41}$$

Let

$$\delta^1(t) = \delta(t), \qquad \delta^{k+1}(t) = \delta(\delta^k(t)), \qquad k = 1, \dots, n.$$

Then by interchanging the order of integration, we have

$$\int_{T^*}^{N}p(t)\int_{\delta(t)}^{t}\lambda_{n-1}(s)\,ds\,dt \geq \int_{T^*}^{\delta(N)}\lambda_{n-1}(t)\int_{t}^{\delta^{-1}(t)}p(s)\,ds\,dt$$
$$= \int_{T^*}^{\delta(N)}p(t)p_1(t)\int_{\delta(t)}^{t}\lambda_{n-2}(s)\,ds\,dt$$
$$\geq \int_{T^*}^{\delta^2(N)}\lambda_{n-2}(t)\int_{t}^{\delta^{-1}(t)}p(s)p_1(s)\,ds\,dt$$
$$= \int_{T^*}^{\delta^2(N)}p(t)p_2(t)\int_{\delta(t)}^{t}\lambda_{n-3}(s)\,ds\,dt$$
$$\cdots$$
$$\geq \int_{T^*}^{\delta^n(N)}\lambda(t)p_n(t)\,dt.$$

From this and (3.41) we obtain

$$\int_{\delta^n(N)}^{N} \lambda(t)p_n(t)\,dt \geq e^{1-n}\int_{T^*}^{N} p(t)\ln(e^n p_n(t))\,dt - \int_{T^*}^{N} p_n(t)D(t)\,dt, \tag{3.42}$$

which together with (3.23) yields

$$2^n \int_{\delta^n(N)}^{N} \lambda(t)\,dt \geq e^{1-n}\int_{T^*}^{N} p(t)\ln(e^n p_n(t))\,dt - 2^n\int_{T^*}^{N} D(t)\,dt,$$

or

$$\ln\frac{x(\delta^n(N))}{x(N)} \geq 2^{-n}e^{1-n}\int_{T^*}^{N} p(t)\ln(e^n p_n(t))\,dt - \int_{T^*}^{N} D(t)\,dt. \tag{3.43}$$

In view of (1.16) and (3.38), we have

$$\lim_{N\to\infty}\frac{x(\delta^n(N))}{x(N)} = \infty. \tag{3.44}$$

On the other hand, (3.24) implies that

$$\frac{x(\delta^n(N))}{x(N)} = \frac{x(\delta^1(N))}{x(N)}\cdot\frac{x(\delta^2(N))}{x(\delta^1(N))}\cdots\frac{x(\delta^n(N))}{x(\delta^{n-1}(N))} \leq M^n.$$

This contradicts (3.44) and completes the proof.

4. Examples

EXAMPLE 4.1. Consider the delay differential equation

$$x'(t) + p(t)f(x(\tau(t))) = 0, \qquad t \geq 3, \tag{4.1}$$

where

$$p(t) = \frac{1}{et\ln 2} + \frac{1}{t\ln t}, \qquad \tau(t) = \frac{t}{2},$$

and $f(u)$ and $g(u)$ are defined by (1.10) and (1.11).

As pointed out in Section 1, assumption (H) holds but (1.4) does not. We check that the conditions (1.13) and (1.14) in Theorem 1.1 hold. In fact, for $t \geq 3$,

$$\int_{t/2}^{t} p(s)\,ds = \int_{t/2}^{t}\left(\frac{1}{es\ln 2} + \frac{1}{s\ln s}\right)ds = e^{-1} - \ln\left(1 - \frac{\ln 2}{\ln t}\right) \geq e^{-1},$$

and

$$\lim_{t\to\infty}\int_{t/2}^{t} p(s)\,ds = e^{-1},$$

and

$$\int_3^\infty p(t)\Big(\int_{t/2}^t p(s)\,ds - e^{-1}\Big)\,dt \geq -\frac{1}{e\ln 2}\int_3^\infty \frac{1}{t}\ln\left(1-\frac{\ln 2}{\ln t}\right)dt = \infty,$$

because

$$\int_3^\infty \frac{1}{t\ln t}\,dt = \infty \quad \text{and} \quad \lim_{t\to\infty}(\ln t)\ln\left(1-\frac{\ln 2}{\ln t}\right) = -\ln 2.$$

By Theorem 1.1 every solution of (4.1) oscillates.

EXAMPLE 4.2. Consider the delay differential eqaution

$$x'(t) + f(t, x(\tau(t))) = 0, \qquad t \geq 0, \tag{4.2}$$

where $\tau(t) = t-1$ and $f(t,u) = [\exp 3(\sin t - 1) + |u|]^{1/3}u$.

Let $p(t) = \exp(\sin t - 1)$ and $g(u) = e^2|u|^{1/3}$. It is easy to see that assumption (H) holds. Clearly

$$\liminf_{t\to\infty}\int_{t-1}^t p(s)\,ds < e^{-1}.$$

By Jensen's inequality,

$$\begin{aligned}\int_0^\infty p(t)\ln\left(e\int_t^{t+1} p(s)\,ds\right)dt &\geq \int_0^\infty p(t)\int_t^{t+1}\sin s\,ds\,dt\\ &= \frac{2\sin 2^{-1}}{e}\int_0^\infty \exp(\sin t)\sin\left(t+\frac{1}{2}\right)dt.\end{aligned}$$

On the other hand, it is easy to see that $\int_0^t \exp(\sin s)\cos s\,ds$ is bounded and $\int_0^{2\pi}\exp(\sin t)\sin t\,dt > 0$. Thus

$$\int_0^\infty p(t)\ln\left(e\int_t^{t+1} p(s)\,ds\right)dt = \infty.$$

By Corollary 1.1, every solution of (4.2) oscillates.

EXAMPLE 4.3. Consider the delay differential equation

$$x'(t) + p(t)f(x(\tau(t))) = 0, \qquad t \geq t_0 \tag{4.3}$$

where $p(t)$ and $\tau(t)$ satisfy (1.2) and

$$f(u) = e^u - 1. \tag{4.4}$$

By setting

$$y(t) = K\exp(x(t)), \tag{4.5}$$

where K is a constant, (4.3) is reduced to the nonautonomous delay-logistic equation

$$y'(t) = p(t)y(t)\left[1 - \frac{y(\tau(t))}{K}\right]. \tag{4.6}$$

In [9], Zhang and Gopalsamy showed that every solution of (4.6) oscillates about K if

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\,ds > e^{-1}. \tag{4.7}$$

It is easy to prove that (4.7) is equivalent to

$$\liminf_{t\to\infty} \int_{t}^{\delta^{-1}(t)} p(s)\,ds > e^{-1}. \tag{4.8}$$

Clearly, $y(t)$ oscillates about K if and only if $x(t)$ oscillates. As pointed out in [7], conditions (1.5) and (1.6), weaker than (4.7), also can guarantee that $y(t)$ is oscillatory about K. Obviously $f(u)$ given by (4.4) satisfies (1.3) and (1.4) and so also satisfies conditions (i)–(iv) in (H) as pointed out in Remark 1.1. Thus, the above-mentioned results may be improved by using Theorem 1.2 (or Corollary 1.2). To see this, we let

$$p(t) = \frac{1}{2e}(1+\cos t), \quad \tau(t) = t - \pi, \quad t \ge 0. \tag{4.9}$$

Then, for $t \ge \pi$,

$$\int_{t-\pi}^{t} \frac{1}{2e}(1+\cos s)\,ds = \frac{1}{2e}(\pi + 2\sin t),$$

$$\liminf_{t\to\infty} \int_{t-\pi}^{t} \frac{1}{2e}(1+\cos s)\,ds = \frac{1}{2e}(\pi - 2) < e^{-1}.$$

This shows that (1.5) and (4.7) do not hold. But from the example in [8] we know that

$$\begin{aligned}\liminf_{t\to\infty} p_4(t) &= \frac{1}{16e^4}\left[\pi^4 - 4\pi^2 - 2\sqrt{(\pi^3 - 6\pi)^2 + 4(\pi^2-4)^2}\,\right]\\ &> \frac{22}{16e^4} > \frac{1}{e^4}.\end{aligned}$$

Thus, by Corollary 1.2 every solution of (4.3) oscillates, and so every solution of (4.6) oscillates about K when $p(t)$ and $\tau(t)$ satisfy (4.9). On the other hand, one can easily see that condition (4.7) or (4.8) implies (1.13), (1.14), (1.17) and (1.18).

REFERENCES

[1] A. Elbert and I. P. Stavroulakis, *Oscillation and nonoscillation criteria for delay differential equations*, Proc. Amer. Math. Soc. 123 (1995), 1503–1510.
[2] L. H. Erbe, Q. K. Kong and B. G. Zhang, *Oscillation Theory for Functional Differential Equations*, Dekker, New York, 1995.
[3] I. Győri and G. Ladas, *Oscillation Theory of Delay Differential Equations with Applications*, Clarendon Press, Oxford, 1991.
[4] G. S. Ladde, V. Lakshmikantham and B. G. Zhang, *Oscillation Theory of Differential Equations with Deviating Arguments*, Dekker, New York, 1987.
[5] B. T. Li, *Oscillations of delay differential equations with variable coefficients*, J. Math. Anal. Appl. 192 (1995), 312–321.
[6] —, *Oscillation of first order delay differential equations*, Proc. Amer. Math. Soc. 124 (1996), 3729–3737.
[7] B. T. Li and Y. Kuang, *Sharp conditions for oscillations in some nonlinear nonautonomous delay differential equations*, Nonlinear Anal. 29 (1997), 1265–1276.
[8] X. H. Tang and J. H. Shen, *Oscillation of first order delay differential equations with variable coefficients*, J. Math. Anal. Appl. 217 (1998), 32–42.
[9] B. G. Zhang and K. Gopalsamy, *Oscillation and nonoscillation in a nonautonomous delay-logistic equation*, Quart. Appl. Math. 46 (1988), 267–273.

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