COLLOQUIUM MATHEMATICUM

VOL. 83 2000 NO. 1

ON THE MAXIMAL SPECTRUM OF COMMUTATIVE SEMIPRIMITIVE RINGS

BY

K. SAMEI (HAMADAN)

Abstract. The space of maximal ideals is studied on semiprimitive rings and reduced rings, and the relation between topological properties of Max(R) and algebric properties of the ring R are investigated. The socle of semiprimitive rings is characterized homologically, and it is shown that the socle is a direct sum of its localizations with respect to isolated maximal ideals. We observe that the Goldie dimension of a semiprimitive ring R is equal to the Suslin number of Max(R).

1. Introduction. Throughout this paper, R is a commutative ring with identity. We write Spec(R), Max(R) and Min(R) for the spaces of prime ideals, maximal ideals and minimal prime ideals of R, respectively. The topology of these spaces is the Zariski topology (see [2], [4], [5] and [7]). Also we denote by $\mathcal{P}_0(R)$, $\mathcal{M}_0(R)$ and $\mathcal{I}_0(R)$ the sets of isolated points of the spaces Spec(R), Max(R) and Min(R), respectively. We say R is *semiprimitive* if $\bigcap \mathrm{Max}(R) = (0)$. For a semiprimitive Gelfand ring R, we show that

$$\mathcal{P}_0(R) = \mathcal{M}_0(R) = \mathcal{I}_0(R) = \mathrm{Ass}(R).$$

A nonzero ideal in a commutative ring is said to be *essential* if it intersects every nonzero ideal nontrivially, and the intersection of all essential ideals, or the sum of all minimal ideals, is called the *socle* (see [9]). We characterize the socle of semiprimitive rings in two ways: in terms of maximal ideals and in terms of localizations with respect to maximal ideals. We denote the socle of R by $S(R)$ or S and the Jacobson radical of R by $J(R)$.

We know that the infinite intersection of essential ideals in any ring may not be an essential ideal. We shall show that in a semiprimitive ring, every intersection of essential ideals is an essential ideal if and only if $\mathcal{M}_0(R)$ is dense in Max(R).

A set $\{B_i\}_{i\in I}$ of nonzero ideals in R is said to be *independent* if $B_i \cap (\sum_{i\neq j\in I} B_j) = (0)$, i.e., $\sum_{i\in I} B_i = \bigoplus_{i\in I} B_i$. We say R has a *finite Goldie dimension* if every independent set of nonzero ideals is finite, and if R does not have a finite Goldie dimension, then the *Goldie dimension* of R, denoted

2000 *Mathematics Subject Classification*: 13A18, 13B24, 13C10, 13C11, 54C40.

by $\dim R$, is the smallest cardinal number c such that every independent set of nonzero ideals in R has cardinality less than or equal to c. Also the smallest cardinal c such that every family of pairwise disjoint nonempty open subsets of a topological space X has cardinality less than or equal to c is called the *Suslin number* or *cellularity* of the space X and is denoted by $\mathcal{S}(X)$ (see [3]). We show that for any semiprimitive ring R, the Suslin number of $\mathrm{Max}(R)$ is equal to the Goldie dimension of R.

2. The socle of semiprimitive rings

DEFINITION. Let $\mathcal{M}(a) = \{M \in \mathrm{Max}(R) : a \in M\}$ for all $a \in R$, and $\mathcal{M}(I) = \{\mathcal{M}(a) : a \in I\}$ for all ideals I of R. An ideal $M \in \mathrm{Max}(R)$ is called *trivial* if M is generated by an idempotent element, i.e., $M = (e)$, where $e^2 = e$.

LEMMA 2.1. *Suppose* $\mathrm{Nil}(R) = J(R)$ *and* M *is a maximal ideal of* R. *Then* $M = \sqrt{(e)}$, *where* e *is an idempotent element if and only if* M *is an isolated point of* $\mathrm{Max}(R)$. *Furthermore in this case*, *if* $M = (e)$ *and* $e \neq 0$, *then* $I = (1-e)$ *is a nonzero minimal ideal.*

Proof. Let $M = \sqrt{(e)}$, where $e^2 = e$. Then $\{M\} = \mathrm{Max}(R) - \mathcal{M}(1-e)$, i.e., M is an isolated point of $\mathrm{Max}(R)$. Conversely, suppose $\{M\}$ is open in $\mathrm{Max}(R)$. If $\mathrm{Max}(R) = \{M\}$, then $M = \sqrt{(0)}$. Otherwise, there exist $a, b, r \in R$ such that $a \in \bigcap_{M' \in \mathrm{Max}(R) - \{M\}} M' - M$, $b \in M$ and $ar + b = 1$. Obviously, $ab \in J$, hence $(ab)^n = 0$ for some $n > 0$. We have $1 = (ar+b)^{2n} = a^n x_1 + b^n x_2$. Let $e = b^n x_2$. Then $e(1-e) = 0$ and this means that e is an idempotent element of R. Also for every $m \in M$, there is $n > 0$ such that $[(1-e)m]^n = 0$, so $m^n \in (e)$, i.e., $M = \sqrt{(e)}$. ∎

The following proposition is proved in [8, 1.6].

LEMMA 2.2. *If* R *is a semiprimitive ring then* I *is a nonzero minimal ideal of* R *if and only if* I *is contained in every maximal ideal except one*, *i.e.*, $|\mathcal{M}(I)| = 2$.

In [7], it is proved that the socle of $C(X)$ (the ring of continuous functions) consists of all functions that vanish everywhere except at a finite number of points of X. We give a generalization of this fact.

THEOREM 2.3. *In a semiprimitive ring* R, *the socle* $S = S(R)$ *is exactly the set of all elements which belong to every maximal ideal except for a finite number. In fact*,

$$S = \{a \in R : \mathrm{Max}(R) - \mathcal{M}(a) \textit{ is finite}\}.$$

Proof. Suppose $a \in S$. If $a = 0$, then $\mathrm{Max}(R) - \mathcal{M}(a) = \emptyset$. Otherwise, $a = a_1 + \ldots + a_n$, where each a_i belongs to some idempotent minimal ideal in R. Thus by 2.2, $a_1 + \ldots + a_n$ belongs to every maximal ideal except for a finite number. It follows that $\mathrm{Max}(R) - \mathcal{M}(a)$ is finite.

Conversely, let $\mathrm{Max}(R) - \mathcal{M}(a)$ be a finite set. We have to show that $a \in S$. Let $\mathrm{Max}(R) - \mathcal{M}(a) = \{M_1, \ldots, M_n\}$. We claim that each M_k, $k = 1, \ldots, n$, is an isolated point of $\mathrm{Max}(R)$. Indeed, for every $i \neq k$ and $1 \leq i \leq n$ there exists $a_i \in M_i - M_k$. Set $b = aa_1 \ldots a_{k-1}a_{k+1} \ldots a_n$. Then $\mathcal{M}(b) = \mathrm{Max}(R) - \{M_k\}$, so $\{M_k\}$ is open, i.e., M_k is isolated. Now by 2.1, for each M_k, there exists a minimal ideal I_k such that $R = M_k \oplus I_k$ and $I_k = (e_k)$, where e_k is an idempotent element of R. Let $b = a - ae_1 - \ldots - ae_n$. Then for every $1 \leq k \leq n$, $e_k b \in J(R) = 0$, and consequently $\{M_1, \ldots, M_n\} \subseteq \mathcal{M}(b)$. On the other hand, $\mathcal{M}(a) \subseteq \mathcal{M}(b)$, hence $b = 0$, therefore $a = ae_1 + \ldots + ae_n \in I_1 + \ldots + I_n \subseteq S$. ■

Now we give a characterization of the socle of semiprimitive rings by localizations with respect to maximal ideals.

THEOREM 2.4. *Let R be a semiprimitive ring and let I be an ideal of R. Then $I \subseteq S$ if and only if the sequence*

$$0 \to I \xrightarrow{\phi} \bigoplus_{M \in \mathrm{Max}(R)} I_M \xrightarrow{\pi} \bigoplus_{M \in \mathrm{Max}(R) - \mathcal{M}_0(R)} I_M \to 0$$

is exact, where ϕ is the natural map and π is the projection map and I_M is the localization of I. Furthermore, the socle is the unique ideal with this property.

Proof. ($\Rightarrow$) Suppose $I \subseteq S$, and consider the natural map $\phi : I \to \bigoplus_{M \in \mathrm{Max}(R)} I_M$ such that $\forall a \in I$, $\phi(a) = (a/1)_{M \in \mathrm{Max}(R)}$. Now suppose $a \in I$. Then by 2.3, $\mathrm{Max}(R) - \mathcal{M}(a) = \{M_1, \ldots, M_n\}$. Hence for each $1 \leq k \leq n$, there exists e_k such that $M_k = (e_k)$. Put $b = e_1 \ldots e_n$. It is evident that $ab \in J = (0)$ and $b \in R - M$ for each $M \in \mathcal{M}(a)$, so $a/1 = 0$ in I_M. Hence ϕ is a well defined homomorphism. Also

$$\mathrm{Ker}\,\phi = \{a \in I : \forall M,\ \exists t \in R - M \text{ such that } ta = 0\} \subseteq J = (0),$$

thus ϕ is one-to-one. Now we show that $\mathrm{Im}\,\phi = \mathrm{Ker}\,\pi$. Suppose $(b/t)_{M \in \mathrm{Max}(R)} \in \mathrm{Im}\,\phi$. Then there is $a \in I$ such that $(a/1)_{M \in \mathrm{Max}(R)} = (b/t)_{M \in \mathrm{Max}(R)}$. Obviously, $a/1 = 0$ in I_M for every $M \in \mathrm{Max}(R) - \mathcal{M}_0(R)$. (Since $\mathrm{Max}(R) - \mathcal{M}(a) = \{M_1, \ldots, M_n\}$, for each $1 \leq k \leq n$ there is $t_k \in M_k - M$. Let $t = t_1, \ldots t_n$. Then $t \in R - M$ and $at \in J = (0)$.) Thus $\phi(a) \in \mathrm{Ker}\,\pi$ and consequently, $\mathrm{Im}\,\phi \subseteq \mathrm{Ker}\,\pi$. To prove $\mathrm{Im}\,\phi \supseteq \mathrm{Ker}\,\pi$, it is enough to show that if $0 \neq b/t \in I_M$, where $M \in \mathcal{M}_0(R)$, then there exists $a \in I$ such that the M-component of $\phi(a)$ is b/t and all the other components are zero. To see this, we note that $b, t \notin M$ and $M = (e)$ where e is an idempotent element of R. So there exists $t' \in R$ such that $tt' - 1 \in M$. Let $a = (1-e)t'b$ and $s = 1 - e$. We have $s(at - b) = sb(tt' - 1 - tt'e) \in J = (0)$. So $a/1 = b/t$

in M-components, and also $ea = 0$, hence $a/1 = 0$ for all other components. Thus the sequence is exact.

($\Leftarrow$) Let $a \in I$ and suppose the sequence

$$0 \to I \xrightarrow{\phi} \bigoplus_{M \in \mathrm{Max}(R)} I_M \xrightarrow{\pi} \bigoplus_{M \in \mathrm{Max}(R) - \mathcal{M}_0(R)} I_M \to 0$$

is exact. Since $\phi(a)$ is well defined, every component of $\phi(a)$ is zero except for a finite numbers of components $M_1, \ldots, M_n$. Clearly, $\mathrm{Max}(R) - \mathcal{M}(a) \subseteq \{M_1, \ldots, M_n\}$. Thus $a \in S$, i.e., $I \subseteq S$.

Finally, if S' is an ideal of R that satisfies the conditions of the theorem, then the exact sequences

$$0 \to S \xrightarrow{\phi} \bigoplus_{M \in \mathrm{Max}(R)} S_M \xrightarrow{\pi} \bigoplus_{M \in \mathrm{Max}(R) - \mathcal{M}_0(R)} S_M \to 0,$$

$$0 \to S' \xrightarrow{\phi} \bigoplus_{M \in \mathrm{Max}(R)} S'_M \xrightarrow{\pi} \bigoplus_{M \in \mathrm{Max}(R) - \mathcal{M}_0(R)} S'_M \to 0$$

yield $S \subseteq S'$ and $S' \subseteq S$, respectively. Consequently, $S = S'$. ∎

COROLLARY 2.5. *In a semiprimitive ring R, for every ideal $I \subseteq S$ we have $I \cong \bigoplus_{M \in \mathcal{M}_0(R)} I_M$. In particular,*

$$S \cong \bigoplus_{M \in \mathcal{M}_0(R)} S_M.$$

We note that minimal ideals in a semiprimitive ring R are projective, so every ideal contained in the socle of R is projective. Next we have the following result.

COROLLARY 2.6. *Let R be a semiprimitive ring and let $I \subseteq S$ be an ideal. Then for each R-module A and $n \geq 2$, we have*

$$\prod_{M \in \mathrm{Max}(R)} \mathrm{Ext}^n_R(I_M, A) \cong \prod_{M \in \mathrm{Max}(R) - \mathcal{M}_0(R)} \mathrm{Ext}^n_R(I_M, A).$$

Proof. The exact sequence

$$0 \to I \xrightarrow{\phi} \bigoplus_{M \in \mathrm{Max}(R)} I_M \xrightarrow{\pi} \bigoplus_{M \in \mathrm{Max}(R) - \mathcal{M}_0(R)} I_M \to 0$$

yields the exact sequence

$$0 = \mathrm{Ext}^{n-1}_R(I, A) \to \mathrm{Ext}^n_R\Big(\bigoplus_{M \in \mathrm{Max}(R) - \mathcal{M}_0(R)} I_M, A\Big) \to \mathrm{Ext}^n_R\Big(\bigoplus_{M \in \mathrm{Max}(R)} I_M, A\Big) \to \mathrm{Ext}^n_R(I, A) = 0.$$

So we have

$$\prod_{M\in \operatorname{Max}(R)} \operatorname{Ext}^n_R(I_M, A) \cong \operatorname{Ext}^n_R\Big(\bigoplus_{M\in \operatorname{Max}(R)-\mathcal{M}_0(R)} I_M, A\Big)$$
$$\cong \prod_{M\in \operatorname{Max}(R)-\mathcal{M}_0(R)} \operatorname{Ext}^n_R(I_M, A). \ \blacksquare$$

3. Essential ideals and space of maximal ideals. The following theorem characterizes essential ideals of a semiprimitive ring R via a topological property.

THEOREM 3.1. *If I is a nonzero ideal in a semiprimitive ring R, then the following are equivalent.*

(i) *I is an essential ideal in R.*
(ii) *$\bigcap \mathcal{M}(I)$ is a nowhere dense subset of* $\operatorname{Max}(R)$, *i.e.*, $\operatorname{int}\bigcap \mathcal{M}(I) = \emptyset$.

Proof. (i)⇒(ii). Suppose that the interior of $\bigcap \mathcal{M}(I)$ is nonempty; denote it by U. Let $M \in U$. Since $\operatorname{Max}(R) - U$ is closed, there exists $a \in \bigcap_{M'\in \operatorname{Max}(R)-U} M' - M$. Thus $ab = 0$ for every $b \in I$, i.e., $\operatorname{Ann}(I) \neq (0)$, a contradiction.

(ii)⇒(i). Let K be a nonzero ideal in R and $0 \neq b \in K$. Then $\operatorname{Max}(R) - \mathcal{M}(b)$ is an open set and clearly $(\operatorname{Max}(R)-\mathcal{M}(b))\cap(\operatorname{Max}(R)-\bigcap\mathcal{M}(I)) \neq \emptyset$. This implies that there is $a \in I$ such that $(\operatorname{Max}(R) - \mathcal{M}(b)) \cap (\operatorname{Max}(R) - \bigcap \mathcal{M}(a)) \neq \emptyset$, so $\mathcal{M}(ab) \neq \operatorname{Max}(R)$, i.e., $0 \neq ab \in K \cap I$. ■

It is easy to see that a finite intersection of essential ideals in any commutative ring is an essential ideal. But a countable intersection of essential ideals need not be an essential ideal. The following result gives a necessary and sufficient condition for essentiality of each intersection of essential ideals in semiprimitive rings.

THEOREM 3.3. *In a semiprimitive ring R, the following are equivalent.*

(i) *Every intersection of essential ideals of R is an essential ideal.*
(ii) $\bigcap_{M\in\mathcal{M}_0(R)} M = (0)$, *i.e.*, *$\mathcal{M}_0(R)$ is dense in* $\operatorname{Max}(R)$.

Proof. (i)⇒(ii). By hypothesis, $\operatorname{Ann}(S)=(0)$. Now if $a \in \bigcap_{M\in\mathcal{M}_0(R)} M$, then for every minimal ideal I of R, $aI = (0)$, so $a \in \operatorname{Ann}(S)$ and this implies $a = 0$.

(ii)⇒(i). Clearly every minimal ideal of R is generated by an idempotent, hence $S=\bigoplus_{e\in E}(e)$, where E is a set of idempotents in R. We note that (e) is minimal if and only if $(1-e)$ is a trivial maximal ideal, and $\operatorname{Ann}(e) = (1-e)$. But

$$\operatorname{Ann}(S) = \bigcap_{e\in E} \operatorname{Ann}(e) = \bigcap_{e\in E} (1-e) = \bigcap_{M\in\mathcal{M}_0(R)} M = 0.$$

This means that S is essential. ■

THEOREM 3.4. *In a semiprimitive ring R, the socle $S = S(R)$ is finitely generated if and only if the number of trivial maximal ideals is finite, i.e., $\mathcal{M}_0(R)$ is finite. In particular, if R is a noetherian ring then $\mathcal{M}_0(R)$ is finite.*

Proof. ($\Rightarrow$) Without loss of generality we can suppose $S = (a, b)$. Assume $\mathcal{M}_0(R) = \{M_i : i \in I\}$ is infinite. We know that for every $i \in I$, $M_i = (e'_i)$, where e'_i is an idempotent element of R. Set $e_i = 1 - e'_i$ and $T = \{e_i : i \in I\}$. Now we have $a = r_{i_1}e_{i_1} + \ldots + r_{i_k}e_{i_k}$ and $b = r_{j_1}e_{j_1} + \ldots + r_{j_s}e_{j_s}$ for some $r_i, r_j \in R$. On the other hand there exists $e \in T - \{e_{i_1}, \ldots, e_{i_k}, e_{j_1}, \ldots, e_{j_s}\}$, so $e = ra + r'b$, where $r, r' \in R$. Also $ee_i \in J = (0)$ for every $e_i \neq e$, so $e = e^2 = rae + rbe = 0$, a contradiction.

($\Leftarrow$) Trivial. ∎

The following theorem characterizes the Goldie dimension of semiprimitive rings via a topological property.

THEOREM 3.5. *In a semiprimitive ring R,* $\dim R = \mathcal{S}(\mathrm{Max}(R))$.

Proof. Let $\dim R = c$ and $\bigoplus_{i\in I} B_i$ be a direct sum of ideals in R, where $|I|$, the cardinality of I, is less than or equal to c. Now for each $i \in I$, let $0 \neq a_i \in B_i$; then $a_ia_j = 0$ when $i \neq j$. Hence $(\mathrm{Max}(R) - \mathcal{M}(a_i)) \cap (\mathrm{Max}(R) - \mathcal{M}(a_j)) = \emptyset$, and this implies that $F = \{\mathrm{Max}(R) - \mathcal{M}(a_i) : i \in I\}$ is a collection of disjoint open sets in $\mathrm{Max}(R)$, i.e., $\mathcal{S}(\mathrm{Max}(R)) \geq c$. Now let $\{G_i : i \in I\}$ be any collection of disjoint open sets in $\mathrm{Max}(R)$. Then for all $i \in I$, there exists $0 \neq a_i \in R$ such that $a_i \in \bigcap_{\mathrm{Max}(R) - G_i} M$. Now we put $B_i = (a_i)$ for all $i \in I$ and claim that $\{B_i\}_{i\in I}$ is an independent set of nonzero ideals in R. To see this, we show that $B_i \cap (\sum_{i\neq r\in I} B_r) = (0)$. Let $a \in B_i \cap (\sum_{i\neq r\in I} B_r)$. Then $a = a_ib = a_{r_1}b_1 + \ldots + a_{r_n}b_n$, where $b, b_k \in R$, $a_i \in B_i$ and $a_{r_k} \in B_{r_k}$ and $i \neq r_k$, for all $k = 1, \ldots, n$. But clearly $a_ia_{r_k} \in J = (0)$ for every $k = 1, \ldots, n$ and this implies that $a_i^2 b = 0$, i.e., $a^2 = 0$ and therefore $a = 0$. This means that $\dim R = c \geq |I|$, i.e., $c \geq \mathcal{S}(\mathrm{Max}(R))$. ∎

The following proposition gives a characterization of essential ideals in a reduced ring R (i.e., R has no nonzero nilpotent element) when $\mathrm{Ass}(R)$ is dense in $\mathrm{Spec}(R)$.

PROPOSITION 3.6. *Let R be a reduced ring, and let E be an ideal of R. Then the following are equivalent:*

(i) $\mathrm{Ass}(R)$ *is dense in* $\mathrm{Spec}(R)$.

(ii) *E is an essential ideal in R if and only if $E \not\subseteq P$ for every $P \in \mathrm{Ass}(R)$.*

Proof. (i)$\Rightarrow$(ii). Suppose E is an essential ideal of R and $P \in \mathrm{Ass}(R)$. Since P is not essential we have $E \not\subset P$. Conversely, suppose $E \not\subset P$ for every $P \in \mathrm{Ass}(R)$. If E is not essential then there is $0 \neq a \in R$ such that $aE = (0) = \bigcap_{P \in \mathrm{Ass}(R)} P$, so $a = 0$, a contradiction.

(ii)⇒(i) . For every $P \in \mathrm{Ass}(R)$, there exists $a_p \in \bigcap_{Q \in \mathrm{Min}(R)-\{P\}} Q - P$. Suppose E is the ideal generated by the a_p's, i.e., $E = \langle a_p : P \in \mathrm{Ass}(R)\rangle$. Observe that $E \not\subset P$ for any $P \in \mathrm{Ass}(R)$, hence E is essential. Now if $a \in \bigcap_{P \in \mathrm{Ass}(R)} P$, then $aa_p = 0$ for every $P \in \mathrm{Ass}(R)$, hence $aE = (0)$. Since E is essential, $a = 0$, therefore $\bigcap_{P \in \mathrm{Ass}(R)} P = (0)$. This yields that $\mathrm{Ass}(R)$ is dense in $\mathrm{Spec}(R)$. ■

The following proposition characterizes the isolated points of the spaces of maximal ideals and minimal prime ideals in a reduced ring R.

PROPOSITION 3.7. *Let R be a reduced ring.*

(i) *If $\mathcal{T} \subseteq \mathrm{Min}(R)$ is dense in $\mathrm{Min}(R)$, then $\mathrm{Ass}(R) \subseteq \mathcal{T}$.*

(ii) *$P \in \mathcal{P}_0(R)$ if and only if $P \in \mathcal{I}_0(R)$ and P is not the intersection of the prime ideals which contain it strictly.*

(iii) *$\mathcal{I}_0(R) = \mathrm{Ass}(R)$.*

In particular, if R is semiprimitive, we have

(iv) *$\mathcal{P}_0(R) = \mathcal{M}_0(R)$.*

Proof. (i) Suppose $P \in \mathrm{Ass}(R)$, hence $P = \mathrm{ann}(a)$ for some $a \in R$. Therefore $P = \bigcap_{Q \in \mathcal{T}-V(a)} Q$, where $V(a) = \{P \in \mathrm{Spec}(R) : a \in P\}$. This implies that $P = Q$ for some $Q \in \mathcal{T}$.

(ii) Suppose $P \in \mathcal{P}_0(R)$. Then clearly $P \in \mathcal{I}_0(R)$. Now if $P = \bigcap_{Q \in V(P)-\{P\}} Q$, where $V(P) = \{Q \in \mathrm{Spec}(R) : P \subseteq Q\}$, then we have $\bigcap_{Q \in \mathrm{Spec}(R)-\{P\}} Q \subseteq P$, i.e., $P \notin \mathcal{P}_0(R)$, a contradiction. Conversely, suppose that $P \in \mathcal{I}_0(R)$ and $P \neq \bigcap_{Q \in V(P)-\{P\}} Q$. Then there exist $a \in \bigcap_{Q \in \mathrm{Min}(R)-\{P\}} Q - P$ and $b \in \bigcap_{Q \in V(P)-\{P\}} Q - P$, thus we have $ab \in \bigcap_{Q \in \mathrm{Spec}(R)-\{P\}} Q - P$, i.e., $P \in \mathcal{P}_0(R)$.

(iii) Assume that $P \in \mathcal{I}_0(R)$. Then there exists $a \in \bigcap_{Q \in \mathrm{Min}(R)-\{P\}} Q - P$, hence $P = \mathrm{ann}(a) \in \mathrm{Ass}(R)$. Conversely, let $P \in \mathrm{Ass}(R)$ so $P = \mathrm{ann}(a)$ for some $a \in R$. Suppose $P \notin \mathcal{I}_0(R)$; put $\mathcal{T} = \mathrm{Min}(R) - \{P\}$. Since $\bigcap_{Q \in \mathcal{T}} Q = (0)$, it follows that $\mathcal{T}$ is dense in $\mathrm{Min}(R)$ and (i) implies that $\mathrm{Ass}(R) \subseteq \mathcal{T}$; consequently, $P \in \mathcal{T}$, a contradiction.

(iv) Suppose $M \in \mathcal{M}_0(R)$. Then $M = (e)$, where e is an idempotent element of R. Hence for any $M \neq P \in \mathrm{Spec}(R)$, $1-e \in P$. This means that $\bigcap_{P \in \mathrm{Spec}(R)-\{M\}} P \not\subset M$, i.e., $M \in \mathcal{P}_0(R)$, and therefore $\mathcal{M}_0(R) \subseteq \mathcal{P}_0(R)$. The opposite inclusion is trivial. ■

4. Gelfand rings. A ring is called a *Gelfand ring* (or a *pm ring*) if each prime ideal is contained in a unique maximal ideal. For a commutative ring R, De Marco and Orsatti [2] show: R is Gelfand if and only if $\mathrm{Max}(R)$ is

Hausdorff, and if and only if Spec(R) is normal. For each $M \in \mathrm{Max}(R)$, let $O_M = \bigcap_{P \subseteq M} P$, where P ranges over all prime ideals contained in M. One can easily see that in a semiprimitive Gelfand ring R, $O_M = \{a \in R : M \in \operatorname{int} \mathcal{M}(a)\}$ and for any $P \in \mathrm{Spec}(R)$, $P \subseteq M$ if and only if $O_M \subseteq P$ (int is the interior in the space Max(R)).

PROPOSITION 4.1. *If R is a semiprimitive Gelfand ring, then*

$$\mathcal{P}_0(R) = \mathcal{M}_0(R) = \mathcal{I}_0(R) = \mathrm{Ass}(R).$$

Proof. By 3.7 it is sufficient to prove $\mathcal{M}_0(R) = \mathcal{I}_0(R)$. Let $P \in \mathcal{I}_0(R)$. Then $P \subseteq M'$ for a unique maximal ideal $M' \in \mathrm{Max}(R)$, therefore $\bigcap_{M \in \mathrm{Max}(R) - \{M'\}} O_M \not\subseteq P$. This means that $\bigcap_{M \in \mathrm{Max}(R) - \{M'\}} O_M \neq (0)$, hence there exists $0 \neq e \in \bigcap_{M \in \mathrm{Max}(R) - \{M'\}} M$. Observe that $e \notin M'$, thus M' is an isolated point of Max(R), and consequently $P = M' \in \mathcal{M}_0(R)$. ■

COROLLARY 4.2. *In a semiprimitive Gelfand ring R every prime ideal is either an essential ideal or an isolated maximal ideal. In particular*,

$$\mathrm{Ass}(R) = \{M \in \mathrm{Max}(R) : M = (e),\ \textit{where } e \textit{ is an idempotent}\}.$$

Proof. Evident by 2.1 and 4.1. ■

The following result shows that in a semiprimitive Gelfand ring, the set of uniform ideals and the set of minimal ideals coincide.

PROPOSITION 4.3. *Let R be a semiprimitive Gelfand ring and I be an ideal in R. Then the following are equivalent.*

(i) *I is a uniform ideal.*
(ii) *For any two nonzero elements $a, b \in I$, $ab \neq 0$.*
(iii) *I is a minimal ideal.*

Proof. (i)⇒(ii). Since $(a) \cap (b) \neq 0$, there exist $c_1, c_2 \in R$ such that $ac_1 = bc_2 \neq 0$. This shows that $abc_1c_2 \neq 0$ and therefore $ab \neq 0$.

(ii)⇒(iii). By 2.2, it is sufficient to show that there is a fixed isolated point $M \in \mathcal{M}_0(R)$ such that $\mathrm{Max}(R) - \{M\} \subseteq \mathcal{M}(a)$ for all $a \in I$. Now let $0 \neq a \in I$, and let M' and M'' be two distinct elements in $\mathrm{Max}(R) - \mathcal{M}(a)$ and G, H be two disjoint open sets containing M', M'' respectively. Then there are $b_1 \in \bigcap_{M \in \mathrm{Max}(R) - G} M - M'$ and $b_2 \in \bigcap_{M \in \mathrm{Max}(R) - H} M - M''$. Clearly ab_1 and ab_2 are nonzero elements of R and $ab_1ab_2 \in \bigcap_{M \in \mathrm{Max}(R)} M = 0$, a contradiction. Next suppose that for distinct nonzero elements $a_1, a_2 \in I$ there are distinct elements $M_1, M_2 \in \mathrm{Max}(R)$ such that $\mathrm{Max}(R) - \{M_1\} \subseteq \mathcal{M}(a_1)$ and $\mathrm{Max}(R) - \{M_2\} \subseteq \mathcal{M}(a_2)$. Then we have $a_1a_2 = 0$, which contradicts (ii).

(iii)⇒(i). Trivial. ■

REFERENCES

[1] M. Contessa, *On PM-rings*, Comm. Algebra 10 (1982), 93–108.

[2] G. De Marco and A. Orsatti, *Commutative rings in which every prime ideal is contained in a unique maximal ideal*, Proc. Amer. Math. Soc. 30 (1971), 459–466.

[3] R. Engelking, *General Topology*, PWN–Polish Sci. Publ., 1977.

[4] L. Gillman, *Rings with Hausdorff structure space*, Fund. Math. 45 (1957), 1–16.

[5] M. Henriksen and M. Jerison, *The space of minimal prime ideals of a commutative ring*, Trans. Amer. Math. Soc. 115 (1965), 110–130.

[6] O. A. S. Karamzadeh and M. Rostami, *On the intrinsic topology and some related ideals of* $C(X)$, Proc. Amer. Math. Soc. 93 (1985), 179–184.

[7] C. W. Kohls, *The space of prime ideals of a ring*, Fund. Math. 45 (1957), 17–27.

[8] G. Mason, *z-ideals and prime ideals*, J. Algebra 26 (1973), 280–297.

[9] J. C. McConnell and J. C. Robson, *Noncommutative Noetherian Rings*, Wiley Interscience, New York, 1987.

[10] S. H. Sun, *Noncommutative rings in which every prime ideal is contained in a unique maximal ideal*, J. Pure Appl. Algebra 76 (1991), 179–192.

[11] —, *Rings in which every prime ideal is contained in a unique maximal ideal*, ibid. 78 (1992), 183–194.

Department of Mathematics
University of Bu-Ali Sina
Hamadan, Iran
E-mail: samei@sadaf.basu.ac.ir

Received 15 January 1999;
revised 19 July 1999 (3692)