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A GENERAL DIFFERENTIATION THEOREM FOR SUPERADDITIVE PROCESSES

BY

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To the memory of my parents Hidetsugu and Shin Sato

Abstract. Let L be a Banach lattice of real-valued measurable functions on a σ-finite measure space and $T = \{T_t : t > 0\}$ be a strongly continuous semigroup of positive linear operators on the Banach lattice L. Under some suitable norm conditions on L we prove a general differentiation theorem for superadditive processes in L with respect to the semigroup T.

Introduction. In this paper a differentiation theorem is proved for superadditive processes in a Banach lattice of functions having an absolutely continuous norm.

Let (Ω, Σ, μ) be a σ-finite measure space and L a vector lattice of real-valued measurable functions on (Ω, Σ, μ) under pointwise operations. Thus we understand that if $f \in L$ then the function $f^+(\omega) = \max\{f(\omega), 0\}$ is also in L, and two functions f and g in L are not distinguished provided that $f(\omega) = g(\omega)$ for almost all $\omega \in \Omega$. We let $|f|(\omega) = \max\{f(\omega), -f(\omega)\}$. Hereafter all statements and relations are assumed to hold modulo sets of measure zero. We further assume that L becomes a Banach space under the norm $\|\cdot\|$, and suppose the following properties:

(I) If $f, g \in L$ and $|f|(\omega) \le |g|(\omega)$ a.e. on Ω then $\|f\| \le \|g\|$.

(II) If g is a real-valued measurable function on Ω such that $|g|(\omega) \le |f|(\omega)$ a.e. on Ω for some $f \in L$ then $g \in L$.

(III) If $E_n \in \Sigma$, $E_n \supset E_{n+1}$ for each $n \ge 1$ and $\bigcap_{n=1}^{\infty} E_n = \emptyset$ then for any $f \in L$ we have

$$\lim_{n\to\infty} \|f \cdot \chi_{E_n}\| = 0,$$

where χ_{E_n} denotes the characteristic function of E_n.

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An operator $S : L \to L$ is called *positive* if $Sf(\omega) \geq 0$ a.e. on Ω for all $f \in L^+ = \{f \in L : f(\omega) \geq 0$ a.e. on $\Omega\}$. Let $T = \{T_t\} = \{T_t\}_{t>0}$ be a strongly continuous semigroup of positive linear operators on L; thus $T_{t+s} = T_t T_s$ and $\lim_{t\to s} \|T_t f - T_s f\| = 0$ for all $t, s > 0$ and $f \in L$. T is called *locally strongly integrable* if for each $f \in L$ the vector-valued function $t \mapsto T_t f$ is Bochner integrable on every finite interval with respect to the Lebesgue measure. By a *process* in L we mean a family $F = \{F_t\} = \{F_t\}_{t>0}$ of functions in L. A process F is called *positive* if $F_t \in L^+$ for all $t > 0$, *increasing* if $F_t(\omega) \leq F_s(\omega)$ a.e. on Ω for all $s > t > 0$, *linearly bounded* if

$$\sup\{\|F_t\|/t : 0 < t < s\} < \infty$$

for some $s > 0$, and *superadditive* [resp. *additive*] (with respect to $T = \{T_t\}$) if

$$F_{t+s}(\omega) \geq F_t(\omega) + T_t F_s(\omega) \text{ a.e.} \quad [\text{resp. } F_{t+s}(\omega) = F_t(\omega) + T_t F_s(\omega) \text{ a.e.}]$$

on Ω for all $t, s > 0$.

By an easy computation, if $T = \{T_t\}$ is locally strongly integrable and if $F = \{F_t\}$ is additive (with respect to T) and such that the vector-valued function $t \mapsto F_t$ is Bochner integrable on the unit interval $(0, 1)$ with respect to the Lebesgue measure, then we observe that

$$F_t = (I - T_t) \int_0^1 F_s \, ds + \int_0^t T_s F_1 \, ds$$

for all $t > 0$. We note that the Bochner integrability of the function $t \mapsto F_t$ on the interval $(0, 1)$ follows from property (III) if F is positive.

If $A \in \Sigma$, then we let $L(A) = \{f \in L : f(\omega) = 0$ a.e. on $\Omega \setminus A\}$ and $L^+(A) = L(A) \cap L^+$. It is easily seen (see e.g. [10]) that Ω decomposes under a positive semigroup $T = \{T_t\}$ into two sets P and N in Σ with the following properties:

(i) if $f \in L(N)$ then $\|T_t f\| = 0$ for all $t > 0$,
(ii) if $0 \neq f \in L^+(P)$ then $\|T_t f\| > 0$ for some $t > 0$.

Since $T = \{T_t\}$ is zero on $L(N)$, it may be readily seen that there are many positive superadditive processes in $L(N)$ for which the limit $\text{q-lim}_{t\to 0} \frac{1}{t} F_t(\omega)$ fails to exist a.e. on N, where $\text{q-lim}_{t\to 0}$ means that the limit is taken as t approaches zero through a countable dense subset in the interval $(0, 1)$. However, the situation is different on P, and we shall prove the following

THEOREM. *Let $T = \{T_t\}$ be a strongly continuous semigroup of positive linear operators on L. If $F = \{F_t\}$ is a superadditive process in L (with respect to T) and satisfies*

$$\sup\{\|F_t^-\|/t : 0 < t < s\} < \infty$$

for some $s > 0$, *where* $F_t^-(\omega) = \max\{-F_t(\omega), 0\}$, *then the limit*

$$\operatorname{q-lim}_{t\to 0} \frac{1}{t} F_t(\omega)$$

exists and is finite a.e. on P.

Various special cases of this theorem have already been proved; in particular, Wiener [12] has proved his local ergodic theorem for measure preserving flows, and recently many authors have studied differentiation theorems in the setting of strongly continuous semigroups $T = \{T_t\}_{t>0}$ of positive linear operators on L_p with $1 \le p < \infty$ (cf. e.g. [2]–[7], [9]). For this subject we refer the reader to Krengel's book [8] (see especially Chapter 7). The present theorem generalizes a differentiation theorem of [6], where superadditive processes have been considered in L_p-spaces and semigroups $T = \{T_t\}$ have been assumed to be locally strongly integrable. But, besides L_p-spaces, there are many interesting function spaces which satisfy properties (I)–(III). Examples are Lorentz spaces and Orlicz spaces, etc. The purpose of this paper is to generalize the differentiation theorem to such function spaces.

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Preliminaries. In this section we provide some necessary lemmas and propositions. For the sake of completeness we give proofs, although some are standard. L will denote the Banach lattice of functions mentioned in the Introduction.

LEMMA 1. *If* $f_n \in L$ *for* $n \ge 1$ *and* $\sum_{n=1}^\infty \|f_n\| < \infty$ *then* $\sum_{n=1}^\infty |f_n(\omega)| < \infty$ *a.e. on* Ω.

Proof. Let $g_n(\omega) = |f_n|(\omega)$. Then $g_n \in L^+$ and $\|g_n\| = \|f_n\|$ by property (I). Since $\sum_{n=1}^\infty \|g_n\| < \infty$, there is an $s \in L$ such that

$$\lim_{n\to\infty} \Big\| s - \sum_{i=1}^n g_i \Big\| = 0.$$

Then the functions $h_n(\omega) = \sum_{i=1}^n g_i(\omega)$ satisfy $0 \le h_n(\omega) \le h_{n+1}(\omega)$ a.e. on Ω and $\lim_{n\to\infty} \|s - h_n\| = 0$, so that we must have $\lim_{n\to\infty} h_n(\omega) = s(\omega)$ a.e. on Ω. This completes the proof.

LEMMA 2. *Let* $f \in L$ *and* $f_n \in L$ *for* $n \ge 1$. *If* $\lim_{n\to\infty} \|f - f_n\| = 0$, *then there exists a subsequence* (n') *of* (n) *such that* $\lim_{n'\to\infty} f_{n'}(\omega) = f(\omega)$ *a.e. on* Ω.

Proof. Obvious from Lemma 1.

LEMMA 3. *Let $f \in L^+$ and $f_n \in L^+$ for $n \geq 1$. If $f(\omega) \geq f_n(\omega) \geq f_{n+1}(\omega)$ a.e. on Ω for each $n \geq 1$, and $\lim_{n\to\infty} f_n(\omega) = 0$ a.e. on Ω then $\lim_{n\to\infty} \|f_n\| = 0$.*

Proof. Let $\varepsilon > 0$ be an arbitralily fixed number. Suppose $f \neq 0$, and write

$$A_n = \{\omega : f(\omega) \geq 1/n\} \quad \text{and} \quad B_n = \{\omega : 0 < f(\omega) < 1/n\} \quad \text{for } n \geq 1.$$

Since $f(\omega) \geq \frac{1}{n}\chi_{A_n}(\omega) \geq 0$ on Ω, it follows from property (II) that $\chi_{A_n} \in L$. Since $A_n \uparrow \{\omega : f(\omega) > 0\}$ as $n \to \infty$, there exists an $M \geq 1$ such that $\mu(A_M) > 0$. Then $\|\chi_{A_M}\| > 0$, and so we can put

$$\alpha = \frac{\varepsilon}{4\|\chi_{A_M}\|}.$$

Since $B_n \downarrow \emptyset$ as $n \to \infty$, it follows from property (III) that

$$\lim_{n\to\infty} \|f \cdot \chi_{B_n}\| = 0.$$

Therefore we may suppose without loss of generality that the above M is such that

$$\|f \cdot \chi_{B_n}\| \leq \|f \cdot \chi_{B_M}\| < \varepsilon/2 \quad \text{for all } n \geq M.$$

Then, since $f_n = f_n \cdot \chi_{A_M} + f_n \cdot \chi_{B_M}$, we have

$$\begin{aligned}\|f_n\| &\leq \|f_n \cdot \chi_{A_M}\| + \|f_n \cdot \chi_{B_M}\| \\ &\leq \|f_n \cdot \chi_{A_M}\| + \|f \cdot \chi_{B_M}\| < \|f_n \cdot \chi_{A_M}\| + \varepsilon/2,\end{aligned}$$

and

$$\begin{aligned}\|f_n \cdot \chi_{A_M}\| &\leq \|f_n \cdot \chi_{A_M \cap \{\omega : f_n(\omega) > \alpha\}}\| + \|f_n \cdot \chi_{A_M \setminus \{\omega : f_n(\omega) > \alpha\}}\| \\ &\leq \|f \cdot \chi_{A_M \cap \{\omega : f_n(\omega) > \alpha\}}\| + \alpha\|\chi_{A_M}\| \\ &\leq \|f \cdot \chi_{A_M \cap \{\omega : f_n(\omega) > \alpha\}}\| + \varepsilon/4.\end{aligned}$$

Since $A_M \cap \{\omega : f_n(\omega) > \alpha\} \downarrow \emptyset$ as $n \to \infty$, property (III) implies

$$\lim_{n\to\infty} \|f \cdot \chi_{A_M \cap \{\omega : f_n(\omega) > \alpha\}}\| = 0;$$

consequently, we can find an $n_0 \geq 1$ such that if $n \geq n_0$ then

$$\|f_n\| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon.$$

This completes the proof.

LEMMA 4. *Let $S : L \to L$ be a positive linear operator. Then $|Sf|(\omega) \leq S|f|(\omega)$ a.e. on Ω for any $f \in L$, and $\|S\| < \infty$.*

Proof. Since $-|f|(\omega) \leq f(\omega) \leq |f|(\omega)$ on Ω, the positivity of S implies that $-S|f|(\omega) \leq Sf(\omega) \leq S|f|(\omega)$ a.e. on Ω. Next, to prove $\|S\| < \infty$, suppose the contrary: $\|S\| = \infty$. Then for each $n \ge 1$ there exists $f_n \in L^+$ such that $\|f_n\| = 1$ and $\|Sf_n\| > n^3$. Then the function

$$f = \sum_{n=1}^{\infty} n^{-2} f_n \quad (\in L^+)$$

satisfies $Sf(\omega) \ge \sum_{i=1}^{n} i^{-2} Sf_i(\omega) \ge n^{-2} Sf_n(\omega) \ge 0$ a.e. on Ω, so that we apply property (I) to infer that $\|Sf\| \ge n^{-2}\|Sf_n\| > n$ for each $n \ge 1$. But this is a contradiction, since $Sf \in L$.

LEMMA 5. *There exists a measurable function w on Ω, with $w(\omega) > 0$ a.e. on Ω, such that*

$$\int_{\Omega} fw \, d\mu < \infty \quad \textit{for all } f \in L^+.$$

Proof. By an easy argument, it suffices to consider the case where there exists an increasing sequence (f_n) of functions in L^+ such that $f_n(\omega) \uparrow \infty$ a.e. on Ω as $n \to \infty$. Let $A_n = \{\omega : f_n(\omega) \ge 1\}$. Then we have

$$\chi_{A_n} \in L \quad \text{by property (II)} \quad \text{and} \quad \Omega = \bigcup_{n=1}^{\infty} A_n.$$

First, fix an $n \ge 1$. If $f \in L(A_n)$ and $f \ne 0$, take a continuous linear functional φ on L by the Hahn–Banach theorem such that $\varphi(f) \ne 0$. Define

$$\nu(E) = \varphi(\chi_E) \quad \text{for } E \in \Sigma(A_n),$$

where $\Sigma(A_n) = \{E \in \Sigma : E \subset A_n\}$. (We note that if $E \in \Sigma(A_n)$ then $\chi_E \in L$ by property (II).) If $E_i \in \Sigma(A_n), E_i \supset E_{i+1}$ for each $i \ge 1$ and $\bigcap_{i=1}^{\infty} E_i = \emptyset$, then $\lim_{i\to\infty} \|\chi_{E_i}\| = 0$ by property (III). It follows that ν is a signed (countably additive) measure on $(A_n, \Sigma(A_n))$. Since ν is absolutely continuous with respect to μ, we then apply the Radon–Nikodym theorem to infer that there exists a real-valued measurable function h on Ω, with $\{\omega : h(\omega) \ne 0\} \subset A_n$, such that for all $E \in \Sigma(A_n)$,

$$\varphi(\chi_E) = \nu(E) = \int_E h \, d\mu.$$

If $g \in L$ then by property (II) there exists a sequence (g_i) of simple functions in L such that $|g_i(\omega)| \le |g(\omega)|$ and $|g(\omega) - g_i(\omega)| \downarrow 0$ a.e. on Ω as $i \to \infty$. Hence $\lim_{i\to\infty} \|g - g_i\| = 0$ by Lemma 3, and we have

$$\varphi(g \cdot \chi_{A_n}) = \lim_{i\to\infty} \int_{A_n} g_i h \, d\mu.$$

Further, using Fatou's lemma, we see that $\int_{A_n} |gh| \, d\mu < \infty$, and thus

$$\varphi(g \cdot \chi_{A_n}) = \int_{A_n} gh \, d\mu = \int gh \, d\mu.$$

Since $\varphi(f)=\int_{A_n} fh\,d\mu \ne 0$, it follows that $h \not\equiv 0$ on A_n. By this fact and the σ-finiteness of μ it is standard from an exhaustion argument (cf. e.g. p. 17 of [8]) to see that there exists a sequence (h_n) of real-valued measurable functions on Ω such that

(i) for all $g \in L$ and $n \ge 1$, $\int_\Omega |gh_n|\,d\mu < \infty$,

(ii) the linear functionals φ_n on L defined by $\varphi_n(g) = \int_\Omega gh_n\,d\mu$ for $g \in L$ are nonzero and continuous,

(iii) $\Omega = \bigcup_{n=1}^\infty \{\omega : |h_n(\omega)| > 0\}$.

Since the positive linear functionals η_n on L defined by $\eta_n(g) = \int_\Omega g|h_n|\,d\mu$ for $g \in L$ satisfy $\|\eta_n\| = \|\varphi_n\|$ by property (I), the bounded linear functional

$$\eta = \sum_{n=1}^\infty \frac{\eta_n}{2^n\|\eta_n\|}$$

on L has the representation

$$\eta(g) = \int_\Omega g(\omega)\Big(\sum_{n=1}^\infty \frac{|h_n(\omega)|}{2^n\|\varphi_n\|}\Big)d\mu \quad \text{for } g \in L.$$

It follows that the function

$$w(\omega) = \sum_{n=1}^\infty \frac{|h_n(\omega)|}{2^n\|\varphi_n\|} \quad \text{for } \omega \in \Omega$$

satisfies the desired properties of the lemma, and the proof is complete.

LEMMA 6. *Let $S : L \to L$ be a positive linear operator and w be a nonnegative measurable function on Ω such that $\int_\Omega |f|w\,d\mu < \infty$ for all $f \in L$. Then there exists a nonnegative measurable function v on Ω, written as $v = S^*w$, such that*

$$\int_\Omega (Sf)w\,d\mu = \int_\Omega fv\,d\mu \quad \textit{for all } f \in L.$$

Proof. As in Lemma 5, we may assume that there exists a sequence (A_n) of sets in Σ such that

(i) $\chi_{A_n} \in L$ for each $n \ge 1$,

(ii) $A_n \cap A_m = \emptyset$ for $n \ne m$,

(iii) $\Omega = \bigcup_{n=1}^\infty A_n$.

By Lemma 4, S is bounded. Similarly we observe that the positive linear functional φ on L defined by $\varphi(f) = \int_\Omega (Sf)w\,d\mu$ is bounded. It follows from the proof of Lemma 5 that for each $n \ge 1$ there exists a nonnegative measurable function v_n on Ω, with $\{\omega : v_n(\omega) \ne 0\} \subset A_n$, such that

$$\int_\Omega (Sf)w\,d\mu = \int_\Omega fv_n\,d\mu \quad \text{for all } f \in L(A_n).$$

Letting $v(\omega) = v_n(\omega)$ for $\omega \in A_n$, we have a nonnegative measurable function v on Ω, and letting $B_n = \bigcup_{i=1}^n A_i$, we see from property (III) that for any $f \in L$,

$$\int_{\Omega} (Sf)w\,d\mu = \lim_{n\to\infty} \int_{\Omega} [S(f\cdot\chi_{B_n})]w\,d\mu = \lim_{n\to\infty} \int_{B_n} fv\,d\mu = \int_{\Omega} fv\,d\mu,$$

which completes the proof.

LEMMA 7. *Let $T = \{T_t\}_{t>0}$ be a strongly continuous semigroup of positive linear operators on L. Then Ω decomposes under T into two sets C and D in Σ with the properties that*

(i) *for some $h \in L^+$, $C = \bigcup_{n=1}^\infty \{\omega : T_{1/n}h(\omega) > 0\}$,*
(ii) *for any $t > 0$ and $f \in L, T_t f(\omega) = 0$ a.e. on D.*

Proof. As is easily seen, it suffices to consider the case where there exists an $h \in L^+$ with $h(\omega) > 0$ on Ω. If g is another function in L^+ with $g(\omega) > 0$ on Ω then, letting $g_k(\omega) = \min\{kg(\omega), h(\omega)\}$, we see that $g_k \in L^+$ by property (II) and $0 \le g_k(\omega) \uparrow h(\omega)$ as $k \to \infty$ for each $\omega \in \Omega$. Since $h - g_k \in L^+$, it follows from Lemma 3 that $\lim_{k\to\infty} \|h - g_k\| = 0$, and so for any $n \ge 1$ we have $\lim_{k\to\infty} \|T_{1/n}h - T_{1/n}g_k\| = 0$. Thus by Lemma 2 it follows that

$$\{\omega : T_{1/n}h(\omega) > 0\} \subset \bigcup_{k=1}^{\infty} \{\omega : T_{1/n}g_k(\omega) > 0\},$$

which together with the fact that $T_{1/n}g_k \le kT_{1/n}g$ on Ω implies $\{\omega : T_{1/n}h(\omega) > 0\} \subset \{\omega : T_{1/n}g(\omega) > 0\}$; consequently,

$$\bigcup_{n=1}^{\infty} \{\omega : T_{1/n}h(\omega) > 0\} \subset \bigcup_{n=1}^{\infty} \{\omega : T_{1/n}g(\omega) > 0\}.$$

Since the argument is symmetric, the reverse inclusion also holds, and thus we get

$$\bigcup_{n=1}^{\infty} \{\omega : T_{1/n}h(\omega) > 0\} = \bigcup_{n=1}^{\infty} \{\omega : T_{1/n}g(\omega) > 0\},$$

from which it follows immediately that the sets $C = \bigcup_{n=1}^\infty \{\omega : T_{1/n}h(\omega) > 0\}$ and $D = \Omega \setminus C$ satisfy the desired properties, completing the proof.

We note that the two decompositions $\Omega = P + N$ and $\Omega = C + D$ have no relation in general (cf. e.g. [11] and § 7.1 of [8]). But under some conditions on the semigroup $T = \{T_t\}$ and the norm $\|\cdot\|$ of L we have $C \subset P$.

PROPOSITION 1. *Let $T = \{T_t\}$ be as in Lemma 7. If the strong limit $T_0 =$ strong-$\lim_{t\to 0} T_t$ exists, then $C = \{\omega : T_0h(\omega) > 0\}$ for some $h \in L^+$. In particular, if $\|T_0\| \le 1$ and the norm $\|\cdot\|$ of L is such that $0 \le f(\omega) \le g(\omega)$ a.e. on Ω and $\|f\| = \|g\|$ imply $f = g$, then $C \subset P$.*

Proof. It suffices to consider the case where there exists an $h \in L^+$ with $h(\omega) > 0$ on Ω. Since $T_t = T_t T_0 = T_0 T_t$ for $t \geq 0$, it follows from the proof of Lemma 7 that

$$\{\omega : T_{1/n}h(\omega) > 0\} = \{\omega : T_0 T_{1/n}h(\omega) > 0\} \subset \{\omega : T_0 h(\omega) > 0\}.$$

Thus $C \subset \{\omega : T_0 h(\omega) > 0\}$. On the other hand, as $\lim_{n\to\infty} \|T_0 h - T_{1/n}h\| = 0$, it also follows from Lemma 2 that

$$\{\omega : T_0 h(\omega) > 0\} \subset \bigcup_{n=1}^{\infty} \{\omega : T_{1/n}h(\omega) > 0\} = C.$$

To prove the remainder of the proposition, let $\|T_0\| \leq 1$. It is sufficient to prove that if $g \in L^+(C)$ and $g \neq 0$ then $\|T_t g\| > 0$ for some $t > 0$. To do so, suppose the contrary: $T_t g = 0$ for all $t > 0$. Then $T_0 g = 0$, and thus the function $\widetilde{g}(\omega) = \min\{g(\omega), T_0 h(\omega)\}$ satisfies $T_0 \widetilde{g} = 0$. It follows that

$$0 \leq T_0 h(\omega) - \widetilde{g}(\omega) \leq T_0 h(\omega) \quad \text{a.e.}$$

on Ω and

$$T_0(T_0 h - \widetilde{g}) = T_0 h - T_0 \widetilde{g} = T_0 h.$$

Since $\|T_0\| \leq 1$, we must have $\|T_0 h - \widetilde{g}\| = \|T_0 h\|$. Hence by the hypothesis on the norm $\|\cdot\|$ of L, we get $T_0 h - \widetilde{g} = T_0 h$ and therefore $\widetilde{g} = 0$. But this is a contradiction, since $T_0 h(\omega) > 0$ on C. The proof is complete.

PROPOSITION 2. *Let $T = \{T_t\}$ be as in Lemma 7. If $\|T_t\| \leq 1$ for all $t > 0$, and the norm $\|\cdot\|$ of L is such that $0 \leq f(\omega) \leq g(\omega)$ a.e. on Ω and $\|f\| = \|g\|$ imply $f = g$, then $C \subset P$ and $T_t L(P) \subset L(P)$ for all $t > 0$.*

Proof. As before, let h denote a function in L^+ with $h(\omega) > 0$ on Ω. Then the function $g = \int_0^1 T_t h \, dt \in L^+$ satisfies

$$\|T_t g - g\| \leq 2t\|h\| \to 0 \quad \text{as } t \to 0.$$

Since $T_t g = T_t(g \cdot \chi_P)$, it follows that $\lim_{t\to 0} \|T_t(g \cdot \chi_P) - g\| = 0$. Since $\|T_t\| \leq 1$ for all $t > 0$ by hypothesis, we have $\|g \cdot \chi_P\| = \|g\|$ and hence $g \cdot \chi_P = g$ as in the proof of Proposition 1. That is,

$$\{\omega : g(\omega) > 0\} \subset P.$$

On the other hand, by the strong continuity of $T = \{T_t\}$,

$$\lim_{t\to 0} \left\| T_{1/n}h - \frac{1}{t} \int_0^t T_u T_{1/n} h \, du \right\| = 0 \quad \text{for } n \geq 1.$$

Since $\int_0^t T_u T_{1/n} h \, du \leq \int_0^1 T_u h \, du = g$ for $t + 1/n \leq 1$, we then apply Lemma 2 together with an approximation argument to see that

$$\{\omega : T_{1/n}h(\omega) > 0\} \subset \{\omega : g(\omega) > 0\},$$

so that $C \subset \{\omega : g(\omega) > 0\}$ and consequently $C \subset P$. (Incidentally, we note that $C = \{\omega : g(\omega) > 0\}$. In fact, by Lemma 2, $g(\omega) = 0$ a.e. on D.)

Since $T_t L(P) \subset L(C)$ by Lemma 7, we can use the above result $C \subset P$ to obtain $T_t L(P) \subset L(P)$. This completes the proof of the proposition.

LEMMA 8. *Let $T = \{T_t\}$ be a strongly continuous semigroup of positive linear operators on L. Then there exists a positive real number α and a sequence (v_n) of nonnegative measurable functions on Ω such that*

(i) $0 \le v_1(\omega) \le v_2(\omega) \le \ldots$ *a.e. on* Ω,
(ii) $P = \{\omega : v_n(\omega) > 0$ *for some* $n \ge 1\}$,
(iii) *for each* $f \in L^+$, $t > 0$ *and* $n \ge 1$ *we have*

$$\int_\Omega (T_t f) v_n \, d\mu \le e^{\alpha t} \int_\Omega f v_n \, d\mu < \infty.$$

Proof. Let $\alpha > 0$ be such that $e^{-\alpha}\|T_1\| < 1$. It follows that for any $f \in L$ the vector-valued function $t \mapsto e^{-\alpha t} T_t f$ is Bochner integrable on the interval $(1/n, \infty)$ with respect to the Lebesgue measure for each $n \ge 1$. Define the positive linear operator $S_n : L \to L$ by

$$S_n f = \int_{1/n}^{\infty} e^{-\alpha t} T_t f \, dt \quad \text{for } f \in L.$$

By Lemma 5 there exists a strictly positive measurable function w on Ω such that $\int_\Omega |f| w \, d\mu < \infty$ for all $f \in L$, and by Lemma 6 let

$$v_n = S_n^* w \quad \text{for } n \ge 1.$$

Clearly, $0 \le v_1(\omega) \le v_2(\omega) \le \ldots$ a.e. on Ω, and for $f \in L^+$ we have

$$\begin{aligned} \infty > \int_\Omega f v_n \, d\mu &= \int_\Omega f (S_n^* w) \, d\mu = \int_\Omega (S_n f) w \, d\mu \\ &= \int_\Omega \Big(\int_{1/n}^{\infty} e^{-\alpha t} T_t f \, dt \Big) w \, d\mu = \int_{1/n}^{\infty} \Big(e^{-\alpha t} \int_\Omega (T_t f) w \, d\mu \Big) dt \end{aligned}$$

by Fubini's theorem. Thus if $f \in L^+(N)$ then, since $\|T_t f\| = 0$ for all $t > 0$, we get $\int_\Omega f v_n \, d\mu = 0$. It follows that $v_n(\omega) = 0$ a.e. on N.

On the other hand, if $f \in L^+(P)$ and $\|f\| > 0$ then $\|T_t f\| > 0$ for some $t > 0$, whence $\int_\Omega (T_t f) w \, d\mu > 0$. It follows that $\int_\Omega f v_n \, d\mu > 0$ whenever $1/n < t$. This proves (ii).

To prove (iii), let $f \in L^+$. Then for any $t > 0$ and $n \ge 1$ we have

$$0 \le \int_\Omega (T_t f) v_n \, d\mu = \int_\Omega \Big(\int_{1/n}^{\infty} e^{-\alpha s} T_{t+s} f \, ds \Big) w \, d\mu$$

$$\le \int_{\Omega} \Big(e^{\alpha t} \int_{1/n}^{\infty} e^{-\alpha s} T_s f \, ds\Big) w \, d\mu = e^{\alpha t} \int_{\Omega} f v_n \, d\mu < \infty,$$

whence (iii) follows.

Proof of Theorem. Let α and (v_n) be as in Lemma 8. Put

$$P_n = \{\omega : v_n(\omega) > 0\} \quad \text{and} \quad N_n = \Omega \setminus P_n.$$

By Lemma 8 it follows that for each $n \ge 1$ the process $\{F_t \cdot \chi_{P_n}\}_{t>0}$ in L is also superadditive with respect to the semigroup $T = \{T_t\}$. Further, since $P = \bigcup_{n=1}^{\infty} P_n$, in order to prove the theorem we may assume without loss of generality that $\Omega = P_n$ for some $n \ge 1$. Then define

$$\widetilde{T}_t = e^{-\alpha t} T_t,$$

so that

$$\int_{\Omega} (\widetilde{T}_t f) v_n \, d\mu \le \int_{\Omega} f v_n \, d\mu < \infty$$

for $f \in L^+(\Omega)$, and $v_n(\omega) > 0$ a.e. on $\Omega = P_n$. It follows that $L \subset L_1(v_n d\mu)$, and by an easy approximation argument, for each $t > 0, \widetilde{T}_t$ can be regarded as a positive linear contraction operator on $L_1(v_n d\mu)$. Since the linear functional ϱ on L defined by $\varrho(f) = \int_{\Omega} f v_n \, d\mu$ for $f \in L$ is positive and hence bounded, it follows that for every $f \in L$,

$$\int_{\Omega} |\widetilde{T}_t f - \widetilde{T}_s f| v_n \, d\mu \le \|\widetilde{T}_t f - \widetilde{T}_s f\| \cdot \|\varrho\| < \infty$$

and hence

$$\lim_{t \to s} \int_{\Omega} |\widetilde{T}_t f - \widetilde{T}_s f| v_n \, d\mu = \lim_{t \to s} \|\widetilde{T}_t f - \widetilde{T}_s f\| = 0 \quad \text{ for } s > 0.$$

Thus, since L is a dense subspace of $L_1(v_n \, d\mu)$, $\widetilde{T} = \{\widetilde{T}_t\}$ can be regarded as a strongly continuous semigroup of positive linear contraction operators on $L_1(v_n d\mu)$. If the L_1-norm of $L_1(v_n d\mu)$ is written as $\|\cdot\|_1$ then from the linear boundedness hypothesis on $\{F_t^-\}$ we get

$$\sup\{\|F_t^-\|_1/t : 0 < t < s\} \le \|\varrho\| \cdot \sup\{\|F_t^-\|/t : 0 < t < s\} < \infty$$

for some $s > 0$. On the other hand, by the superadditivity of $F = \{F_t\}$ with respect to $T = \{T_t\}$ we deduce that

$$e^{-\alpha(t+s)} F_{t+s}^-(\omega) \le e^{-\alpha t} F_t^-(\omega) + \widetilde{T}_t(e^{-\alpha s} F_s^-)(\omega) \quad \text{ a.e.}$$

on Ω for all $t, s > 0$. It is now standard (cf. e.g. the proof of Theorem 2.1 of [1]) to construct a positive process $G = \{G_t\}_{t>0}$ in $L_1(v_n \, d\mu)$, additive with respect to $\widetilde{T} = \{\widetilde{T}_t\}$, such that

(i) $e^{-\alpha t} F_t^-(\omega) \le G_t(\omega)$ a.e. on Ω for each $t > 0$,
(ii) $\sup\{t^{-1} \|G_t\|_1 : t > 0\} < \infty$.

Then we have

$$F_t(\omega) + e^{\alpha t} G_t(\omega) \geq 0 \quad \text{a.e.}$$

and

$$\begin{aligned} e^{\alpha(t+s)} G_{t+s}(\omega) &= e^{\alpha(t+s)} G_t(\omega) + e^{\alpha s} T_t G_s(\omega) && (\text{by } \widetilde{T}_t = e^{-\alpha t} T_t) \\ &\geq e^{\alpha t} G_t(\omega) + T_t(e^{\alpha s} G_s)(\omega) \geq 0 \quad \text{a.e.} && (\text{by } G_t(\omega) \geq 0) \end{aligned}$$

on Ω, and thus if we set

$$\widetilde{F}_t(\omega) = F_t(\omega) + e^{\alpha t} G_t(\omega),$$

then $\widetilde{F} = \{\widetilde{F}_t\}$ becomes a positive superadditive process in $L_1(v_n d\mu)$ with respect to the semigroup $T = \{T_t\}$, where each $T_t = e^{\alpha t}\widetilde{T}_t$ is considered to be a positive linear operator on $L_1(v_n d\mu)$. If we define

$$H_t(\omega) = e^{\alpha t} \widetilde{F}_t(\omega),$$

then, using the facts that $\widetilde{F} = \{\widetilde{F}_t\}$ is positive and $e^{\alpha t} T_t = e^{2\alpha t}\widetilde{T}_t \geq \widetilde{T}_t \geq 0$ for $t > 0$, we obtain

$$\begin{aligned} H_{t+s}(\omega) = e^{\alpha(t+s)} \widetilde{F}_{t+s}(\omega) &\geq e^{\alpha(t+s)} [\widetilde{F}_t(\omega) + T_t \widetilde{F}_s(\omega)] \\ &\geq e^{\alpha t} \widetilde{F}_t(\omega) + e^{\alpha t} T_t(e^{\alpha s} \widetilde{F}_s)(\omega) \\ &\geq H_t(\omega) + \widetilde{T}_t H_s(\omega) \geq 0 \quad \text{a.e.} \end{aligned}$$

on Ω, so that $H = \{H_t\}_{t>0}$ is a positive superadditive process in $L_1(v_n d\mu)$ with respect to the positive contraction operator semigroup $\widetilde{T} = \{\widetilde{T}_t\}$ on $L_1(v_n d\mu)$.

Since the decomposition $\Omega = P + N$, mentioned in the Introduction, of the space $\Omega = P_n$ with respect to the semigroup $\widetilde{T} = \{\widetilde{T}_t\}$ on $L_1(v_n d\mu)$ is $P = \Omega = P_n$ and $N = \emptyset$ (cf. the proof of Lemma 8) and since $H_t^- = G_t^- = 0$ a.e. on Ω for all $t > 0$, it follows from the Proposition of [6] that the limits

$$\begin{aligned} \text{q-}\lim_{t\to 0} \frac{H_t(\omega)}{t} &= \text{q-}\lim_{t\to 0} \frac{e^{\alpha t}\widetilde{F}_t(\omega)}{t} \\ &= \text{q-}\lim_{t\to 0} \frac{e^{\alpha t}(F_t(\omega) + e^{\alpha t} G_t(\omega))}{t} \quad \text{and} \quad \text{q-}\lim_{t\to 0} \frac{G_t(\omega)}{t} \end{aligned}$$

exist and are finite a.e. on $\Omega = P_n$, which completes the proof of the theorem.

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