COLLOQUIUM MATHEMATICUM

VOL. 83 2000 NO. 1

INVARIANTS OF LIE COLOR ALGEBRAS ACTING ON GRADED ALGEBRAS

BY

JEFFREY BERGEN (CHICAGO, IL) AND
PIOTR GRZESZCZUK (BIAŁYSTOK)

Abstract. We prove a series of "going-up" theorems contrasting the structure of semiprime algebras and their subalgebras of invariants under the actions of Lie color algebras.

1. Introduction and terminology. Over the last 25 years, there have been many papers analyzing the invariants and actions of groups and Lie algebras. Actions of groups and Lie algebras correspond, respectively, to actions of group algebras and enveloping algebras, both of which are cocommutative Hopf algebras. However, recent work on quantum groups has increased the interest in Hopf algebras which are neither commutative nor cocommutative.

In a recent paper [BG1], we examined Lie superalgebras L and their actions on associative algebras R and proved a series of "going-up" theorems relating the structure of the subalgebra of invariants R^L to the original algebra R. Actions of Lie superalgebras L on $\mathbb{Z}_2$-graded algebras correspond to actions of Hopf algebras $H = U(L) * G$, where $U(L)$ is the enveloping algebra of L and G is a group of order two. Hopf algebras of this form are of particular interest as they provide a large class of examples of Hopf algebras which are neither commutative nor cocommutative.

Lie superalgebras can be considered as part of a larger class of nonassociative algebras known as Lie color algebras. Lie color algebras L are graded by abelian groups G and the homogeneous elements of L act on G-graded algebras as skew derivations. Lie color algebras L also have an enveloping

2000 *Mathematics Subject Classification*: 17B40, 17A36, 16W50, 17D05.

The first author was supported by the Faculty Research and Development Fund of the College of Liberal Arts & Sciences at DePaul University. Both authors were supported by the University Research Council at DePaul University and by Polish KBN Grant 2 P03A 039 14. Much of this work was done when the first author was a visitor at University of Białystok and when the second author was a visitor at DePaul University. We would like to thank both Universities for their hospitality.

algebra $U(L)$ on which the group G acts. Actions of L then correspond to actions of $H = U(L) * G$ and H is once again a noncommutative, noncocommutative Hopf algebra. Therefore, in an attempt to better understand the actions of noncommutative, noncocommutative Hopf algebras, it is reasonable to look at the actions of Lie color algebras.

In [Sc], Scheunert shows that if L is a Lie color algebra and G is finitely generated, then the multiplication in L can be twisted by a 2-cocycle to obtain a new algebra $\widetilde{L}$ which is actually a Lie superalgebra. Motivated by Scheunert's work, we show in Theorem 3 that whenever a Lie color algebra L acts on a G-graded algebra A, then A can be twisted to obtain a new algebra $\widetilde{A}$ on which the Lie superalgebra $\widetilde{L}$ acts. By contrasting the structure of the algebras A and $\widetilde{A}$ and their subalgebras of invariants, we are in a position to extend our results of [BG1] from Lie superalgebras to Lie color algebras.

Our main result in this direction, which we prove in Section 4, is

THEOREM 16. *Let* $R = \bigoplus_{g\in G} R_g$ *be a semiprime* K*-algebra graded by a finitely generated abelian group* G *and suppose* R *is acted on by a finite-dimensional nilpotent Lie color algebra* $L = \bigoplus_{g\in G} L_g$ *such that if* $\operatorname{char} K = p$ (*the characteristic of* K) *then* L *is restricted and if* $\operatorname{char} K = 0$ *then* L *acts by algebraic transformations.*

(i) *If* R^L *is right Noetherian*, *then* R *is a Noetherian right* R^L*-module. In particular*, R *is Noetherian and finitely generated as a right* R^L*-module.*

(ii) *If* R^L *is right Artinian*, *then* R *is an Artinian right* R^L*-module. In particular*, R *is Artinian and finitely generated as a right* R^L*-module.*

(iii) *If* R^L *is finite-dimensional over* K, *then* R *is finite-dimensional over* K.

(iv) *If* R^L *has finite Goldie dimension as a right* R^L*-module*, *then* R *has finite Goldie dimension as a right* R*-module.*

(v) *If* R^L *has Krull dimension* α *as a right* R^L*-module*, *then* R *has Krull dimension* α *as a right* R^L*-module. Thus* R *has Krull dimension at most* α *as a right* R*-module.*

We begin by defining many of the terms we will use throughout this paper. L will be a vector space over a field K of characteristic different from 2. G will be an abelian group and we call a map $\varepsilon : G \times G \to K^*$ a *bicharacter* if $\varepsilon(g, hk) = \varepsilon(g, h)\varepsilon(g, k)$ and $\varepsilon(g, h) = \varepsilon(h, g)^{-1}$ for all $g, h, k \in G$. L is said to be a *G-graded algebra* if there exist K-subspaces L_g such that $L = \bigoplus_{g\in G} L_g$ and L has a K-linear multiplication $[\,,\,]$ such that $[L_g, L_h] \subseteq L_{gh}$ for all $g, h \in G$.

L is a *Lie color algebra* over the field K if L is a G-graded algebra and there exists a bicharacter $\varepsilon : G \times G \to K^*$ such that

$$[x, y] = -\varepsilon(g, h)[y, x] \quad \text{and} \quad [[x, y], z] = [x, [y, z]] - \varepsilon(g, h)[y, [x, z]]$$

for all $x \in L_g$, $y \in L_h$, and $z \in L$. Observe that if $G = \mathbb{Z}_2$ and ε_0 is the bicharacter given by $\varepsilon_0(i,j) = (-1)^{ij}$ then $L = L_0 \oplus L_1$ is an ordinary Lie superalgebra.

The elements of $\bigcup_{g\in G} L_g$ are known as the *homogeneous* elements of L. If $g \in G$, then it is easy to check that either $\varepsilon(g,g) = 1$ or -1. If we let $G_+ = \{g \in G \mid \varepsilon(g,g) = 1\}$ and $G_- = \{g \in G \mid \varepsilon(g,g) = -1\}$, then we can let $L_+ = \bigcup_{g\in G_+} L_g$ and $L_- = \bigcup_{g\in G_-} L_g$. In addition, if $\operatorname{char} K = 3$, then we also require that $[x,[x,x]] = 0$ for all $x \in L_-$.

If $\operatorname{char} K = p > 2$, there is one additional structure we can add. We say that L is a *restricted Lie color algebra* over a field K of characteristic $p > 2$ if L is a Lie color algebra with a pth power map $L_+ \to L_+$, denoted by ${}^{[p]}$, satisfying

(i) $(\alpha x)^{[p]} = \alpha^p x^{[p]}$ for all $\alpha \in K$ and $x \in L_+$;

(ii) $[x^{[p]}, y] = (\mathrm{ad}_x)^p(y)$ for all $x \in L_+$ and $y \in L$;

(iii) $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x,y)$ for all $x, y \in L_+$;

where $\mathrm{ad}_x(y) = [x,y]$ and $is_i(x,y)$ is the coefficient of t^{i-1} in $(\mathrm{ad}_{tx+y})^{p-1}(x)$.

We now give several important examples of Lie color algebras.

1. Let $R = \bigoplus_{g\in G} R_g$ be an associative G-graded algebra and let $\varepsilon : G \times G \to K^*$ be a bicharacter. Putting

$$[x,y] = xy - \varepsilon(g,h)yx$$

for all $x \in R_g$, $y \in R_h$, we obtain a Lie color algebra $R^{(\varepsilon)}$ and call it the *adjoint Lie color algebra*.

2. Let $V = \bigoplus_{g\in G} V_g$ be a G-graded vector space over a field K. For any $g \in G$, consider a subspace $E_g \subseteq \mathrm{End}_K(V)$ consisting of all linear endomorphisms of degree g (i.e., endomorphisms mapping every V_h into V_{gh}). Then $\bigoplus_{g\in G} E_g$ has a natural structure of an associative G-graded algebra. If ε is a bicharacter on G, then we call the adjoint Lie color algebra $(\bigoplus_{g\in G} E_g)^{(\varepsilon)}$ the *general Lie color algebra* and denote it by $\mathfrak{gl}(V,G,\varepsilon)$.

3. Let $A = \bigoplus_{g\in G} A_g$ be an arbitrary G-graded algebra over K (not necessarily associative), and let ε be a bicharacter. It is easy to see that ε induces an action $\Phi : G \to \mathrm{Aut}_K(A)$ as K-linear automorphisms of A. Indeed, the mapping $\Phi(g) : A \to A$ given by $\Phi(g)(x_h) = \varepsilon(g,h)x_h$, for all $x_h \in A_h$, is an automorphism of A and it is easy to verify that Φ is a group homomorphism. Consider the subspace $\mathfrak{Der}(A,\varepsilon) = \bigoplus_{g\in G} D_g$ in $\mathfrak{gl}(A,G,\varepsilon)$, where $D_g \subset E_g$ is the set of all $\Phi(g)$-derivations of A of degree g. Recall that if σ is an automorphism of A, then a K-linear endomorphism δ is said to be a σ-*derivation* if $\delta(xy) = \delta(x)y + x^\sigma \delta(y)$ for all $x, y \in A$. It is easy to check that $\mathfrak{Der}(A,\varepsilon)$ is a Lie color subalgebra of $\mathfrak{gl}(A,G,\varepsilon)$. Furthermore, if

K has a positive characteristic $p > 2$, then the pth power map

$$^p : \mathfrak{Der}(A,\varepsilon)_+ \to \mathfrak{Der}(A,\varepsilon)_+$$

gives $\mathfrak{Der}(A,\varepsilon)$ the structure of a restricted Lie color algebra.

Now we are able to define the action of a Lie color algebra on a graded algebra. If $L = \bigoplus_{g\in G} L_g$ is a Lie color algebra, we say that L acts on $A = \bigoplus_{g\in G} A_g$ if there is a homomorphism of Lie color algebras $\Psi : L \to \mathfrak{Der}(A,\varepsilon)$. In the characteristic $p > 2$ case, we additionally assume that L is restricted and that Ψ also satisfies $\Psi(l^{[p]}) = \Psi(l)^p$, where $^{[p]}$ is the pth power map and $l \in L_+$. When L acts on A, we define the subalgebra of invariants A^L to be $\{a \in A \mid \delta(a) = 0 \text{ for all } \delta \in \Psi(L)\}$.

Note that if $L = \bigoplus_{g\in G} L_g$ acts on $A = \bigoplus_{g\in G} A_g$, then we can assume that L is *faithfully graded* by G, i.e., the subgroup $H = \{g \in G \mid \varepsilon(g,G) = 1\}$ is trivial. Indeed, the grading of L and A by G induces a grading by the quotient group G/H and no information about the structure of L and $\mathfrak{Der}(A,\varepsilon)$ is lost when we consider L and A as G/H-graded algebras. Moreover, if the characteristic of K is $p > 0$ and L is faithfully G-graded, then the group G has no elements of order p. Indeed, if $g^p = 1$, then for any $x \in G$ we have $\varepsilon(g,x)^p = \varepsilon(g^p,x) = \varepsilon(1,x) = 1$, so $\varepsilon(g,x) = 1$. Consequently, we can always assume that when the characteristic of K is positive, then the orders of all the elements of G are relatively prime to the characteristic.

2. Twisted algebras. Let G be an abelian group and let $A = \bigoplus_{g\in G} A_g$ be a G-graded algebra over a field K. Let $f \in Z^2(G,K^*)$ be a 2-cocycle with respect to the trivial action of G on K^*, that is, $f : G \times G \to K^*$ is a function such that $f(xy,z)f(x,y) = f(x,yz)f(y,z)$ for all $x,y,z \in G$. From A we obtain a new algebra $\widetilde{A}$ which is identical to A as a K-vector space but whose multiplication $*$ is defined as

$$x * y = f(g,h)xy,$$

where $x \in A_g$, $y \in A_h$. If $\delta : A \to A$ is a linear endomorphism of degree g, we define $\widetilde{\delta}$ to be the endomorphism

$$\widetilde{\delta}(x) = f(g,h)\delta(x)$$

where $x \in A_h$.

For any $g \in G$ and $n \geq 1$, let

$$\lambda(g,n) = \prod_{j=1}^{n-1} f(g,g^j).$$

Next, we define the function $\mu : G \times G \to K^*$ by

$$\mu(g,h) = \frac{f(g,h)}{f(h,g)}.$$

LEMMA 1. *Under the above notation,*

(i) *For any $g, h \in G$ and $n \geq 1$, $\prod_{j=0}^{n-1} f(g, g^j h) = \lambda(g, n) f(g^n, h)$.*
(ii) *The function μ is a bicharacter on G.*

Proof. (i) If $j \geq 1$, we have

$$f(g, g^j h) = \frac{f(g^{j+1}, h) f(g, g^j)}{f(g^j, h)}$$

and it therefore follows that

$$\begin{aligned}\prod_{j=0}^{n-1} f(g, g^j h) &= \prod_{j=0}^{n-1} \frac{f(g^{j+1}, h) f(g, g^j)}{f(g^j, h)} \\ &= f(g^n, h) \prod_{j=1}^{n-1} f(g, g^j) = \lambda(g, n) f(g^n, h),\end{aligned}$$

as required.

(ii) Since G is abelian, for $a, b, c \in G$,

$$\begin{aligned}\mu(ab, c) = \frac{f(ab, c)}{f(c, ab)} &= \frac{f(a, bc) f(b, c)}{f(a, b)} \cdot \frac{f(a, b)}{f(ca, b) f(c, a)} = \frac{f(ab, c)}{f(ac, b)} \cdot \frac{f(b, c)}{f(c, a)} \\ &= \frac{f(a, bc) f(a, c) f(b, c)}{f(a, cb) f(c, b) f(c, a)} = \mu(a, c)\mu(b, c).\end{aligned}$$

Clearly, $\mu(a, b) = \mu(b, a)^{-1}$, thus μ is a bicharacter. ■

Now we will examine properties of the functor $\tilde{}$ for a large class of algebras including associative, alternative, Lie and Lie color algebras. Following [BG2], recall that an algebra A over a field K is said to be a *left* (α, β, γ)-*algebra* if there exists a multiplicatively closed set $S = S(A)$ which generates A as a vector space and there exist functions $\alpha, \beta, \gamma : S \times S \to k$ such that

$$x(yz) = \alpha(x, y) y(xz) + \beta(x, y)(xy)z + \gamma(x, y)(yx)z$$

for all $x, y \in S$ and $z \in A$.

Analogously, A is a *right* (α, β, γ)-*algebra* if it satisfies the identity

$$(zx)y = \alpha(x, y)(zy)x + \beta(x, y) z(xy) + \gamma(x, y) z(yx)$$

for all $x, y \in S$ and $z \in A$.

Algebras which are both left and right (α, β, γ)-algebras are simply called (α, β, γ)-*algebras.*

We will say that an (α, β, γ)-algebra A is G-*graded* if $A = \bigoplus_{g \in G} A_g$ is graded in the ordinary sense and additionally $S(A) = \bigcup_{g \in G} S_g$, where $S_g = S(A) \cap A_g$.

Observe that:

1. Associative algebras are (α,β,γ)-algebras with $S = A$ and $\alpha \equiv 0$, $\beta \equiv 1$, $\gamma \equiv 0$.

2. Antiassociative algebras are (α,β,γ)-algebras with $S = A$ and $\alpha \equiv 0$, $\beta \equiv -1$, $\gamma \equiv 0$.

3. Left alternative algebras are left (α,β,γ)-algebras with $S = A$ and $\alpha \equiv -1$, $\beta \equiv 1$, $\gamma \equiv 1$. Analogously, right alternative algebras are right (α,β,γ)-algebras with $S = A$ and $\alpha \equiv -1$, $\beta \equiv 1$, $\gamma \equiv 1$.

4. Lie algebras are (α,β,γ)-algebras with $S = A$ and $\alpha \equiv 1$, $\beta \equiv 1$, $\gamma \equiv 0$.

5. Lie color algebras are (α,β,γ)-algebras with $S = \bigcup_{g\in G} L_g$ and $\alpha = \varepsilon$, $\beta \equiv 1$, $\gamma \equiv 0$.

PROPOSITION 2. *Let $A = \bigoplus_{g\in G} A_g$ be a G-graded K-algebra and let $f \in Z^2(G,K^*)$ be a 2-cocycle. Then*

(i) *If A is a G-graded (α,β,γ)-algebra, then $\widetilde{A}$ is a G-graded $(\widetilde{\alpha},\beta,\widetilde{\gamma})$-algebra, where $\widetilde{\alpha}(x,y) = \mu(g,h)\alpha(x,y)$ and $\widetilde{\gamma}(x,y) = \mu(g,h)\gamma(x,y)$ for all $x \in S_g$, $y \in S_h$. In particular, if $L = \bigoplus_{g\in G} L_g$ is an ε-Lie color algebra, then $\widetilde{L}$ is an $\widetilde{\varepsilon}$-Lie color algebra, where $\widetilde{\varepsilon} = \mu\varepsilon$. If $\operatorname{char} K = p > 2$ and L is restricted, then $\widetilde{L}$ is also restricted with respect to the pth power map ${}^{[p]_*} : \widetilde{L}_+ \to \widetilde{L}_+$ given by $x^{[p]_*} = \lambda(g,p)x^{[p]}$, where $x \in L_g$.*

(ii) *If $\delta_a, \delta_b : A \to A$ are K-linear endomorphisms of degrees a and b respectively, then*

$$\widetilde{\delta_a\delta_b} = \frac{1}{f(a,b)}\widetilde{\delta}_a\widetilde{\delta}_b, \quad \textit{hence} \quad \widetilde{\delta_a^n} = \frac{1}{\lambda(a,n)}\widetilde{\delta}_a^n.$$

(iii) *If $\delta : A \to A$ is a K-linear algebraic endomorphism of degree g, then $\widetilde{\delta}$ is also an algebraic endomorphism of degree g.*

(iv) *Let σ be an automorphism of A such that $\sigma(A_h) \subseteq A_h$ for all $h \in G$ and let δ be a σ-derivation of A of degree g. Then $\widetilde{\delta}$ is a $\widehat{\sigma}$-derivation of $\widetilde{A}$ of degree g, where $\widehat{\sigma}(x) = \mu(g,h)\sigma(x)$ for all $x \in A_h$ and $h \in G$.*

Proof. (i) Let $a,b,c \in G$, $x \in S_a$, $y \in S_b$, $z \in S_c$ and let $\alpha = \alpha(x,y)$, $\beta = \beta(x,y)$, $\gamma = \gamma(x,y)$. Note that

$$\begin{aligned} f(b,c)f(a,bc) &= f(a,b)f(ab,c) = \frac{f(a,b)}{f(b,a)}f(ba,c)f(b,a) \\ &= \mu(a,b)f(b,ac)f(a,c) = \mu(a,b)f(ba,c)f(b,a). \end{aligned}$$

Hence

$$
\begin{aligned}
x*(y*z) &= f(b,c)f(a,bc)\cdot x(yz)\\
&= f(b,c)f(a,bc)(\alpha\cdot y(xz)+\beta\cdot(xy)z+\gamma\cdot(yx)z)\\
&= \frac{f(b,c)f(a,bc)}{f(a,c)f(b,ac)}\alpha\cdot y*(x*z)+\frac{f(b,c)f(a,bc)}{f(a,b)f(ab,c)}\beta\cdot(x*y)*z\\
&\quad+\frac{f(b,c)f(a,bc)}{f(b,a)f(ba,c)}\gamma\cdot(y*x)*z\\
&= \frac{f(a,b)}{f(b,a)}\alpha\cdot y*(x*z)+\beta\cdot(x*y)*z+\frac{f(a,b)}{f(b,a)}\gamma\cdot(y*x)*z\\
&= \mu(a,b)\alpha\cdot y*(x*z)+\beta\cdot(x*y)*z+\mu(a,b)\gamma\cdot(y*x)*z.
\end{aligned}
$$

Consequently, $\widetilde{A}$ is a left $(\widetilde{\alpha},\beta,\widetilde{\gamma})$-algebra, with the same spanning set S as A. Analogously one can check the right condition.

We will prove that if $\operatorname{char} K=p>2$ and L is a restricted Lie color algebra, then $\widetilde{L}$ is also restricted. Let $[\,,\,]_*$ denote the Lie color bracket in $\widetilde{L}$ and let $x,y\in L_g, z\in L_h$, where $g\in G_+$, $h\in G$. It is also easy to check that $(\cdot)^{[p]_*}$ satisfies the first axiom of a restricted Lie color algebra.

Since $x^{[p]_*}\in L_{g^p}$, it follows from Lemma 1 that

$$
\begin{aligned}
[x^{[p]_*},z]_* &= f(g^p,h)\lambda(g,p)[x^{[p]},z]=f(g^p,h)\lambda(g,p)(\mathrm{ad}_x)^p(z)\\
&= \frac{f(g^p,h)\lambda(g,p)}{\prod_{j=0}^{p-1}f(g,g^jh)}(\widetilde{\mathrm{ad}_x})^p(z)=(\widetilde{\mathrm{ad}_x})^p(z).
\end{aligned}
$$

Next note that

$$
(\widetilde{\mathrm{ad}_{tx+y}})^{p-1}(x)=\prod_{j=1}^{p-1}f(g,g^j)(\mathrm{ad}_{tx+y})^{p-1}(x)=\lambda(g,p)(\mathrm{ad}_{tx+y})^{p-1}(x).
$$

Therefore the coefficient $i\widetilde{s_i}(x,y)$ of t^{i-1} in $(\mathrm{ad}_{tx+y})^{p-1}(x)$ is equal to $\lambda(g,p)is_i(x,y)$ and consequently

$$
\begin{aligned}
(x+y)^{[p]_*} &= \lambda(g,p)(x+y)^{[p]}=\lambda(g,p)\Big(x^{[p]}+y^{[p]}+\sum_{i=1}^{p-1}s_i(x,y)\Big)\\
&= x^{[p]_*}+y^{[p]_*}+\sum_{i=1}^{p-1}\widetilde{s}_i(x,y).
\end{aligned}
$$

Thus $\widetilde{L}$ is indeed a restricted Lie color algebra.

(ii) Let $x\in L_c$, $c\in G$; since $\delta_a\delta_b$ is of degree ab, we have

$$
\begin{aligned}
\widetilde{\delta_a\delta_b}(x) &= f(ab,c)\delta_a\delta_b(x)=\frac{f(ab,c)}{f(b,c)f(a,bc)}\widetilde{\delta}_a\widetilde{\delta}_b(x)\\
&= \frac{f(ab,c)}{f(ab,c)f(a,b)}\widetilde{\delta}_a\widetilde{\delta}_b(x)=\frac{1}{f(a,b)}\widetilde{\delta}_a\widetilde{\delta}_b(x),
\end{aligned}
$$

as required.

(iii) We will consider separately the cases where g is of finite and infinite order. If $g^n = 1$ then, for any $x \in A_h$, $h \in G$,

$$\begin{aligned}\widetilde{\delta}^n(x) &= \prod_{j=0}^{n-1} f(g, g^j h)\delta^n(x) = \lambda(g,n) f(g^n, h)\delta^n(x) \\ &= \lambda(g,n) f(1,h)\delta^n(x) = \lambda(g,n) f(1,1)\delta^n(x).\end{aligned}$$

This means that there exists a scalar $\theta \in K$ such that, for any $x \in A$, $\widetilde{\delta}^n(x) = \theta\delta^n(x)$. Thus $\widetilde{\delta}$ is an algebraic endomorphism, provided δ is algebraic.

Now suppose that g is of infinite order and δ satisfies an identity

$$\alpha_0 + \alpha_1\delta + \ldots + \alpha_{m-1}\delta^{m-1} + \delta^m = 0.$$

For any $x \in A_h$, $h \in G$, and $j \geq 0$, we have $\delta^j(x) \in A_{g^j h}$, so $\delta^m(x) \in A_{g^m h} \cap (A_g \oplus \ldots \oplus A_{g^{m-1}h}) = 0$. Hence δ must be a nilpotent transformation and clearly $\widetilde{\delta}$ must also be nilpotent.

Note that in both cases, $\widetilde{\delta}$ is algebraic provided δ is algebraic.

(iv) Let $x \in A_a$, $y \in A_b$, $a, b \in G$; then $x * y \in A_{ab}$ and

$$\begin{aligned}\widetilde{\delta}(x * y) &= f(g, ab) f(a,b)\delta(xy) = f(g,ab) f(a,b)(\delta(x)y + \sigma(x)\delta(y)) \\ &= \frac{f(g,ab)f(a,b)}{f(g,a)f(ga,b)}\widetilde{\delta}(x) * y + \frac{f(g,ab)f(a,b)}{f(a,gb)f(g,b)}\sigma(x) * \widetilde{\delta}(y) \\ &= \widetilde{\delta}(x) * y + \widehat{\sigma}(x) * \widetilde{\delta}(y).\end{aligned}$$

Hence $\widetilde{\delta}$ is a $\widehat{\sigma}$-derivation of $\widetilde{A}$. ■

We can summarize our above considerations in the following

THEOREM 3. *Let* $A = \bigoplus_{g \in G} A_g$ *be a* G*-graded algebra acted on by a Lie color algebra* $L = \bigoplus_{g \in G} L_g$ *and let* $f \in Z^2(G, K^*)$. *Then the action* $\Psi : L \to \mathfrak{Der}(A, \varepsilon)$ *induces a unique action* $\widetilde{\Psi} : \widetilde{L} \to \mathfrak{Der}(\widetilde{A}, \widetilde{\varepsilon})$ *such that the diagram*

$$\begin{array}{ccc} L & \xrightarrow{\ \Psi\ } & \mathfrak{Der}(A,\varepsilon) \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \sim} \\ \widetilde{L} & \xrightarrow{\ \widetilde{\Psi}\ } & \mathfrak{Der}(\widetilde{A},\widetilde{\varepsilon}) \end{array}$$

is commutative. In this situation, the subalgebras of invariants $\widetilde{A}^{\widetilde{L}}$ *and* A^L *are equal as sets. More precisely,* $A^L = \bigoplus_{g \in G} A_g^L$ *is* G*-graded and* $\widetilde{A}^{\widetilde{L}}$ *can be obtained from* A^L *by deformation using* f, *i.e.* $\widetilde{A}^{\widetilde{L}} = \widetilde{A^L}$. *Furthermore, if* L *acts by algebraic transformations, then* $\widetilde{L}$ *also acts on* $\widetilde{A}$ *by algebraic transformations. In the positive characteristic case, if* L *acts as a restricted Lie color algebra, then* $\widetilde{L}$ *also acts as a restricted Lie color algebra.*

Proof. It suffices to put $\widetilde{\Psi}(x) = \widetilde{\Psi(x)}$. Indeed, if $x \in L_a$, $y \in L_b$ where $a, b \in G$ then

$$\begin{aligned}\widetilde{\Psi}([x,y]_*) &= f(a,b)\Psi(\widetilde{[x,y]}) = f(a,b)(\widetilde{\Psi(x)\Psi(y)} - \varepsilon(a,b)\widetilde{\Psi(y)\Psi(x)}) \\ &= f(a,b)\left(\frac{1}{f(a,b)}\widetilde{\Psi}(x)\widetilde{\Psi}(y)\frac{\varepsilon(a,b)}{f(b,a)}\widetilde{\Psi}(y)\widetilde{\Psi}(x)\right) \\ &= \widetilde{\Psi}(x)\widetilde{\Psi}(y) - \widetilde{\varepsilon}(a,b)\widetilde{\Psi}(y)\widetilde{\Psi}(x) = [\widetilde{\Psi}(x), \widetilde{\Psi}(y)]_*.\end{aligned}$$

If L is restricted then, by Proposition 2,

$$\widetilde{\Psi}(x^{[p]_*}) = \lambda(a,p)\widetilde{\Psi}(x^{[p]}) = \lambda(a,p)\widetilde{\Psi(x)^p} = (\widetilde{\Psi(x)})^p = \widetilde{\Psi}(x)^p.$$

The equalities $\widetilde{A}^{\widetilde{L}} = A^L$ (as sets) and $\widetilde{A}^{\widetilde{L}} = \widetilde{A^L}$ (as algebras) follow immediately since both A^L and $\widetilde{A}^{\widetilde{L}}$ can be computed as invariants of homogeneous elements from $\Psi(L)$ and $\widetilde{\Psi}(\widetilde{L})$, respectively. ∎

Now assume that the group G is finitely generated. We can then consider the bicharacter $\widetilde{\varepsilon} : G \times G \to K^*$ such that $\widetilde{\varepsilon}(g,h) = (-1)^{i_g i_h}$, where $i_g = 0$ if $g \in G_+$ and $i_g = 1$ if $g \in G_-$. By [Sc] there exists a 2-cocycle $f \in Z^2(G, K^*)$ such that $\widetilde{\varepsilon} = \mu\varepsilon$. Then $\widetilde{L} = \widetilde{L}_0 \oplus \widetilde{L}_1$ becomes an ordinary Lie superalgebra, where $\widetilde{L}_0 = \bigoplus_{g \in G_+} L_g, \widetilde{L}_1 = \bigoplus_{g \in G_-} L_g$. We now have

COROLLARY 4. *Let $A = \bigoplus_{g \in G} A_g$ be a G-graded algebra acted on by a Lie color algebra $L = \bigoplus_{g \in G} L_g$, where G is finitely generated, and let $A_0 = \bigoplus_{g \in G_+} A_g$, $A_1 = \bigoplus_{g \in G_-} A_g$. Then there exists a 2-cocycle $f \in Z^2(G, K^*)$ such that $\widetilde{L}$ is a Lie superalgebra acting on the $\mathbb{Z}_2$-graded algebra $\widetilde{A} = \widetilde{A}_0 \oplus \widetilde{A}_1$. Moreover, the subalgebras of invariants $\widetilde{A}^{\widetilde{L}}$ and A^L are equal as sets. If L acts by algebraic transformations, then $\widetilde{L}$ also acts on $\widetilde{A}$ by algebraic transformations. In the positive characteristic case, if L acts as a restricted Lie color algebra, then $\widetilde{L}$ acts as a restricted Lie superalgebra.*

3. Actions of nilpotent Lie color algebras. In this section we begin our investigation of the invariants of the actions of nilpotent Lie color algebras on algebras which are not necessarily associative.

PROPOSITION 5. *Let $L = L_0 \oplus L_1$ be a finite-dimensional nilpotent Lie superalgebra acting on an algebra $A = A_0 \oplus A_1$. Then $A^L \neq 0$ if and only if $A^{L_0} \neq 0$.*

Proof. It is clear that if $A^L \neq 0$, then $A^{L_0} \neq 0$. Suppose that $A^{L_0} \neq 0$; since L is nilpotent we can choose a chain of subalgebras $L(0) \subseteq L(1) \subseteq \ldots \ldots \subseteq L(m) = L$ such that $L(0) = L_0$, $\dim L(j+1) = \dim L(j) + 1$, and $[L(j), L(j)] \subseteq L(j-1)$. We will prove, by induction on j, that $A^{L(j)} \neq 0$. To this end, let $x \in L_1$ be homogeneous such that $L(j) = L(j-1) \oplus Kx$, and let $\delta = \Psi(x)$. Thus $\delta^2 = \frac{1}{2}\Psi([x,x]) \in \Psi(L_0)$ and $\delta(\delta(A^{L(j-1)})) = \delta^2(A^{L(j-1)}) \subseteq$

$\Psi(L_0)(A^{L(j-1)}) = 0$. This means that $\delta(A^{L(j-1)}) \subseteq A^{Kx}$. However, for any $\partial \in \Psi(L(j-1))$, we have $[\partial, \delta] \in L(j-1)$ and

$$\partial\delta(A^{L(j-1)}) = \delta\partial(A^{L(j-1)}) + [\partial, \delta](A^{L(j-1)}) = 0,$$

hence $\delta(A^{L(j-1)}) \subseteq A^{L(j)}$.

On the other hand, if $\delta(A^{L(j-1)}) = 0$ then clearly $A^{L(j)} = A^{L(j-1)}$. ∎

We now extend Proposition 5 from Lie superalgebras to Lie color algebras.

COROLLARY 6. *Let $L = \bigoplus_{g\in G} L_g$ be a finite-dimensional nilpotent Lie color algebra acting on an algebra $A = \bigoplus_{g\in G} A_g$. Then $A^L \neq 0$ if and only if $A^{L_+} \neq 0$.*

Proof. Since L is finite-dimensional, the subgroup of G generated by the support of L is finitely generated. Therefore there exists a finitely generated subgroup H of G which contains the support of L and $A^{L_+} \cap \bigoplus_{h\in H} A_h \neq 0$. Consequently, we may assume that G is finitely generated. By Corollary 4, we may assume without loss of generality that L is a nilpotent Lie superalgebra acting on the $\mathbb{Z}_2$-graded algebra $A = A_0 \oplus A_1$. The result now immediately follows from Proposition 5. ∎

In [J], Jacobson showed that if D is a nilpotent Lie algebra of derivations of a finite-dimensional Lie algebra of characteristic 0 such that $L^D = 0$, then L is nilpotent. In [BG2, Corollary 3.3], we generalized this result to Lie color algebras of arbitrary dimension. More precisely, we proved that if $L = \bigoplus_{g\in G} L_g$ is a Lie color algebra over a field K of characteristic 0 acted on by a finite-dimensional nilpotent Lie algebra D of homogeneous derivations of L which are algebraic as K-linear transformations of L such that $L^D = 0$, then L is nilpotent with the index of nilpotency depending only on the dimension of the action. We can now extend that result.

THEOREM 7. *Let $L = \bigoplus_{g\in G} L_g$ be a Lie color algebra over a field K of characteristic zero and let $D \subseteq \mathfrak{Der}(L, \varepsilon)$ be a finite-dimensional nilpotent Lie color algebra of skew derivations of L which are algebraic as K-linear transformations of L. If $L^D = 0$, then L is nilpotent.*

Proof. Let G_D denote the subgroup of G generated by the support of D. Clearly, G_D is finitely generated. Let H be any finitely generated subgroup of G containing G_D and let $L_H = \bigoplus_{h\in H} L_h$. Clearly, L_H is a Lie color subalgebra of L (with the same bicharacter) and D can be viewed as a subalgebra of $\mathfrak{Der}(L_H, \varepsilon)$. By Corollary 4, there exists a 2-cocycle $f \in Z^2(H, K^*)$ such that $\widetilde{D} = \widetilde{D}_0 \oplus \widetilde{D}_1$ becomes an ordinary Lie superalgebra, where $\widetilde{D}_0 = \bigoplus_{h\in H_+} D_h$, $\widetilde{D}_1 = \bigoplus_{h\in H_-} D_h$. Furthermore, $\widetilde{D}$ acts on $\widetilde{L}_H$ by algebraic transformations. From Proposition 5 it follows that $\widetilde{L}_H^{\widetilde{D}_0}=0$. Hence, by Corollary 3.3 from [BG2], the algebra $\widetilde{L}_H$ is nilpotent and the index of nilpotency of $\widetilde{L}_H$ is bounded by some integer N which does not depend on H. Note that L_H must also be nilpotent with the same index of nilpotency $\leq N$. If $x_1,\ldots,x_N$ are homogeneous elements of L then there exits a finitely generated subgroup H which contains G_D such that $\{x_1,\ldots,x_N\}\subseteq L_H$. This implies that

$$[x_1,[x_2,[\ldots[x_{N-1},x_N]\ldots]]]=0.$$

Thus L is nilpotent. ∎

If L if finite-dimensional, we can prove a characteristic p version of the previous result.

PROPOSITION 8. *Let $L=\bigoplus_{g\in G}L_g$ be a restricted Lie color algebra finite-dimensional over a field K of characteristic $p>0$ and let $D\subseteq \mathfrak{Der}(L,\varepsilon)$ be a nilpotent restricted Lie color algebra of skew derivations of L. If $L^D=0$, then L is nilpotent.*

Proof. Since L is finite-dimensional, we may assume that G is finitely generated. Using Corollaries 4 and 6 we can reduce the problem to the case when L is a restricted Lie superalgebra acted on by a restricted nilpotent Lie algebra D of homogeneous derivations. As in the proof of Lemma 3.2 in [BG1], for any $n\geq 0$, let $Z_{(n)}$ denote the K-linear span of the set $\{z^{[p^n]}\mid z\in D \text{ and } [z^{[p^n]},D]=0\}$. If we take N such that $Z_{(N)}=Z_{(N+1)}$, then clearly $I=Z_{(N)}$ is an abelian restricted ideal of D. Moreover, it is easy to see that if $\{z_1,\ldots,z_m\}$ is a basis of I, then $\{z_1^{[p]},\ldots,z_m^{[p]}\}$ is also a basis of I. Therefore, by Hochschild's theorem (see [M, Theorem 2.3.3]) on the semisimplicity of restricted enveloping algebras, the restricted enveloping algebra $u(I)$ is semisimple. By [M, Corollary 2.3.5], there exists a finite separable extension $E\supseteq K$ such that $u(I)\otimes_K E$ is isomorphic to the dual of the group algebra $E[(\mathbb{Z}_p)^n]$. Let $L'=L\otimes_K E$, $D'=D\otimes_K E$ and $I'=I\otimes_K E$. The K-linear action of D on L extends to an E-linear action of D' on L' with $(L')^{D'}=L^D\otimes_K E$. By the construction of I, the restriction of any derivation of D to L^I is nilpotent. Indeed, since D is nilpotent, there exists a $k\geq 1$ such that $z^{[p^k]}$ is central for all $z\in D$. Therefore $z^{[p^k]}$ is in I, for all $z\in D$, and D acts nilpotently on L^I. Now it is easy to see that $L^I=0$. Consequently, we may assume that $D=I$, and L can be viewed as a $(\mathbb{Z}_p)^n$-graded algebra with zero identity component. Now if $x\in L_0$ is a homogeneous element with respect to the $(\mathbb{Z}_p)^n$-grading, then $(\mathrm{ad}_x)^p$ is a mapping preserving the $(\mathbb{Z}_p)^n$-grading of L and from the identity $(\mathrm{ad}_x)^p=\mathrm{ad}_{x^{[p]}}$ it follows that $x^{[p]}$ belongs to the center of L. Thus $(\mathrm{ad}_x)^p=0$. On the other hand, if x is a homogeneous element of L_1, then from the Jacobi identity it follows that $(\mathrm{ad}_x)^2=\frac{1}{2}\mathrm{ad}_{[x,x]}$ and clearly $[x,x]\in L_0$. Consequently, in this case

$(\mathrm{ad}_x)^{2p} = 0$. Applying the Engel–Jacobson theorem on weakly nil sets, we deduce that L must be nilpotent. ■

In light of Theorem 7 and Proposition 8, it is natural to ask:

QUESTION. *Does the conclusion of Proposition* 8 *still hold if we no longer assume that* L *is finite-dimensional?*

4. Actions on associative algebras. Let $R = \bigoplus_{g\in G} R_g$ be an associative G-graded K-algebra, where G is a finitely generated abelian group (denoted multiplicatively), and let $f : G\times G \to K^*$ be a 2-cocycle. In addition, suppose that G acts on R by K-linear homogeneous automorphisms, that is, we have a group homomorphism $\sigma : G \to \mathbf{Aut}(R)$ such that $R_h^{\sigma(g)} \subseteq R_h$ for all $g, h \in G$. We let $R(G, \sigma, f)$ denote the G-graded ring which is identical to R as a K-vector space but whose multiplication $\circ$ is defined as

$$x_g \circ x_h = f(g, h)x_g^{\sigma(h)} x_h,$$

where $x_g \in R_g$, $x_h \in R_h$. In particular, $\widetilde{R} = R(G, 1, f)$. We let $R *_{(\sigma,f)} G$ denote the crossed product of G over R, i.e. each element $r \in R *_{(\sigma,f)} G$ is uniquely expressed as the finite sum $r = \sum_{g\in G} \overline{g} r_g$ where the elements $\{\overline{g} \mid g \in G\}$ are generators of $R *_{(\sigma,f)} G$ as a free left and right R-module satisfying the conditions

$$\overline{g}\,\overline{h} = \overline{gh} f(g, h), \qquad r\overline{g} = \overline{g} r^{\sigma(g)}$$

for all $g, h \in G$ and $r \in R$.

Since G is abelian, it can be easily checked that we have a homogeneous action $\widehat{\sigma} : G \to \mathbf{Aut}(R(G, \sigma, f))$ given by

$$r_h^{\widehat{\sigma}(g)} = \frac{f(h, g)}{f(g, h)} r_h^{\sigma(g)}.$$

Thus we can form a skew group ring $R(G, \sigma, f) * G$ with respect to the action $\widehat{\sigma}$.

PROPOSITION 9. *The crossed product* $R *_{(\sigma,f)} G$ *and the skew group ring* $R(G, \sigma, f) * G$ *are isomorphic.*

Proof. The mapping $\sum_{g\in G} r_g \mapsto \sum_{g\in G} \overline{g} r_g$ provides an embedding of $R(G, \sigma, f)$ into $R *_{(\sigma,f)} G$. Thus we can identify $R(G, \sigma, f)$ with $\overline{R} = \bigoplus_{g\in G} \overline{g} R_g \subseteq R *_{(\sigma,f)} G$. In order to show that $R *_{(\sigma,f)} G$ is a free right (left) $\overline{R}$-module, suppose that $\sum_{g\in G} \overline{g}\,\widetilde{r}_g = 0$, where $\widetilde{r}_g = \sum_{x\in G} \overline{x} r_x^{(g)} \in \overline{R}$ and $r_x^{(g)} \in R_x$. Then

$$0 = \sum_{g\in G} \overline{g}\Big(\sum_{x\in G} \overline{x} r_x^{(g)}\Big) = \sum_{g,x\in G} \overline{gx} f(g, x) r_x^{(g)} = \sum_{h\in G} \overline{h}\Big(\sum_{g\in G} f(g, g^{-1}h) r_{g^{-1}h}^{(g)}\Big).$$

Hence $\sum_{g\in G} f(g,g^{-1}h)r^{(g)}_{g^{-1}h} = 0$ for all $h \in G$. As a result, $r^{(g)}_x = 0$ for all $x, g \in G$, and consequently $\widetilde{r}_g = 0$.

Now let $r = \sum_{x\in G} r_x \in R$; therefore if $g \in G$, we have

$$\overline{g}r = \sum_{x\in G} \overline{gx^{-1}}\,\overline{x}r_x f(gx^{-1},x)^{-1} \in \bigoplus_{x\in G} \overline{x}\overline{R}.$$

Thus $R *_{(\sigma,f)} G = \bigoplus_{g\in G} \overline{g}\overline{R}$ as a right $\overline{R}$-module. Furthermore, for any $g, x \in G$ and $r_x \in R_x$ (G is abelian),

$$(\overline{x}r_x)\overline{g} = \overline{x}\cdot\overline{g}r_x^{\sigma(g)} = \overline{xg}f(x,g)r_x^{\sigma(g)} = \overline{g}\cdot\overline{x}\frac{f(x,g)}{f(g,x)}r_x^{\sigma(g)} = \overline{g}(\overline{x}r_x)^{\widehat{\sigma}(g)}. \ \blacksquare$$

If G is finite, then the above proposition shows that for the rings R and $R(G,\sigma,f)$, there exists a common overring S which can be viewed as a free finite normalizing extension of both R and $R(G,\sigma,f)$. Thus we have

COROLLARY 10. *Let $R = \bigoplus_{g\in G} R_g$ be a G-graded K-algebra, where G is a finite abelian group. Let $f \in Z^2(G,K^*)$ and suppose there is a homogeneous action $\sigma : G \to \mathbf{Aut}(R)$. Then*

(i) *R is right Artinian if and only if $R(G,\sigma,f)$ is right Artinian.*

(ii) *R is right Noetherian if and only if $R(G,\sigma,f)$ is right Noetherian.*

(iii) *R has finite Goldie dimension if and only if $R(G,\sigma,f)$ has finite Goldie dimension.*

(iv) *R has right Krull dimension α if and only if $R(G,\sigma,f)$ has right Krull dimension α.*

The next proposition will allow us to consider separately the cases where our groups are finite and torsion free. First, note that if $G = A \oplus B$ then R has a natural structure as an A-graded ring (and clearly as a B-graded ring). To this end, it suffices to let $S_a = \bigoplus_{b\in B} R_{ab}$ and $R = \bigoplus_{a\in A} S_a$. If we let $f_A, f_B, \sigma_A, \sigma_B$ denote the restrictions of f and σ to A and B respectively, then we can form the ring $R(A,\sigma_A,f_A)$. On the other hand, the ring $R(A,\sigma_A,f_A)$ is both G-graded and B-graded, therefore we can form the ring $R(A,\sigma_A,f_A)(B,\overline{\sigma}_B,f_B)$, where

$$r_{ab}^{\overline{\sigma}_B(x)} = \frac{f(a,x)}{f(x,a)}r_{ab}^{\sigma_B(x)}.$$

We have the following

PROPOSITION 11. *The rings $R(G,\sigma,f)$ and $R(A,\sigma_A,f_A)(B,\overline{\sigma}_B,f_B)$ are isomorphic.*

P r o o f. First note that if the 2-cocycles f, f' determine the same element in the second cohomology group $H^2(G,K^*)$, then the rings $R(G,\sigma,f)$ and $R(G,\sigma,f')$ are isomorphic. Indeed, we have a 2-coboundary $\beta : G\times G \to K^*$ such that $f' = \beta f$. In this case $\beta(g,h) = \phi(g)\phi(h)/\phi(gh)$ for some

$\phi : G \to K^*$. Consider the K-linear mapping $\alpha : R(G,\sigma,f) \to R(G,\sigma,f')$ given by $\alpha(r_g) = (1/\phi(g))r_g$. Therefore

$$\begin{aligned}\alpha(r_g \circ r_h) &= \frac{f(g,h)}{\phi(gh)} r_g^{\sigma(h)} r_h = \frac{\phi(g)\phi(h)}{\phi(gh)} f(g,h)\alpha(r_g^{\sigma(h)})\alpha(r_h)\\ &= \alpha(r_g)^{\sigma(h)} \circ \alpha(r_h)\end{aligned}$$

and α is a ring isomorphism.

Now let $a_1, a_2 \in A$ and $b_1, b_2 \in B$. Applying the 2-cocycle rule we have

$$\begin{aligned}f(a_1b_1, a_2b_2) &= \frac{f(a_1b_1b_2, a_2)f(a_1b_1, b_2)}{f(b_2, a_2)}\\ &= \frac{f(b_1b_2, a_1a_2)f(a_1, a_2)}{f(b_1b_2, a_1)} \cdot \frac{f(a_1, b_1b_2)f(b_1, b_2)}{f(a_1, b_1)f(b_2, a_2)}\\ &= \frac{f(b_1b_2, a_1a_2)f(a_1, a_2)f(b_1, b_2)}{f(a_1, b_1)f(b_2, a_2)} \cdot \frac{f(a_1, b_1)f(a_1, b_2)}{f(b_1, a_1)f(b_2, a_1)}\\ &= f(a_1, a_2)f(b_1, b_2)\frac{f(b_1b_2, a_1a_2)}{f(b_1, a_1)f(b_2, a_2)} \cdot \frac{f(a_1, b_2)}{f(b_2, a_1)}.\end{aligned}$$

Letting $\phi(ab) = f(b,a)$, we have a 2-coboundary β connected with ϕ such that

$$f(a_1b_1, a_2b_2) = \beta(a_1b_1, a_2b_2)f_A(a_1, a_2)f_B(b_1, b_2)\frac{f(a_1, b_2)}{f(b_2, a_1)}.$$

By the foregoing we have an isomorphism of rings $R(G,\sigma,f) \approx R(G,\sigma,\bar{f})$, where

$$\bar{f}(a_1b_1, a_2b_2) = f_A(a_1, a_2)f_B(b_1, b_2)\frac{f(a_1, b_2)}{f(b_2, a_1)}.$$

Next we need to prove that

$$R(G,\sigma,\bar{f}) = R(A,\sigma_A,f_A)(B,\overline{\sigma}_B,f_B).$$

To this end, let $\circ_G$, $\circ_A$, $\circ_B$ denote the multiplications in $R(G,\sigma,\bar{f})$, $R(A,\sigma_A,f_A)$ and $R(A,\sigma_A,f_A)(B,\overline{\sigma}_B,f_B)$, respectively. If $r_{a_1b_1}, r_{a_2b_2} \in R$, we have

$$\begin{aligned}r_{a_1b_1} \circ_B r_{a_2b_2} &= f_B(b_1, b_2)r_{a_1b_1}^{\overline{\sigma}(b_2)} \circ_A r_{a_2b_2}\\ &= f_B(b_1, b_2)\frac{f(a_1, b_2)}{f(b_2, a_1)} r_{a_1b_1}^{\sigma(b_2)} \circ_A r_{a_2b_2}\\ &= f_B(b_1, b_2)f_A(a_1, a_2)\frac{f(a_1, b_2)}{f(b_2, a_1)}(r_{a_1b_1}^{\sigma(b_2)})^{\sigma(a_2)} r_{a_2b_2}\\ &= \bar{f}(a_1b_1, a_2b_2)r_{a_1b_1}^{\sigma(a_2b_2)} r_{a_2b_2} = r_{a_1b_1} \circ_G r_{a_2b_2}. \ \blacksquare\end{aligned}$$

We now contrast the structure of our algebras R to their twisted counterparts $\widetilde{R}$.

Corollary 12. *Let* $R = \bigoplus_{g\in G} R_g$ *be a* G*-graded* K*-algebra, where* G *is a finitely generated abelian group, and let* $f \in Z^2(G, K^*)$. *Then*

(i) *If the orders of all the elements of* G *are either infinite or invertible in* K *and* R *is semiprime, then* $\widetilde{R}$ *is semiprime.*

(ii) R *is right Noetherian if and only if* $\widetilde{R}$ *is right Noetherian.*

(iii) R *has finite right Goldie dimension if and only if* $\widetilde{R}$ *has finite right Goldie dimension.*

Proof. Since G is finitely generated, $G = T \oplus F$, where T is the torsion part of G and F is a torsion-free finitely generated subgroup of G. By Proposition 11, the ring $\widetilde{R} = R(G, 1, f)$ is isomorphic to $R(T, 1, f_T)(F, \overline{\sigma}_F, f_F)$. To prove part (i), note that the order $|T|$ is invertible in K, hence the crossed product $R*_{(1,f_T)} T$ is semiprime. Proposition 9 now implies that $R(T, 1, f_T)$ is semiprime. Therefore we can continue the proof with the same arguments for a torsion-free group F.

For parts (ii) and (iii), we note that Corollary 10 implies that if R either is right Noetherian or has finite right Goldie dimension, then the same properties hold for the ring $R(T, 1, f_T)$. The result now follows by the Hilbert basis theorem and the well known fact that a skew polynomial ring of automorphism type has the same Goldie dimension as the ring of coefficients. ■

Remark. It is not true that if R is prime then $\widetilde{R}$ is prime. Clearly, the field $\mathbb{C}$ of complex numbers is $\mathbb{Z}_2$-graded, where $\mathbb{C}_0 = \mathbb{R}$ and $\mathbb{C}_1 = \mathbb{R}i$. Consider the 2-cocycle f given by $f(k, l) = (-1)^{kl}$ for $k, l \in \mathbb{Z}_2$. It is easy to see that $\widetilde{\mathbb{C}}$ is isomorphic to the group algebra $\mathbb{R}[\mathbb{Z}_2]$, which is not prime.

For a subset X of G, let $L_X = \bigoplus_{x\in X} L_x$. Clearly, if H is a subgroup of G, then L_H can be viewed as a H-graded Lie color algebra with respect to the bicharacter ε_H, the restriction of ε to $H\times H$. We will need the following

Lemma 13. *Let* $L = \bigoplus_{g\in G} L_g$ *be a nilpotent* G*-graded Lie color algebra and* H *a subgroup of* G *such that* $L_{G\setminus H} \neq 0$. *Then* L *contains an ideal* M *containing* L_H *and a homogeneous element* $x \in L_{G\setminus H}$ *such that* $L = M \oplus Kx$.

Proof. We claim that if $L_{G\setminus H} \subseteq [L, L]$, then $L_{G\setminus H} = 0$. To see this, note that $L = L_H + L_{G\setminus H}$, $[L_H, L_H] \subseteq L_H$ and $[L_H, L_{G\setminus H}] \subseteq L_{G\setminus H}$. Hence if $L_{G\setminus H} \subseteq [L, L]$, then

$$L_{G\setminus H} \subseteq [L_{G\setminus H}, L_H] + [L_{G\setminus H}, L_{G\setminus H}] \subseteq [L_{G\setminus H}, L].$$

By iterating the above formula, we see that

$$L_{G\setminus H} \subseteq [\dots[[L_{G\setminus H}, L], L], \dots, L] = 0,$$

as claimed. Since $L/[L,L]$ is abelian, any graded subspace is automatically an ideal, thus we can find a subspace N of codimension one containing $L_H + [L,L]/[L,L]$ and complementary to a homogeneous element $\overline{x} \in L_{G\setminus H} + [L,L]/[L,L]$. Therefore if we let M be the inverse image of N, then M is an ideal with the desired property. ■

In the main result of this section, we will generalize the following result, which was proved in [BG1].

THEOREM 14. *Let R be a semiprime K-algebra acted on by a finite-dimensional nilpotent Lie superalgebra L such that if* $\operatorname{char} K = p$ *then L is restricted and if* $\operatorname{char} K = 0$ *then L acts on R as algebraic derivations and algebraic superderivations.*

(i) *If R^L is right Noetherian, then R is a Noetherian right R^L-module. In particular, R is right Noetherian and is a finitely generated right R^L-module.*

(ii) *If R^L is right Artinian, then R is an Artinian right R^L-module. In particular, R is right Artinian and is a finitely generated right R^L-module.*

(iii) *If R^L is finite-dimensional over K then R is also finite-dimensional over K.*

(iv) *If R^L has finite Goldie dimension as a right R^L-module, then R has finite Goldie dimension as a right R-module.*

(v) *If R^L has Krull dimension α as a right R^L-module, then R has Krull dimension α as a right R^L-module. Thus R has Krull dimension at most α as a right R-module.*

Before proving our main result, we first need a lemma which is actually part of Proposition 2.2 from [BG1].

LEMMA 15. *Suppose δ is a nilpotent skew derivation of a K-algebra R and let $R^{(\delta)} = \{r \in R \mid \delta(r) = 0\}$.*

(i) *If $R^{(\delta)}$ is right Artinian, then R is an Artinian right $R^{(\delta)}$-module. In particular, R is right Artinian and is a finitely generated right $R^{(\delta)}$-module.*

(ii) *If A is a subring of $R^{(\delta)}$ such that $R^{(\delta)}$ has Krull dimension α as a right A-module, then R has Krull dimension α as a right A-module.*

We can now prove the main result of this section.

THEOREM 16. *Let $R = \bigoplus_{g\in G} R_g$ be a semiprime K-algebra graded by a finitely generated abelian group G and suppose R is acted on by a finite-dimensional nilpotent Lie color algebra $L = \bigoplus_{g\in G} L_g$ such that if* $\operatorname{char} K = p$ *then L is restricted and if* $\operatorname{char} K = 0$ *then L acts by algebraic transformations.*

(i) *If R^L is right Noetherian, then R is a Noetherian right R^L-module. In particular, R is Noetherian and finitely generated as a right R^L-module.*

(ii) *If R^L is right Artinian, then R is an Artinian right R^L-module. In particular, R is Artinian and finitely generated as a right R^L-module.*

(iii) *If R^L is finite-dimensional over K, then R is finite-dimensional over K.*

(iv) *If R^L has finite Goldie dimension as a right R^L-module, then R has finite Goldie dimension as a right R-module.*

(v) *If R^L has Krull dimension α as a right R^L-module, then R has Krull dimension α as a right R^L-module. Thus R has Krull dimension at most α as a right R-module.*

Proof. In light of Corollary 12, we can immediately generalize parts (i), (iii), and (iv) of Theorem 14 from the action of Lie superalgebras to Lie color algebras. Therefore it remains to prove parts (ii) and (v). To this end, we have a decomposition $G = T \oplus F$, where T is the torsion part of G and F a torsion-free subgroup of G. If n is the order of T, let $F^n = \{g^n \mid g \in F\}$ and $H = T \oplus F^n$. Clearly, H is a subgroup of G of finite index, therefore G/H is finite. Now we consider the action of L_H on R_H. By Lemmas 13 and 15, it follows that if R^L is Artinian (or has Krull dimension), then R^{L_H} has the same property. (Recall that every homogeneous $x \in L_{G\setminus H}$ acts as a nilpotent skew derivation.) Note that R^{L_H} is G-graded and $R_H^{L_H} = R_H \cap R^{L_H}$, thus $R_H^{L_H}$ is also Artinian (or has Krull dimension). We now claim that R_H has the same property. Observe that the subgroups T and F^n are orthogonal with respect to ε, that is, $\varepsilon(T, F^n) = 1$. This implies that L_T acts on R_H viewed as a T-graded algebra. Moreover, we can again apply Lemmas 13 and 15 to see that $R_H^{L_T}$ is Artinian (or has Krull dimension). By Theorem 14 and Corollary 15, we deduce that R_H is semiprime Artinian (or with Krull dimension). On the other hand, R can be viewed as a semiprime G/H-graded algebra with identity component R_H. Therefore, by the result of Cohen–Rowen [CR, Theorem 1.7], R is a finitely generated R_H-module, and so R is Artinian (or has Krull dimension). ∎

REFERENCES

[BG1] J. Bergen and P. Grzeszczuk, *Invariants of Lie superalgebras acting on associative algebras*, Israel J. Math. 94 (1996), 403–428.

[BG2] —, —, *Gradings, derivations, and automorphisms of some algebras which are nearly associative*, J. Algebra 179 (1996), 732–750.

[CR] M. Cohen and L. Rowen, *Group graded rings*, Comm. Algebra 11 (1983), 1253–1270.

[J] N. Jacobson, *A note on automorphisms and derivations of Lie algebras*, Proc. Amer. Math. Soc. 6 (1955), 281–283.

[M] S. Montgomery, *Hopf Algebras and Their Actions on Rings*, CBMS Regional Conf. Ser. in Math. 82, Amer. Math. Soc., Providence, RI, 1993.
[Sc] M. Scheunert, *Generalized Lie algebras*, J. Math. Phys. 20 (1979), 712–720.

Department of Mathematics
DePaul University
Chicago, IL 60614, U.S.A.
E-mail: jbergen@condor.depaul.edu

Institute of Mathematics
University of Białystok
Akademicka 2
15-267 Białystok, Poland
E-mail: piotrgr@cksr.ac.bialystok.pl

Received 1 June 1999;
revised 17 September 1999 (3774)