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ON A PAPER OF GUTHRIE AND NYMANN ON SUBSUMS OF INFINITE SERIES

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Abstract. In 1988 the first author and J. A. Guthrie published a theorem which characterizes the topological structure of the set of subsums of an infinite series. In 1998, while attempting to generalize this result, the second author noticed the proof of the original theorem was not complete and perhaps not correct. The present paper presents a complete and correct proof of this theorem.

In [1] the following theorem was presented.

THEOREM 1. *If E is the set of subsums of a positive term convergent series $\sum a_n$, then E is one of the following*:

(i) *a finite union of closed intervals*;
(ii) *homeomorphic to the Cantor set*;
(iii) *homeomorphic to the set T of subsums of $\sum \beta_n$ where $\beta_{2n-1} = 3/4^n$ and $\beta_{2n} = 2/4^n$ $(n = 1, 2, \ldots)$.*

(*Note that in* [2] *the requirement that $\sum a_n$ have only positive terms is removed.*)

While the present authors were attempting to generalize this theorem to the set of P-sums (see [2] for definitions) the second author raised serious questions about the proof of Theorem 1 given in [1]. The authors are now convinced that the earlier proof is certainly not complete and perhaps not correct. The purpose of the present note is to give a complete and correct proof of Theorem 1.

We shall follow the notation of [1] rather than [2]. Let $\sum a_n$ be a convergent series with $0 < a_{n+1} \le a_n$ for all n and let

$$E = \Big\{ \sum \varepsilon_n a_n : \varepsilon_n = 0 \text{ or } 1 \ (n = 1, 2, \ldots) \Big\}$$

denote its set of subsums. Also, let

$$r_n = \sum_{k=n+1}^{\infty} a_k$$

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denote the sum of the nth "tail" of the series and let s denote the sum of the series.

The following three facts about the set E are well known:

A. E is a perfect set.

B. E is a finite union of intervals if and only if $a_n \le r_n$ for n sufficiently large. (Also, E is an interval if and only if $a_n \le r_n$ for all n.)

C. If $a_n > r_n$ for n sufficiently large, then E is homeomorphic to the Cantor set.

We also introduce the following notation:

$$E_k = \Big\{ \sum_{n=k+1}^{\infty} \varepsilon_n a_n : \varepsilon_n = 0 \text{ or } 1 \ (n = 1, 2, \ldots) \Big\}$$

denotes the set of subsums of the k-tail of $\sum a_n$ and

$$F_k = \Big\{ \sum_{n=1}^{k} \varepsilon_n a_n : \varepsilon_n = 0 \text{ or } 1 \ (n = 1, \ldots, k) \Big\}$$

denotes the set of k-finite sums of $\sum a_n$.

Remark 1. Using the above notation the following decomposition for E is easy to see:

$$E = \bigcup_{f \in F_k} (f + E_k).$$

This tells us that E is a finite union of translates of the set of subsums of any tail of $\sum a_n$.

We will call a component of E an *interval* of E and a component of $[0, s] \setminus E$ will be called a *gap* of E.

The following lemma will be used in the proof of Theorem 1.

Lemma 2. *If (a, b) is a gap of E, then for some $\varepsilon > 0$ and $\varepsilon' > 0$,*

$$b + ([0, \varepsilon] \cap E) = [b, b + \varepsilon] \cap E$$

and

$$[s - \varepsilon', s] \cap E = (s - a) + ([a - \varepsilon', a] \cap E).$$

Proof. It is not difficult to see that b must be a finite subsum. Suppose

$$b = \sum_{i=1}^{k} \varepsilon_i a_i \in F_k$$

with $\varepsilon_k = 1$. Now suppose that the elements of F_k, in order, are

$$0 = f_1 < \ldots < f_t = \sum_{i=1}^{k} a_i$$

and suppose $b = f_j$.

Recall that

$$E = \bigcup_{i=1}^{t} (f_i + E_k).$$

From this we see that $a = f_{j-1} + r_k$.

Let $\varepsilon = \min(a_k, r_k, f_{j+1} - f_j)$. Then

$$b + ([0, \varepsilon] \cap E) = b + ([0, \varepsilon] \cap E_k) = [b, b + \varepsilon] \cap (b + E_k) = [b, b + \varepsilon] \cap E.$$

Now let $\varepsilon' = \min(a_k, r_k, f_{j-1} - f_{j-2})$. Since $f_{t-1} + r_k = s - a_k$,

$$[s - \varepsilon', s] \cap E = [s - \varepsilon', s] \cap (f_t + E_k).$$

Also

$$[a - \varepsilon', a] \cap E = [a - \varepsilon', a] \cap (f_{j-1} + E_k).$$

The conclusion follows after noting that $s - a = f_t - f_{j-1}$.

Proof of Theorem 1. Suppose that E is neither a finite union of intervals nor homeomorphic to the Cantor set. Then it is clear that E has infinitely many gaps. E must contain infinitely many intervals as well for if there were only finitely many, then either $[0, \varepsilon] \subset E$ for some $\varepsilon > 0$ or $E \cap [0, \varepsilon]$ contains no interval for some $\varepsilon > 0$. If the former is true, E would be a finite union of intervals, contrary to our initial supposition. (If E is not a finite union of intervals, then by B above, there is some k such that $a_k > r_k$ and $r_k < \varepsilon$. Thus $r_k < a_k \leq a_{k-1}$ and this implies $E \cap (r_k, a_k) = \emptyset$, a contradiction with $[0, \varepsilon] \subset E$.) If the latter holds, then $E \cap [0, \varepsilon]$ is homeomorphic to the Cantor set. Hence, there is some k such that E_k is homeomorphic to the Cantor set and hence E is homeomorphic to the Cantor set, contrary to our initial assumption.

In fact $E \cap [a, b]$ cannot be homeomorphic to the Cantor set for any $a, b \in [0, s]$ since for every k, E_k must contain (infinitely many) intervals. Suppose then that for some $x \in E$,

$$E \cap (x, x + \varepsilon) = \emptyset \quad \text{for some } \varepsilon > 0.$$

Then, since E is perfect, $E \cap (x - \varepsilon, x) \neq \emptyset$ for every $\varepsilon > 0$, and therefore there are intervals in E arbitrarily close to x; i.e. the union of the intervals of E is dense in E.

We now define a strictly increasing mapping f from the union of all intervals of T onto all intervals of E and also from all the gaps of T onto all the gaps of E. We define the mapping inductively. Begin by mapping the longest interval of T in a strictly increasing way onto the longest interval of E. There can be at most finitely many intervals of the same length in either set, so in case no one interval is the longest, we may choose the left-most interval. Denote this interval in T by I. Also, as part of the first induction step, we map the longest gap of T to the left (respectively right) of I in a strictly increasing way onto the longest gap of E to the left (respectively right) of $f[I]$. If more than one gap has the same length, we again use the "left-most rule" discussed above. This completes step one of the inductive definition of f.

After the nth step, $(4^n - 1)/3$ intervals of T will have been mapped in a strictly increasing way onto $(4^n - 1)/3$ intervals of E, and $2(4^n - 1)/3$ gaps of T will have been mapped in a strictly increasing way onto $2(4^n - 1)/3$ gaps of E. We now apply the process of the first induction step to each of the spaces between any two adjacent intervals and/or gaps (of which there are $4^n - 1$) and to the space between 0 and the left-most of the gaps and to the space between the right-most of these gaps and 5/3. (Note that $5/3 = \sum \beta_n$.) To be sure we can carry out step $n+1$ for $n = 1, 2, \ldots$, we need to know that in each of the 4^n spaces of T (and E) to which we apply the first inductive step, there are infinitely many intervals and gaps of T (and E). Lemma 2 guarantees this since there are infinitely many intervals and gaps of T (respectively E) in $[0, \varepsilon]$ and $[5/3 - \varepsilon, 5/3]$ (respectively $[0, \varepsilon]$ and $[s - \varepsilon, \varepsilon]$).

When f is defined in this way, it is a strictly increasing mapping of the union of all the intervals of T onto the union of all the intervals of E, and of the union of all the gaps of T onto the union of all the gaps of E. Earlier in the proof it was shown that the union of all the intervals of T (respectively E) is dense in T (respectively E). Hence the union of all the intervals and gaps of T (respectively E) is dense in $[0, 5/3]$ (respectively $[0, s]$). Thus f can be extended in a unique way to a strictly increasing mapping of $[0, 5/3]$ onto $[0, s]$ which maps T onto E. Then f restricted to T is the desired homeomorphism. (In the simpler inductive construction of f in [1], if x is a point of T not in an interval of T and $\langle x_n \rangle$ and $\langle x'_n \rangle$ are sequences in the intervals of T such that $x_n \nearrow x$ and $x'_n \searrow x$, then $\lim f(x_n)$ and $\lim f(x'_n)$ could be endpoints of a gap of E and the extension of f is not well defined. This is one of the major problems of the proof in [1].)

REFERENCES

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