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DIRECTING COMPONENTS FOR QUASITILTED ALGEBRAS

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Abstract. We show here that a directing component of the Auslander–Reiten quiver of a quasitilted algebra is either postprojective or preinjective or a connecting component.

Tilting theory, introduced in the 80's by Brenner–Butler [2] and Happel–Ringel [9], has been very important in the recent developments of the representation theory of algebras, the class of tilted algebras playing a central role. More recently, Happel–Reiten–Smalø [8] introduced the class of quasitilted algebras, which contains the tilted algebras, in order to give a general tilting theory for abelian categories and since then it has been the object of much investigation [3, 4, 5, 6, 8, 10, 13].

On the other hand, the Auslander–Reiten quiver Γ_A of an algebra A records much of the information on the category $\operatorname{mod} A$ of finitely generated A-modules, whence the importance of studying it (see [1]). Although the structure of the Auslander–Reiten quiver of a tilted algebra is well known (see for instance [11]), the same cannot be said yet for the quasitilted case. Partial results in this direction have been proven in [3, 6]. Also, the characterization of the tame quasitilted algebras given in [13] provides us with a complete description of the Auslander–Reiten quiver in this case.

The purpose of this note is to give a description of the components of the Auslander–Reiten quiver of a quasitilted algebra consisting of directing indecomposable modules, that is, modules which do not lie in oriented cycles. For tilted algebras, the directing components can be of the following three types: either postprojective or preinjective or a connecting component (see below for further definitions). It is shown here that the same holds for quasitilted algebras. As a consequence, if A is a quasitilted algebra which is not tilted, then the directing components are either postprojective or preinjective. The proof of this result will be given in the next section, after recalling some basic notions on quasitilted algebras.

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1. The result

1.1. Along this note, all algebras are finite-dimensional algebras over a fixed field k. For an algebra A, denote by $\operatorname{mod} A$ the category of finitely generated left A-modules, and by $\operatorname{ind} A$ the subcategory of $\operatorname{mod} A$ with one representative of each isoclass of indecomposable modules. Given an A-module X, denote by $\operatorname{pd}_A X$ and by $\operatorname{id}_A X$ its projective and injective dimension, respectively. The global dimension $\operatorname{gldim} A$ of A is the supremum of $\operatorname{pd}_A X$ with $X \in \operatorname{mod} A$. Let now X and Y be two indecomposable modules. A *path* from Y to X is a chain

$$Y = Y_1 \xrightarrow{f_1} Y_2 \to \ldots \to Y_{s-1} \xrightarrow{f_{s-1}} Y_s = X$$

with $s \geq 1$, and where for each i, f_i is a nonzero nonisomorphism and $Y_i \in \operatorname{ind} A$. In this case, we say that X is a *successor* of Y and that Y is a *predecessor* of X (observe that each indecomposable module is a successor and a predecessor of itself). An indecomposable module is called *directing* if it does not lie in an oriented cycle, that is, on a nontrivial path starting and ending at the same indecomposable module.

1.2. The Auslander–Reiten quiver Γ_A of an algebra A is defined as follows. The vertices are in one-to-one correspondence with the isoclasses of indecomposable A-modules (and so we shall not distinguish indecomposable modules and vertices of Γ_A). There is an arrow from X to Y in Γ_A if and only if there is an irreducible map from X to Y in $\operatorname{mod} A$. To each arrow there is assigned a pair of natural numbers (we shall not define them here) making Γ_A a valued quiver (see [1]). This quiver is also endowed with a translation τ_A, known as the Auslander–Reiten translation. A component Γ of Γ_A is called *postprojective* (respectively, *preinjective*) provided it does not contain oriented cycles and each module belonging to Γ lies in the τ_A-orbit of a projective (respectively, injective) module. These components are *directing*, that is, all their modules are directing. We refer the reader to [1] for unexplained notions in representation theory of algebras.

1.3. Recall that an algebra A is called *quasitilted* provided:

(i) $\operatorname{gldim} A \leq 2$; and
(ii) for an indecomposable module X, either $\operatorname{pd}_A X \leq 1$ or $\operatorname{id}_A X \leq 1$.

See [8] for an equivalent definition envolving tilting objects in abelian categories. Tilted algebras, as introduced by Happel–Ringel [9], satisfy these conditions. This latter class of algebras can be characterized by the existence of a component which contains a complete slice (see [9]). Such a component is directing and it is called *connecting*.

Denote by $\mathcal{L}_A$ (respectively, by $\mathcal{R}_A$) the full subcategory of mod A formed by those indecomposable modules X such that every predecessor (respectively, successor) Y of X has $\mathrm{pd}_A Y \le 1$ (respectively, $\mathrm{id}_A X \le 1$). It has been shown in [8] that if A is quasitilted, then the subcategories $\mathcal{L}_A$ and $\mathcal{R}_A$ induce a trisection in mod$A$, that is, the following properties hold:

(i) $\mathrm{ind}\, A = (\mathcal{L}_A \setminus \mathcal{R}_A) \cup (\mathcal{L}_A \cap \mathcal{R}_A) \cup (\mathcal{R}_A \setminus \mathcal{L}_A)$; and

(ii) $\mathrm{Hom}_A((\mathcal{R}_A \setminus \mathcal{L}_A), \mathcal{L}_A) = 0 = \mathrm{Hom}_A((\mathcal{L}_A \cap \mathcal{R}_A), (\mathcal{L}_A \setminus \mathcal{R}_A))$.

For details on tilted and quasitilted algebras, we refer the reader to [8, 9].

1.4. For the proof of our main result we shall need the following lemma proven in [4](3.1). We also mention that a particular case of it was proven in [8](II.3.1).

LEMMA. *Let A be a tilted algebra with a connecting component Γ which is neither postprojective nor preinjective. Then $\mathcal{L}_A \cap \mathcal{R}_A \subset \Gamma$.*

1.5. Our main result is as follows.

THEOREM. *Let A be a quasitilted algebra and let Γ be a directing component of Γ_A. Then Γ is either postprojective or preinjective or a connecting component.*

Proof. We use induction on the number n of simple A-modules, the case $n = 1$ being trivially true.

Suppose $n > 1$. If A is tilted, then there is nothing to prove [11]. So assume that A is not tilted. It then follows from the results of [6] that Γ is entirely contained in one of the subcategories $\mathcal{L}_A \setminus \mathcal{R}_A$, $\mathcal{L}_A \cap \mathcal{R}_A$ or $\mathcal{R}_A \setminus \mathcal{L}_A$. If $\Gamma \subset \mathcal{L}_A \cap \mathcal{R}_A$, then there exists a directing indecomposable module lying in $\mathcal{L}_A \cap \mathcal{R}_A$, and so, by [12], A would be tilted, a contradiction to our hypothesis.

Suppose now that $\Gamma \subset \mathcal{L}_A \setminus \mathcal{R}_A$. By [6](Theorem D), Γ has no injective modules and so it is right stable, that is, $\tau_A^j X \neq 0$ for each $j < 0$ and each $X \in \Gamma$.

Let $X \in \Gamma$. Since $X \notin \mathcal{R}_A$, there exists a path from X to an indecomposable module of injective dimension 2. Recall that an indecomposable module Y has injective dimension greater than one if and only if $\mathrm{Hom}_A(\tau_A^{-1} Y, A) \neq 0$ (see [1]). Therefore, for each $X \in \Gamma$, there exists a path from X to an indecomposable projective module.

Since Γ is directing and right stable, we infer that there exist indecomposable projective modules lying in components other than Γ which are successors of all modules of Γ. Clearly, there are no paths from such projective modules to any module in Γ.

Let now P be an indecomposable projective module with the above conditions and assume that $\mathrm{Hom}_A(P,Q)=0$ for all indecomposable projective modules Q not isomorphic to P. Observe that one can make such a choice because quasitilted algebras are triangular (see [8](III.1.1)). Hence, A is a one-point extension of an algebra B by the module $M=\operatorname{rad} P$, that is,

$$A=B[M]=\begin{pmatrix} A & M \\ 0 & k \end{pmatrix}.$$

By construction, the number of simple B-modules is n–1. Also, by [8](III.2.3, III.2.4), B is quasitilted and $M\in \operatorname{add}\mathcal{L}_B$. By the choice of the projective P, the component Γ can be identified with a component of Γ_B, which we shall also denote by Γ. Moreover, Γ is a right stable directing component in Γ_B. So, by the induction hypothesis, Γ is either postprojective or preinjective or a connecting component. Observe that Γ cannot be preinjective since it is right stable. On the other hand, if Γ is a postprojective component of Γ_B, then by [7], Γ is a postprojective component of Γ.

The remaining case to be considered is when Γ is a connecting component of Γ_B which is neither postprojective nor preinjective. Then, by 1.4, $\mathcal{L}_B\cap\mathcal{R}_B\subset\Gamma$. Observe that there exists an indecomposable summand M' of M which is a successor of all modules $\Gamma\subset\Gamma_B$ and does not lie in Γ. In particular, $M'\in\mathcal{R}_B$ because $\mathcal{R}_B$ is closed under successors. Now, since $\mathcal{L}_B\cap\mathcal{R}_B\subset\Gamma$, we infer that $M'\notin\mathcal{L}_B$, a contradiction to the fact that $M\in\operatorname{add}\mathcal{L}_B$.

The case $\Gamma\subset\mathcal{R}_A\setminus\mathcal{L}_A$ is similar and the result is proven. ■

References

[1] M. Auslander, I. Reiten and S. O. Smalø, *Representation Theory of Artin Algebras*, Cambridge Univ. Press, 1995.

[2] S. Brenner and M. Butler, *Generalizations of the Bernstein–Gelfand–Ponomarev reflection functors*, in: Proc. ICRA II, Lecture Notes in Math. 832, Springer, 1980, 103–169.

[3] F. U. Coelho and D. Happel, *Quasitilted algebras admit a preprojective component*, Proc. Amer. Math. Soc. 125 (1997), 1283–1291.

[4] F. U. Coelho, Ma. I. R. Martins and J. A. de la Peña, *Quasitilted extensions of algebras I*, Proc. Amer. Math. Soc., to appear.

[5] —, —, —, *Quasitilted extensions of algebras II*, J. Algebra, to appear.

[6] F. U. Coelho and A. Skowroński, *On Auslander–Reiten components for quasitilted algebras*, Fund. Math. 149 (1996), 67–82.

[7] P. Dräxler and J. A. de la Peña, *On the existence of postprojective components in the Auslander–Reiten quiver of an algebra*, Tsukuba J. Math. 20 (1996), 457–469.

[8] D. Happel, I. Reiten and S. Smalø, *Tilting in abelian categories and quasitilted algebras*, Mem. Amer. Math. Soc. 575 (1996).

[9] D. Happel and C. Ringel, *Tilted algebras*, Trans. Amer. Math. Soc. 274 (1982), 399–443.
[10] H. Lenzing and A. Skowroński, *Quasi-tilted algebras of canonical type*, Colloq. Math. 71 (1996), 161–181.
[11] S. Liu, *The connected components of the Auslander–Reiten quiver of a tilted algebra*, J. Algebra 61 (1993), 505–523.
[12] J. A. de la Peña and I. Reiten, *Trisection of module categories*, to appear.
[13] A. Skowroński, *Tame quasi-tilted algebras*, J. Algebra 203 (1998), 470–490.

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