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ASYMPTOTICS OF SUMS OF SUBCOERCIVE OPERATORS

BY

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Abstract. We examine the asymptotic, or large-time, behaviour of the semigroup kernel associated with a finite sum of homogeneous subcoercive operators acting on a connected Lie group of polynomial growth. If the group is nilpotent we prove that the kernel is bounded by a convolution of two Gaussians whose orders correspond to the highest and lowest orders of the homogeneous subcoercive components of the generator. Moreover we establish precise asymptotic estimates on the difference of the kernel and the kernel corresponding to the lowest order homogeneous component. We also prove boundedness of a range of Riesz transforms with the range again determined by the highest and lowest orders. Finally we analyze similar properties on general groups of polynomial growth and establish positive results for local direct products of compact and nilpotent groups.

1. Introduction. There have been two different approaches to the asymptotic analysis of strongly elliptic or subcoercive operators H, the first through bounds on the corresponding semigroup kernels (see [Dav], [Rob] or [VSC] for background information), and the second through asymptotic expansions [NRS]. The first approach has been largely restricted to homogeneous operators with the aim of establishing Gaussian bounds valid for all times. Barbatis and Davies [BaD] pointed out, however, that the kernel of the simplest inhomogeneous operator, the sum of two distinct powers of the Laplacian on $\mathbb{R}^d$, is a convolution of Gaussians. They then established that although the higher order term determines the short-time behaviour the lower order term is important for the long-time distribution. Our aim is to analyze this phenomenon for sums of homogeneous subcoercive operators H_{m_i} of different orders m_i acting on Lie groups G of polynomial growth. If, for example, the group G is nilpotent then we show that the kernel is bounded by a convolution of two Gaussians, the first of order $m = \max m_i$ and the second of order $\underline{m} = \min m_i$: the short and long time behaviours are governed by the orders m and $\underline{m}$, respectively, and the kernel can be bounded by a single Gaussian if, and only if, $m = \underline{m}$, or G is compact. The last result illustrates that asymptotic analysis through simple Gaussian bounds is not suited to the study of inhomogeneous operators.

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The second approach to asymptotic analysis through asymptotic expansions originated with the work of Nagel–Ricci–Stein [NRS] and has been analyzed for nilpotent Lie groups in [DERS]. The method consists in constructing asymptotic approximates, G_∞ and H_∞, of G and H by a scaling limit. In the simplest case $G = G_\infty$ the kernel of H_∞ gives the leading term in an asymptotic expansion for the kernel of H. But the situation is more complicated for non-homogeneous groups with $G \neq G_\infty$. We show, however, that for a general nilpotent group the leading term in the asymptotic expansion can be identified as the kernel associated with the lowest order term $H_{\underline{m}}$ in H. Our argument does not require homogeneity of G and gives an optimal estimate for the remainder in the expansion (see Theorem 2.12).

It was shown in [ERS2] that for homogeneous real symmetric second-order operators the kernel and its derivatives satisfy good large time Gaussian bounds if, and only if, the group G is a local direct product, $G = K \times_l N$, of a compact group K and a nilpotent group N. Then Dungey [Dun] established that the kernels of a large class of homogeneous operators of order four or more have good Gaussian bounds if, and only if, $G = K \times_l N$. Hence it appears appropriate to begin the analysis of inhomogeneous operators on nilpotent groups N and the near nilpotent groups $K \times_l N$.

In Section 2 we consider nilpotent groups, in Section 3 we discuss why some of our conclusions are not necessarily valid for general groups of polynomial growth and in Section 4 we analyze local products $G = K \times_l N$. Since good asymptotic bounds for the derivatives of the kernels of second-order operators are related to boundedness of the Riesz transforms of all orders [ERS2] we also analyze these relationships for the inhomogeneous situation. But then there is a range of Riesz transforms to consider, a range delineated by the order of the singularities, local and global, of the semigroup kernel, i.e., by the parameters m and $\underline{m}$.

Throughout the following G denotes a connected Lie group with polynomial growth, (bi-invariant) Haar measure dg and Lie algebra $\mathfrak{g}$. One can associate a subelliptic right invariant distance $(g,h) \mapsto d'(g\,;h)$ with a fixed algebraic basis $a_1, \ldots, a_{d'}$ of $\mathfrak{g}$. Let $g \mapsto |g|' = d'(g\,;e)$, where e is the identity element of G, denote the corresponding modulus. Then the Haar measure $|B'(g\,;\varrho)|$ of the subelliptic ball $B'(g\,;\varrho) = \{h \in G : |gh^{-1}|' < \varrho\}$ is independent of g. Set $V(\varrho) = |B'(g\,;\varrho)|$. Next, for all $i \in \{1, \ldots, d'\}$ let $A_i = dL(a_i)$ denote the generator of left L translations acting on the classical function spaces in the direction a_i. Multiple derivatives are denoted with multi-index notation, e.g., if $\alpha = (i_1, \ldots, i_n) \in J(d') = \bigcup_{k=0}^{\infty} \{1, \ldots, d'\}^k$ then $A^\alpha = A_{i_1} \ldots A_{i_n}$ and $|\alpha| = n$. If $p \in [1, \infty]$, $n \in \mathbb{N}$ and the function space equals L_p then we set $L'_{p;n} = \bigcap_{|\alpha|=n} D(A^\alpha)$. (In general we adopt the notation of [Rob] and [ElR1].)

Next for all $r \in \mathbb{N}$ let $\mathfrak{g}(d', r)$ denote the nilpotent Lie algebra with d' generators which is free of step r. Thus $\mathfrak{g}(d', r)$ is the quotient of the free Lie algebra with d' generators by the ideal generated by the commutators of order at least $r + 1$. Further let $G(d', r)$ be the connected simply connected Lie group with Lie algebra $\mathfrak{g}(d', r)$. It is automatically a non-compact group. We call $G(d', r)$ the nilpotent Lie group on d' generators free of step r and use the notation $\widetilde{\mathfrak{g}} = \mathfrak{g}(d', r)$, $\widetilde{G} = G(d', r)$ for brevity. Generally we add a tilde to distinguish between quantities associated with $\widetilde{G}$ and those associated with G. For example, we denote the generators of $\widetilde{\mathfrak{g}}$ by $\widetilde{a}_1, \ldots, \widetilde{a}_{d'}$. We also set $L_p = L_p(G\,; dg)$ and $L_{\widetilde{p}} = L_p(\widetilde{G}\,; d\widetilde{g})$ and denote the corresponding norms by $\|\cdot\|_p$ and $\|\cdot\|_{\widetilde{p}}$. Then the norm of an operator X on L_p is denoted by $\|X\|_{p\to p}$ and the norm of an operator $\widetilde{X}$ on $L_{\widetilde{p}}$ by $\|\widetilde{X}\|_{\widetilde{p}\to\widetilde{p}}$. One simple example of this construction is for the Abelian nilpotent group $G = \mathbb{T}^n$. Then $\widetilde{G} = \mathbb{R}^n$.

Let m be an even positive integer and for every multi-index α with $|\alpha| = m$ let $c_\alpha \in \mathbb{C}$. The homogeneous mth order operator

$$H_m = \sum_{|\alpha|=m} c_\alpha A^\alpha ,$$

with domain $D(H_m) = L'_{p;m}$, is defined [ElR1] to be *subcoercive of step* r if the comparison operator

$$\widetilde{H}_m = \sum_{|\alpha|=m} c_\alpha \widetilde{A}^\alpha$$

satisfies a Gårding inequality on $L_{\widetilde{2}}$, i.e., there exists a $\widetilde{\mu}_m > 0$ such that

$$\mathrm{Re}(\widetilde{\varphi}, \widetilde{H}_m \widetilde{\varphi}) \geq \widetilde{\mu}_m \sum_{|\alpha|=m/2} \|\widetilde{A}^\alpha \widetilde{\varphi}\|_{\widetilde{2}}^2 \tag{1}$$

uniformly for all $\widetilde{\varphi} \in C_c^\infty(\widetilde{G})$. We let μ_m denote the largest value of $\widetilde{\mu}_m$ for which this is satisfied and refer to this as the *ellipticity constant*. Note that it follows from this definition that there is a $\theta_m \in \langle 0, \pi/2]$ such that $e^{i\theta} H$ is subcoercive of step r for all $\theta \in \langle -\theta_m, \theta_m \rangle$. It also follows, but this is less evident, that subcoercivity of step r implies subcoercivity of step s for all $s \leq r$ (see [ElR3], Corollary 3.6).

Now let $\{m_j\}_{1 \leq j \leq k}$ be a family of even positive integers with $m = m_1 > \ldots > m_k = \underline{m}$. We consider inhomogeneous operators

$$H = \sum_{j=1}^{k} H_{m_j} ,$$

again with domain $D(H) = L'_{p;m}$, and now H is defined to be *strongly subcoercive of step* r if each of the homogeneous components H_{m_j} is subcoercive of step r. The highest order m and the lowest order $\underline{m}$ of the operators occurring in the sum will play a key role in all subsequent estimates. Then the operator $\overline{H}$ generates a holomorphic semigroup S with a kernel K.

2. Nilpotent groups. In this section we assume that G is a connected nilpotent Lie group of rank r. Our first aim is to prove the following theorem.

THEOREM 2.1. *Assume G is a connected nilpotent Lie group with Lie algebra of rank r and that the inhomogeneous operator H is strongly subcoercive of step r. The following are valid.*

I. *For all $\alpha \in J(d')$ and $j \in \{1, \ldots, k\}$ one has $D(H^{|\alpha|/m_j}) \subseteq D(A^\alpha)$ and there exists a $c > 0$ such that*

$$\|A^\alpha \varphi\|_2 \le c\|H^{|\alpha|/m_j}\varphi\|_2$$

for all $\varphi \in D(H^{|\alpha|/m_j})$. In particular, for all $\alpha \in J(d')$ there exists a $c > 0$ such that

$$\|A^\alpha \varphi\|_2 \le c(\|H^{|\alpha|/m}\varphi\|_2 \wedge \|H^{|\alpha|/\underline{m}}\varphi\|_2)$$

for all $\varphi \in D(H^{|\alpha|/m}) \cap D(H^{|\alpha|/\underline{m}})$.

II. *For all $\alpha \in J(d')$ there exist $b, c > 0$ such that*

$$|(A^\alpha K_t)(g)| \le c(t^{-|\alpha|/m} \wedge t^{-|\alpha|/\underline{m}})(G^{(m)}_{b,t} * G^{(\underline{m})}_{b,t})(g)$$

for all $g \in G$ and $t > 0$, where $G^{(n)}_{b,t}(g) = V(t)^{-1/n} e^{-b((|g|')^n t^{-1})^{1/(n-1)}}$. Alternatively, for all $\alpha \in J(d')$ there exist $b, c > 0$ such that

$$|(A^\alpha K_t)(g)| \le c(t^{-|\alpha|/m} \wedge t^{-|\alpha|/\underline{m}})(V(t)^{-1/m} \wedge V(t)^{-1/\underline{m}})(e^{(m)}_{b,t}(g) \vee e^{(\underline{m})}_{b,t}(g))$$

for all $g \in G$ and $t > 0$, where $e^{(n)}_{b,t}(g) = e^{-bt(|g|' t^{-1})^{n/(n-1)}}$

REMARK 2.2. The Barbatis–Davies estimates, [BaD], Proposition 5.1, for sums of powers of the Laplacian on $\mathbb{R}^d$ correspond to bounds

$$|K_t(x)| \le V(t)^{-1/m}(e^{(m)}_{b,t}(g) \vee e^{(\underline{m})}_{b,t}(g)).$$

The last statement of the theorem optimizes the large time decay of these bounds.

Subsequently, in Theorem 2.12, we establish that the lowest order part $H_{\underline{m}}$ of H determines its asymptotic behaviour by deriving good large t estimates on the difference $K_t - K^{(\underline{m})}_t$, where $K^{(\underline{m})}$ is the kernel of the semigroup generated by $H_{\underline{m}}$. Despite the fact that K_t approaches $K^{(\underline{m})}_t$ asymptotically it is not usually bounded by a Gaussian of order $\underline{m}$, or any other order, uniformly for all t. This statement is made precise in Proposition 2.15.

The proof of Theorem 2.1 will be given in a series of lemmas, propositions and corollaries which give extra detail on the asymptotics. For example, Proposition 2.11 gives several alternative formulations of the kernel bounds.

The first useful observation is that subcoercivity combined with nilpotency implies the strong Gårding inequality.

LEMMA 2.3. *If H_m is a homogeneous subcoercive operator of step r and order m with ellipticity constant μ_m then*

$$\operatorname{Re}(\varphi, H_m\varphi) \geq \mu_m \sum_{|\alpha|=m/2} \|A^\alpha\varphi\|_2^2$$

for all $\varphi \in L'_{2;m}(G\,;dg)$.

Proof. Let $\Delta_m = \sum_{|\alpha|=m/2}(A^\alpha)^* A^\alpha$ on $L_2(G)$ and $\widetilde{\Delta}_m$ be the comparable operator on $L_2(\widetilde{G})$. If $\mu \in \langle 0, \mu_m\rangle$ then the Gårding inequality (1) for $\widetilde{H}_m$ on $L_2(\widetilde{G})$ implies that $\operatorname{Re}(\widetilde{H}_m - \mu\widetilde{\Delta}_m) \geq 0$. Thus the semigroup generated by $\widetilde{H}_m - \mu\widetilde{\Delta}_m$ is contractive. But then it follows from the transference arguments of [ERS1], Theorem 2.1 and Lemma 3.2, that the semigroup generated by $H_m - \mu\Delta_m$ is also contractive. Hence $\operatorname{Re}(H_m - \mu\Delta_m) \geq 0$ and the lemma follows. ■

It follows straightforwardly from this lemma that there are $\mu, \underline{\mu} > 0$ such that the strongly subcoercive, inhomogeneous operator H satisfies the estimates

$$\operatorname{Re}(\varphi, H\varphi) \geq \mu \sum_{|\alpha|=m/2} \|A^\alpha\varphi\|_2^2 + \underline{\mu} \sum_{|\alpha|=\underline{m}/2} \|A^\alpha\varphi\|_2^2 \tag{2}$$

for all $\varphi \in L'_{2;m}(G\,;dg)$. We call (2) the *strong Gårding inequality* for the (inhomogeneous) operator H.

The main idea in the subsequent analysis of the inhomogeneous operator H is the introduction of a second comparison system consisting of k copies of the original system weighted in such a way that H is a weighted homogeneous operator. To this end we introduce a family $c_1, \ldots, c_{kd'}$ of elements of $\mathfrak{g}$ which contains k copies of the algebraic basis $a_1, \ldots, a_{d'}$. Then for $l \in \{1, \ldots, kd'\}$ we consider the elements c_l of $\mathfrak{g}$ with weights w_l defined by $c_{(j-1)d'+i} = a_i$ and $w_{(j-1)d'+i} = m/m_j$ for $i \in \{1, \ldots, d'\}$ and $j \in \{1, \ldots, k\}$. The weighted length $\|\alpha\|$ of the multi-index $\alpha = (l_1, \ldots, l_n) \in J(kd')$ is defined by $\|\alpha\| = \sum_{p=1}^n w_{l_p}$. Then with these definitions one can write H in the form

$$H = \sum_{\|\alpha\|=m} b_\alpha C^\alpha.$$

The component H_{m_j} of H is expressed in terms of the jth copy of the algebraic basis $a_1, \ldots, a_{d'}$ and although it has unweighted order m_j it has weighted order m. Thus H is homogeneous with respect to the weighted structure with weighted order m.

Next consider the nilpotent Lie algebra $\widetilde{\mathfrak{g}}_k$ which is free of (unweighted) step r and with generators $\widetilde{c}_1, \ldots, \widetilde{c}_{kd'}$. Thus if $\alpha = (l_1, \ldots, l_n)$ then the (unweighted) order of the commutator

$$\widetilde{c}_{[\alpha]} = [\widetilde{c}_{l_1}, [\ldots [\widetilde{c}_{l_{n-1}}, \widetilde{c}_{l_n}] \ldots]]$$

is defined to be n and all commutators in $\widetilde{\mathfrak{g}}_k$ of order greater than or equal to $r+1$ are assumed to vanish. Thus $\widetilde{\mathfrak{g}}_k$ is the quotient of the free Lie algebra with kd' generators $\widetilde{c}_l$, with weights w_l, by the ideal generated by the commutators of unweighted order at least $r+1$. Note that the maps $\widetilde{\gamma}_t(\widetilde{c}_l) = t^{w_l}\widetilde{c}_l$, with $t > 0$, extend to dilations on $\widetilde{\mathfrak{g}}_k$. (Cf. [ElR4], Example 2.7.) Let $\widetilde{G}_k$ denote the connected simply connected homogeneous Lie group with Lie algebra $\widetilde{\mathfrak{g}}_k$ and $|\cdot|'$ the modulus on $\widetilde{G}_k$ associated with the algebraic basis $\widetilde{c}_1, \ldots, \widetilde{c}_{kd'}$ and weights $w_1, \ldots, w_{kd'}$.

One can now define the natural extension $\widetilde{H}$ of H to the spaces $L_p(\widetilde{G}_k)$ by

$$\widetilde{H} = \sum_{\|\alpha\|=m} b_\alpha \widetilde{C}^\alpha.$$

The operator $\widetilde{H}$ is again homogeneous with weighted order m and the next lemma states that it is a subcoercive operator on $L_2(\widetilde{G}_k)$.

Lemma 2.4. *If the inhomogeneous operator H is strongly subcoercive of step r then the homogeneous weighted operator $\widetilde{H}$ is weighted subcoercive on $L_2(\widetilde{G}_k)$, i.e., there is a $\mu > 0$ such that $\widetilde{H}$ satisfies the Gårding inequality*

$$\operatorname{Re}(\widetilde{\varphi}, \widetilde{H}\widetilde{\varphi}) \geq \mu \sum_{\|\alpha\|=m/2} \|\widetilde{C}^\alpha \widetilde{\varphi}\|_2^2$$

uniformly for all $\widetilde{\varphi} \in C_c^\infty(\widetilde{G}_k)$.

Proof. The lemma is a weighted version of Lemma 3.10 of [ElR3], using [ElR4], Theorem 9.2.IV, instead of [ElR3], Theorem 3.3.III. ■

The operator H can now be analyzed by examining the homogeneous operator $\widetilde{H}$ on the free group $\widetilde{G}_k$ and then projecting down to G as in [ERS1]. The projection technique requires the introduction of an appropriate homomorphism from $\widetilde{\mathfrak{g}}_k$ to $\mathfrak{g}$. There exists a unique Lie algebra homomorphism $\Lambda : \widetilde{\mathfrak{g}}_k \to \mathfrak{g}$ such that $\Lambda(\widetilde{c}_l) = c_l$ for all $l \in \{1, \ldots, kd'\}$ and this lifts to a homomorphism $\pi : \widetilde{G}_k \to G$ by the exponential map. Explicitly,

$$\pi = \exp \circ \Lambda \circ \widetilde{\exp}^{-1}$$

where $\widetilde{\exp} : \widetilde{\mathfrak{g}}_k \to \widetilde{G}_k$ and $\exp : \mathfrak{g} \to G$. For any finite measure $\widetilde{\mu}$ on $\widetilde{G}_k$ let $\pi_*(\widetilde{\mu})$ denote the image measure on G. Then the map $\pi_* : M(\widetilde{G}) \to M(G)$ is also contractive (see [ERS1], Section 2).

Using transference techniques one can next prove the first statement of Theorem 2.1.

Proof of Theorem 2.1.I. We follow the reasoning of [ERS1], Section 4. First for all $\beta \in J(kd')$ introduce the regularized transforms

$$\widetilde{R}_{\beta;\nu,\varepsilon} = \widetilde{C}^\beta(\nu I + \widetilde{H})^{-\|\beta\|/m}(I + \varepsilon\widetilde{H})^{-N}$$

with $\varepsilon > 0$ and N a suitably large positive integer. The factor $(I + \varepsilon\widetilde{H})^{-N}$ reduces the singularity of the kernels $\widetilde{k}_{\beta;\nu,\varepsilon}$ of these operators by the introduction of a factor $\widetilde{g} \mapsto (|\widetilde{g}|')^{Nm}$. Therefore if N is sufficiently large the kernels are integrable although the norms $\|\widetilde{k}_{\beta;\nu,\varepsilon}\|_1$ diverge as $\nu \downarrow 0$ or $\varepsilon \downarrow 0$. But $\widetilde{R}_{\beta;\nu,\varepsilon}$ is bounded on $L_{\widetilde{2}} = L_2(\widetilde{G}_k)$ uniformly in ν and ε. In particular

$$\|\widetilde{R}_{\beta;\nu,\varepsilon}\|_{\widetilde{2}\to\widetilde{2}} \le \|\widetilde{C}^\beta(\nu I + \widetilde{H})^{-\|\beta\|/m}\|_{\widetilde{2}\to\widetilde{2}} = \|\widetilde{C}^\beta(I + \widetilde{H})^{-\|\beta\|/m}\|_{\widetilde{2}\to\widetilde{2}}$$

where the estimate follows from contractivity and the equality by scaling.

Now if $k_{\beta;\nu,\varepsilon}$ is the kernel of the operator

$$R_{\beta;\nu,\varepsilon} = C^\beta(\nu I + H)^{-\|\beta\|/m}(I + \varepsilon H)^{-N}$$

one has $k_{\beta;\nu,\varepsilon} = \pi_*(\widetilde{k}_{\beta;\nu,\varepsilon})$, where we identify L_1-functions with complex measures, and hence

$$\|R_{\beta;\nu,\varepsilon}\|_{2\to 2} = \|L_G(k_{\beta;\nu,\varepsilon})\|_{2\to 2} \le \|L_{\widetilde{G}_k}(\widetilde{k}_{\beta;\nu,\varepsilon})\|_{\widetilde{2}\to\widetilde{2}} = \|\widetilde{R}_{\beta;\nu,\varepsilon}\|_{\widetilde{2}\to\widetilde{2}}.$$

So the norm of $R_{\beta;\nu,\varepsilon}$ is bounded uniformly in ν and ε on $L_2(G)$. Then, taking limits as in the proof of Lemma 4.2 of [ERS1], but using Theorem 9.2.IV of [ElR4] instead of Theorem 3.3.III of [ElR3], one deduces that $D(H^{\|\beta\|/m}) \subseteq D(C^\beta)$ and

$$\|C^\beta\varphi\|_2 \le \|\widetilde{C}^\beta(I + \widetilde{H})^{-\|\beta\|/m}\|_{\widetilde{2}\to\widetilde{2}}\|H^{\|\beta\|/m}\varphi\|_2$$

for all $\varphi \in D(H^{\|\beta\|/m})$ and $\beta \in J(kd')$.

Finally let $\alpha = (i_1, \ldots, i_n) \in J(d')$ and $j \in \{1, \ldots, k\}$. Introduce the multi-index β by $\beta = ((j-1)d' + i_1, \ldots, (j-1)d' + i_n)$. Then $D(H^{|\alpha|/m_j}) = D(H^{\|\beta\|/m}) \subseteq D(C^\beta) = D(A^\alpha)$ and

$$\begin{aligned}\|A^\alpha\varphi\|_2 = \|C^\beta\varphi\|_2 &\le \|\widetilde{C}^\beta(I + \widetilde{H})^{-\|\beta\|/m}\|_{\widetilde{2}\to\widetilde{2}}\|H^{\|\beta\|/m}\varphi\|_2 \\ &= \|\widetilde{C}^\beta(I + \widetilde{H})^{-\|\beta\|/m}\|_{\widetilde{2}\to\widetilde{2}}\|H^{|\alpha|/m_j}\varphi\|_2\end{aligned}$$

for all $\varphi \in D(H^{|\alpha|/m_j})$. ∎

The foregoing proof has two immediate corollaries.

COROLLARY 2.5. *For all $n \in \mathbb{N}$ and all $\alpha \in J(d')$ with $n\underline{m} \le |\alpha| \le nm$ there exists a $c > 0$ such that*

$$\|A^\alpha\varphi\|_2 \le c\|H^n\varphi\|_2$$

for all $\varphi \in D(H^n)$.

Proof. It follows as in the proof of Lemma III.3.3 of [Rob] that there exists a $c > 0$ such that

$$\|A^\alpha\varphi\|_2 \le c(\max_{|\beta|=n\underline{m}} \|A^\beta\varphi\|_2 + \max_{|\gamma|=nm} \|A^\gamma\varphi\|_2)$$

for all $\varphi \in C_c^\infty(G)$. Then the corollary follows from Theorem 2.1.I and density. ■

Corollary 2.6. *For all $n \in \mathbb{N}$ and $j \in \{1, \ldots, k\}$ there exists a $c > 0$ such that*

$$\|A^\alpha\varphi\|_2 \le \varepsilon^{nm_j-|\alpha|}\|H^n\varphi\|_2 + c\varepsilon^{-|\alpha|}\|\varphi\|_2$$

for all $\alpha \in J(d')$ with $|\alpha| < nm_j$, $\varepsilon > 0$ and $\varphi \in D(H^n)$.

Proof. This follows from the subelliptic analogue of [Rob], Lemma III.3.3, and Corollary 2.5. ■

Our next aim is to prove the second statement of Theorem 2.1, the kernel bounds, and to this end we examine the Davies perturbation

$$S_t^\varrho = U_\varrho S_t U_\varrho^{-1}$$

of the semigroup S where $\psi \in C_b^\infty(G)$ is real-valued, $\varrho \in \mathbb{R}$ and U_ϱ denotes the operator of multiplication by the function $e^{-\varrho\psi}$ on L_2. Following Dungey [Dun] we consider a one-parameter family $(\psi_R)_{R>0}$ of functions defined by

$$\psi_R = R\eta_R$$

where the η_R are cutoff functions of the type considered in [ERS2], Section 2. These are a family of C^∞-functions $(\eta_R)_{R>0}$ for which there exist $\sigma > 0$ and for all multi-indices α a $c_\alpha > 0$ such that $\operatorname{supp}\eta_R \subset B'_R$, $0 \le \eta_R \le 1$, $\eta_R(g) = 1$ for all $g \in B'_{\sigma R}$ and

$$\|A^\alpha\eta_R\|_\infty \le c_\alpha R^{-|\alpha|} \tag{3}$$

uniformly for $R > 0$ and $\alpha \in J(d')$. (These cutoff functions exist because G is nilpotent, [ERS2], Theorem 4.5.)

Now let H_ϱ denote the corresponding Davies perturbation of H,

$$H_\varrho = U_\varrho H U_\varrho^{-1}$$

where U_ϱ is now the operator of multiplication with $e^{-\varrho\psi_R}$ and, for simplicity, we omit any notational dependence on R. Then for each $n \in \mathbb{N}$ it is clear that $H_\varrho^n - H^n$ is a polynomial in the A_i of (unweighted) order $nm - 1$ with coefficients which are polynomials in ϱ of order at most nm. But

$$U_\varrho A_i U_\varrho^{-1} = A_i + \varrho(A_i\psi_R) = A_i + \varrho R(A_i\eta_R) \tag{4}$$

and the special properties of the cutoff functions lead to the following estimates.

LEMMA 2.7. *There exists a* $c>0$ *such that*

$$|(\varphi,H_\varrho\varphi)-(\varphi,H\varphi)|\le\varepsilon\operatorname{Re}(\varphi,H\varphi)+c\sum_{j=1}^{k}\varepsilon^{-m_j+1}|\varrho|^{m_j}\|\varphi\|_2^2$$

for all $\varphi\in C_c^\infty(G)$ *uniformly for* $\varepsilon\in\langle 0,1]$, $R\in\langle 0,\infty\rangle$ *and* $\varrho\in\mathbb{R}$ *with* $|\varrho|\ge R^{-1}$.

Proof. If $H_{j,\varrho}=U_\varrho H_{m_j}U_\varrho^{-1}$ then it follows as in [Dun], Proposition 4.1, using the strong Gårding inequality (2), that for all $j\in\{1,\ldots,k\}$ there exists a $c>0$ such that

$$|(\varphi,H_{j,\varrho}\varphi)-(\varphi,H_{m_j}\varphi)|\le\varepsilon\operatorname{Re}(\varphi,H\varphi)+c\varepsilon^{-m_j+1}|\varrho|^{m_j}\|\varphi\|_2^2$$

for all $\varphi\in C_c^\infty(G)$ uniformly for $\varepsilon\in\langle 0,1]$, $R\in\langle 0,\infty\rangle$ and $\varrho\in\mathbb{R}$ with $|\varrho|\ge R^{-1}$. Then the lemma follows by addition. ■

COROLLARY 2.8. *There exists a* $c>0$ *such that*

$$\operatorname{Re}(\varphi,H_\varrho\varphi)\ge 2^{-1}\operatorname{Re}(\varphi,H\varphi)-c(|\varrho|^m+|\varrho|^{\underline{m}})\|\varphi\|_2^2$$

and

$$|(\varphi,H_\varrho\varphi)|\le c\operatorname{Re}(\varphi,H\varphi)+c(|\varrho|^m+|\varrho|^{\underline{m}})\|\varphi\|_2^2$$

for all $\varphi\in D(H)$, $R\in\langle 0,\infty\rangle$ *and* $\varrho\in\mathbb{R}$ *with* $|\varrho|\ge R^{-1}$.

Proof. This follows from Lemmas 2.3 and 2.7. ■

Next introduce θ_H by

$$\theta_H=\sup\{\theta\in\langle 0,\pi/2]:\ \forall_{\eta\in[-\theta,\theta]}[e^{i\eta}H \text{ is a strongly subcoercive operator of step } r]\}.$$

Thus θ_H is a lower bound for the angle of the sector on which S is holomorphic.

LEMMA 2.9. *There exist* $c,\omega>0$ *and* $\theta_0\in\langle 0,\theta_H\rangle$ *such that*

$$\|S_z^\varrho\|_{2\to 2}\le e^{\omega(|\varrho|^m+|\varrho|^{\underline{m}})|z|}\quad\text{and}\quad\|H_\varrho S_t^\varrho\|_{2\to 2}\le ct^{-1}e^{\omega(|\varrho|^m+|\varrho|^{\underline{m}})t}$$

for all $t>0$, $z\in\mathbb{C}\setminus\{0\}$ *with* $|\arg z|\le\theta_0$, $R\in\langle 0,\infty\rangle$ *and* $\varrho\in\mathbb{R}$ *with* $|\varrho|\ge R^{-1}$.

Proof. Let $c>0$ be as in Corollary 2.8. Then for all $z\in\mathbb{C}$ with $|\arg z|\le\theta_0=2^{-1}\theta_H\wedge\arctan(2c)^{-1}$ and $\varphi\in L_2$ one has

$$\begin{aligned}\frac{d}{dt}\|S^\varrho_{e^{i\theta}t}\varphi\|_2^2&=-2\operatorname{Re}(S^\varrho_{e^{i\theta}t}\varphi,e^{i\theta}H_\varrho S^\varrho_{e^{i\theta}t}\varphi)\\&\le-2\cos\theta\operatorname{Re}(S^\varrho_{e^{i\theta}t}\varphi,H_\varrho S^\varrho_{e^{i\theta}t}\varphi)+2|\sin\theta|\cdot|(S^\varrho_{e^{i\theta}t}\varphi,H_\varrho S^\varrho_{e^{i\theta}t}\varphi)|\\&\le-2\cos\theta(2^{-1}\operatorname{Re}(S^\varrho_{e^{i\theta}t}\varphi,HS^\varrho_{e^{i\theta}t}\varphi)-c(|\varrho|^m+|\varrho|^{\underline{m}})\|S^\varrho_{e^{i\theta}t}\varphi\|_2^2)\\&\quad+2|\sin\theta|(c\operatorname{Re}(S^\varrho_{e^{i\theta}t}\varphi,HS^\varrho_{e^{i\theta}t}\varphi)+c(|\varrho|^m+|\varrho|^{\underline{m}})\|S^\varrho_{e^{i\theta}t}\varphi\|_2^2)\\&\le 4c(|\varrho|^m+|\varrho|^{\underline{m}})\|\varphi\|_2^2\end{aligned}$$

for all $t > 0$. Hence $\|S_z^\varrho\|_{2\to2} \le e^{2c(|\varrho|^m+|\varrho|^{\underline{m}})|z|}$ uniformly for all $z \in \mathbb{C}\setminus\{0\}$ with $|\arg z| \le \theta_0$, $R \in \langle 0,\infty\rangle$ and $\varrho \in \mathbb{R}$ with $|\varrho| \ge R^{-1}$. Then using the Cauchy integral representation (see, for example, [Rob], Lemma III.4.4, or [Dav], Lemma 2.38) one obtains bounds

$$\|H_\varrho S_t^\varrho\|_{2\to2} \le c't^{-1}(1+\omega(|\varrho|^m+|\varrho|^{\underline{m}})t)e^{\omega(|\varrho|^m+|\varrho|^{\underline{m}})t}$$

uniformly for all $\varrho \in \mathbb{R}$ and all $t > 0$. The estimates of the lemma then follow by slightly increasing the value of ω. ∎

The following lemma is the key to estimating derivatives of the perturbed semigroup.

LEMMA 2.10. *For all $\alpha \in J(d')$ and $j \in \{1,\ldots,k\}$ there exists a $c > 0$ such that*

$$\|A^\alpha S_t^\varrho\|_{2\to2} \le ct^{-|\alpha|/m_j}e^{\omega(|\varrho|^m+|\varrho|^{\underline{m}})t}$$

for all $t > 0$, $R \in \langle 0,\infty\rangle$ and $\varrho \in \mathbb{R}$ with $|\varrho| \ge R^{-1}$.

Proof. Let $n \in \mathbb{N}$ be such that $nm_j > |\alpha|$. It follows by induction from (4) that for all $\beta \in J(d')$ with $n\underline{m} \le |\beta| \le nm$ there are $c_{\beta,\gamma,\gamma_1,\ldots,\gamma_N} \in \mathbb{R}$ such that

$$U_\varrho A^\beta U_\varrho^{-1}\varphi - A^\beta\varphi = \sum c_{\beta,\gamma,\gamma_1,\ldots,\gamma_N}(\varrho R)^N(A^{\gamma_1}\eta_R)\ldots(A^{\gamma_N}\eta_R)A^\gamma\varphi \tag{5}$$

where the sum is over all $N \in \{1,\ldots,|\beta|\}$, all $\gamma \in J(d')$ with $|\gamma| < |\beta|$ and $\gamma_1,\ldots,\gamma_N \in J^+(d')$ with $|\gamma_1|+\ldots+|\gamma_N|+|\gamma| = |\beta|$. Consider one term in this sum. Since $|\gamma_1|+\ldots+|\gamma_N|-N \ge 0$ and $R^{-1} \le |\varrho|$ one has

$$\begin{aligned}|(\varrho R)^N|\cdot\|(A^{\gamma_1}\eta_R)\ldots(A^{\gamma_N}\eta_R)A^\gamma\varphi\|_2 &\le |\varrho|^N c_{\gamma_1}\ldots c_{\gamma_N}R^{-(|\gamma_1|+\ldots+|\gamma_N|-N)}\|A^\gamma\varphi\|_2\\ &\le |\varrho|^{|\gamma_1|+\ldots+|\gamma_N|}c_{\gamma_1}\ldots c_{\gamma_N}\|A^\gamma\varphi\|_2\end{aligned} \tag{6}$$

by (3). But by Corollary 2.6 one has bounds

$$\|A^\gamma\varphi\|_2 \le \varepsilon^{|\beta|-|\gamma|}\|H^n\varphi\|_2 + c\varepsilon^{-|\gamma|}\|\varphi\|_2$$

uniformly for all $\varepsilon > 0$ and $|\gamma| < nm$. Hence

$$\begin{aligned}&|(\varrho R)^N|\cdot\|(A^{\gamma_1}\eta_R)\ldots(A^{\gamma_N}\eta_R)A^\gamma\varphi\|_2\\ &\qquad\le |\varrho|^{|\beta|-|\gamma|}c_{\gamma_1}\ldots c_{\gamma_N}(\varepsilon^{|\beta|-|\gamma|}\|H^n\varphi\|_2 + c\varepsilon^{-|\gamma|}\|\varphi\|_2)\end{aligned}$$

for all $\varepsilon > 0$. Therefore taking $\varepsilon = \delta|\varrho|^{-1}$, and adding the various terms, it follows that there is a $c' > 0$ such that

$$\|(H_\varrho^n - H^n)\varphi\|_2 \le c'(\delta\|H^n\varphi\|_2 + (|\varrho|^{nm}+|\varrho|^{n\underline{m}})\delta^{-nm}\|\varphi\|_2)$$

for all $\varphi \in D(H^n)$ and $\delta \in \langle 0,1]$. Choosing δ appropriately one deduces that there is a $c'' > 0$ such that

$$\|H^n\varphi\|_2 \le 2\|H_\varrho^n\varphi\|_2 + c''(|\varrho|^{nm}+|\varrho|^{n\underline{m}})\|\varphi\|_2$$

for all $\varphi \in D(H^n)$, $R \in \langle 0,\infty\rangle$ and $\varrho \in \mathbb{R}$ with $|\varrho| \ge R^{-1}$.

Next it follows from Corollary 2.6 and Lemma 2.9 that there exist $c, \omega > 0$ such that

$$\begin{aligned}
&\|A^\alpha S_t^\varrho\|_{2\to 2} \\
&\quad\le \varepsilon^{nm_j-|\alpha|}\|H^n S_t^\varrho\|_{2\to 2} + c\varepsilon^{-|\alpha|}\|S_t^\varrho\|_{2\to 2} \\
&\quad\le \varepsilon^{nm_j-|\alpha|}(2\|H_\varrho^n S_t^\varrho\|_{2\to 2} + c''(|\varrho|^{nm} + |\varrho|^{n\underline{m}})\|S_t^\varrho\|_{2\to 2}) \\
&\qquad + c\varepsilon^{-|\alpha|}\|S_t^\varrho\|_{2\to 2} \\
&\quad\le (\varepsilon^{nm_j-|\alpha|}(2(cnt^{-1})^n + c''(|\varrho|^{nm} + |\varrho|^{n\underline{m}})) + c\varepsilon^{-|\alpha|})e^{\omega(|\varrho|^m + |\varrho|^{\underline{m}})t}
\end{aligned}$$

for all $t > 0$, $\varepsilon > 0$, $R \in \langle 0, \infty\rangle$ and $\varrho \in \mathbb{R}$ with $|\varrho| \ge R^{-1}$. Then the lemma follows by setting $\varepsilon = t^{1/m_j}$ and making an elementary estimate. ■

We now have sufficient preparation to prove the second statement of Theorem 2.1, the kernel bounds.

Proof of Theorem 2.1.II. For each $m, n \in \mathbb{N}$, $t > 0$ and $b, \omega > 0$ with $m \ge n$ introduce the functions $G_{b,t}^{(n)}, N_{\omega,t}^{(m,n)}, E_{b,t}^{(m,n)} : G \to \mathbb{R}$ by

$$\begin{aligned}
G_{b,t}^{(n)}(g) &= V(t)^{-1/n} e^{-b((|g|')^n t^{-1})^{1/(n-1)}} = V(t)^{-1/n} e^{-bt(|g|' t^{-1})^{n/(n-1)}}, \\
N_{\omega,t}^{(m,n)}(g) &= (V(t)^{-1/m} \wedge V(t)^{-1/n}) \inf_{\varrho > 0} e^{-\varrho|g|' + \omega(\varrho^m + \varrho^n)t}
\end{aligned}$$

and

$$\begin{aligned}
E_{b,t}^{(m,n)}(g) &= (V(t)^{-1/m} \wedge V(t)^{-1/n}) \\
&\qquad\cdot (e^{-b((|g|')^m t^{-1})^{1/(m-1)}} \vee e^{-b((|g|')^n t^{-1})^{1/(n-1)}}) \\
&= \begin{cases} (V(t)^{-1/m} \wedge V(t)^{-1/n}) e^{-bt(|g|' t^{-1})^{m/(m-1)}} & \text{if } |g|' \ge t, \\ (V(t)^{-1/m} \wedge V(t)^{-1/n}) e^{-bt(|g|' t^{-1})^{n/(n-1)}} & \text{if } |g|' \le t. \end{cases}
\end{aligned}$$

It will be a consequence of Proposition 2.11 that $N_{\omega,t}^{(m,n)}(g) > 0$ and that the four functions $G_{b,t}^{(m)} * G_{b,t}^{(n)}$, $G_{b,t}^{(n)} * G_{b,t}^{(m)}$, $N_{\omega,t}^{(m,n)}$ and $E_{b,t}^{(m,n)}$ are comparable.

We initially prove bounds for the kernel expressed in terms of $N_{\omega,t}^{(m,\underline{m})}$. This is accomplished in two steps. First we derive uniform bounds.

Fix $j \in \{1, \ldots, k\}$ and $n \in \mathbb{N}$ such that $n\underline{m} > (D' \vee D)/2$. Then $nm_j > (D' \vee D)/2$. In the Sobolev inequality ([Dun], Lemma 3.1)

$$\|\varphi\|_\infty \le cV(t)^{-1/(2m_j)}(\|\varphi\|_2 + t^n \max_{|\beta| = nm_j} \|A^\beta \varphi\|_2) \tag{7}$$

one replaces φ by $A^\alpha S_t \varphi$ and notes that one has bounds

$$\|A^\gamma S_t \varphi\|_2 \le c\|H^{|\gamma|/m_j} S_t \varphi\|_2 \le c' t^{-|\gamma|/m_j} \|\varphi\|_2$$

for each $\gamma \in J(d')$ uniformly for all $t > 0$, by Theorem 2.1.I. It follows that there exists a $c > 0$ such that

$$\begin{aligned}\|A^\alpha S_t\|_{2\to\infty} &\le cV(t)^{-1/(2m_j)}(t^{-|\alpha|/m_j} + t^n t^{-(nm_j+|\alpha|)/m_j})\\ &= 2cV(t)^{-1/(2m_j)}t^{-|\alpha|/m_j}.\end{aligned}$$

Repeating the argument with $|\alpha| = 0$ and with H^* and S_t^* replacing H and S_t yields

$$\|S_t\|_{1\to 2} = \|S_t^*\|_{2\to\infty} \le c'V(t)^{-1/(2m_j)}$$

for a suitable $c' > 0$. Hence

$$\|A^\alpha S_{2t}\|_{1\to\infty} \le \|A^\alpha S_t\|_{2\to\infty}\|S_t^*\|_{2\to\infty} \le cc'V(t)^{-1/m_j}t^{-|\alpha|/m_j}$$

uniformly for all $t > 0$. Since this is valid for all j it follows that there is a $c > 0$ such that

$$\|A^\alpha K_t\|_\infty \le c(t^{-|\alpha|/m} \wedge t^{-|\alpha|/\underline{m}})(V(t)^{-1/m} \wedge V(t)^{-1/\underline{m}}) \tag{8}$$

for all $t > 0$.

Next we extend these bounds to establish that there exist $c, \omega > 0$ such that

$$|(A^\alpha K_t)(g)| \le c(t^{-|\alpha|/m} \wedge t^{-|\alpha|/\underline{m}})N^{(m,\underline{m})}_{\omega,t}(g) \tag{9}$$

for all $t > 0$ and $g \in G$. Again, fix $j \in \{1, \ldots, k\}$ and $n \in \mathbb{N}$ with $n\underline{m} > (D' \vee D)/2$. Substituting $A^\alpha S_t^\varrho\varphi$ for φ in the Sobolev inequality (7) yields

$$\|A^\alpha S_t^\varrho\varphi\|_\infty \le cV(t)^{-1/(2m_j)}(\|A^\alpha S_t^\varrho\varphi\|_2 + t^n \max_{|\beta|=nm_j+|\alpha|}\|A^\beta S_t^\varrho\varphi\|_2)$$

and substituting the bounds of Lemma 2.10 gives

$$\|A^\alpha S_t^\varrho\|_{2\to\infty} \le c'V(t)^{-1/(2m_j)}t^{-|\alpha|/m_j}e^{\omega(\varrho^m+\varrho^{\underline{m}})t}.$$

Arguing by duality one obtains

$$\|A^\alpha S_t^\varrho\|_{1\to\infty} \le cV(t)^{-1/m_j}t^{-|\alpha|/m_j}e^{\omega(\varrho^m+\varrho^{\underline{m}})t}.$$

Thus there exist $c, \omega > 0$ such that

$$\|A^\alpha S_t^\varrho\|_{1\to\infty} \le c(V(t)^{-1/m} \wedge V(t)^{-1/\underline{m}})(t^{-|\alpha|/m} \wedge t^{-|\alpha|/\underline{m}})e^{\omega(\varrho^m+\varrho^{\underline{m}})t}$$

for all $t > 0$, $\varrho \in \mathbb{R}$ and $R > 0$ such that $|\varrho| \ge R^{-1}$. Then by a combination with (3), (5) and arguing as in (6) one establishes the estimates

$$\begin{aligned}&\|U_\varrho A^\alpha U_\varrho^{-1}S_t^\varrho\|_{1\to\infty}\\ &\quad\le c'(V(t)^{-1/m} \wedge V(t)^{-1/\underline{m}})\sum_{|\gamma|\le|\alpha|}|\varrho|^{|\alpha|-|\gamma|}(t^{-|\alpha|/m} \wedge t^{-|\alpha|/\underline{m}})e^{\omega'(\varrho^m+\varrho^{\underline{m}})t}\\ &\quad\le c''(V(t)^{-1/m} \wedge V(t)^{-1/\underline{m}})(t^{-|\alpha|/m} \wedge t^{-|\alpha|/\underline{m}})e^{\omega''(\varrho^m+\varrho^{\underline{m}})t}.\end{aligned}$$

Then in particular

$$\begin{aligned}&|(A^\alpha K_t)(g)|\\ &\quad\le c''(V(t)^{-1/m} \wedge V(t)^{-1/\underline{m}})(t^{-|\alpha|/m} \wedge t^{-|\alpha|/\underline{m}})e^{\omega''(\varrho^m+\varrho^{\underline{m}})t}e^{\varrho(\psi_R(g)-\psi_R(e))}\end{aligned}$$

uniformly for all $t > 0$, $g \in G$, $\varrho \in \mathbb{R}$ and $R > 0$ such that $|\varrho| \geq R^{-1}$. Now for $g \neq e$ one sets $R = |g|' > 0$ so that $\psi_R(g) = 0$ and $\psi_R(e) = |g|'$. Then

$$(10) \qquad |(A^\alpha K_t)(g)| \leq c(V(t)^{-1/m} \wedge V(t)^{-1/\underline{m}})(t^{-|\alpha|/m} \wedge t^{-|\alpha|/\underline{m}}) e^{\omega(\varrho^m + \varrho^{\underline{m}})t - \varrho|g|'}$$

whenever $g \in G$ and $\varrho > 0$ are such that $|g|' \geq \varrho^{-1}$. On the other hand, for $g \in G$ and $\varrho > 0$ such that $|g|' \leq \varrho^{-1}$, one has

$$e^{\omega''(\varrho^m + \varrho^{\underline{m}})t - \varrho|g|'} \geq e^{\omega''(\varrho^m + \varrho^{\underline{m}})t - 1} \geq e^{-1}$$

and thus the bounds (10) follow from the uniform bounds (8). Hence (10) holds for all $g \in G$ and $\varrho > 0$, and the proof of the bounds (9) is complete.

The bounds of the theorem now follow from Statement I of the next proposition.

PROPOSITION 2.11. *Let $m, n \in \mathbb{N}$ with $m \geq n \geq 2$.*

I. *For all $b, \omega > 0$ there exist $b', c, \omega' > 0$ such that*

$$(11) \qquad \begin{aligned} N_{\omega,t}^{(m,n)} &\leq E_{b',t}^{(m,n)}, \\ E_{b,t}^{(m,n)} &\leq c G_{b',t}^{(m)} * G_{b',t}^{(n)}, \\ G_{b,t}^{(m)} * G_{b,t}^{(n)} &\leq c G_{b',t}^{(n)} * G_{b',t}^{(m)}, \\ G_{b,t}^{(n)} * G_{b,t}^{(m)} &\leq c N_{\omega',t}^{(m,n)} \end{aligned}$$

for all $t > 0$.

II. *For all $b > 0$ there exist $b', c > 0$ such that*

$$G_{b,t}^{(m)} \leq c G_{b',t}^{(m)} * G_{b',t}^{(n)} \quad \textit{and} \quad G_{b,t}^{(m)} * G_{b,t}^{(n)} \leq c G_{b',t}^{(m)}$$

for all $t \in \langle 0, 1]$.

III. *For all $b > 0$ there exist $b', c > 0$ such that*

$$G_{b,t}^{(n)} \leq c G_{b',t}^{(m)} * G_{b',t}^{(n)}$$

for all $t \geq 1$.

IV. *For all $b > 0$ and $\varepsilon > 0$ there exists a $c > 0$ such that*

$$G_{b,t}^{(m)} * G_{b,s}^{(m)} \leq c G_{b-\varepsilon,t+s}^{(m)}$$

uniformly for all $t, s > 0$.

Proof. Without loss of generality we may assume the normalization $V(1) = 1$.

First let $\omega \geq 1$, $g \in G$ and $t > 0$. If $|g|' \leq t$ then with $\varrho = 2^{-1}(|g|'(n\omega t)^{-1})^{1/(n-1)}$ one has

$$\begin{aligned} &-\varrho|g|' + \omega(\varrho^m + \varrho^n)t \\ &\quad = -((|g|')^n t^{-1})^{1/(n-1)} (n\omega)^{-1/(n-1)} 2^{-1} (1 - 2^{-(n-1)} n^{-1} \varrho^{m-n} - 2^{-(n-1)} n^{-1}) \\ &\quad \leq -4^{-1}(n\omega)^{-1/(n-1)}((|g|')^n t^{-1})^{1/(n-1)}. \end{aligned}$$

Alternatively, if $|g|' \geq t$ then with $\varrho = 2^{-1}(m\omega)^{-m/n}(|g|'(m\omega t)^{-1})^{1/(m-1)}$ one has

$$\begin{aligned}
&-\varrho|g|' + \omega(\varrho^m + \varrho^n)t \\
&\quad = -((|g|')^m t^{-1})^{1/(m-1)}(m\omega)^{-1/(m-1)} 2^{-1}(m\omega)^{-m/n} \\
&\qquad \times (1 - \delta^{m-1}m^{-1} - \delta^{n-1}m^{-1}(m\omega)^{(m-n)/(m-1)}(|g|'t^{-1})^{-(m-n)/(n-1)}) \\
&\quad \leq -4^{-1}(m\omega)^{-m}((|g|')^m t^{-1})^{1/(m-1)},
\end{aligned}$$

where $\delta = 2^{-1}(m\omega)^{-m/n} \leq (4\omega)^{-1}$. So

$$N^{(m,n)}_{\omega,t}(g) \leq E^{(m,n)}_{b',t}(g),$$

where $b' = 4^{-1}(m\omega)^{-m}$.

Secondly, fix $b > 0$. Then for all $g, h \in G$ one has

$$\begin{aligned}
(|gh^{-1}|')^{m/(m-1)} &\leq 2^{m/(m-1)}((|g|')^{m/(m-1)} + (|h|')^{m/(m-1)}) \\
&\leq 2^{n/(n-1)}((|g|')^{m/(m-1)} + (|h|')^{m/(m-1)}),
\end{aligned}$$

so

$$e^{(m)}_{b',t}(gh^{-1}) \geq e^{(m)}_{b,t}(g)e^{(m)}_{b,t}(h)$$

for all $t > 0$, where $b' = 2^{-n/(n-1)}b$,

$$e^{(q)}_{b,t}(g) = e^{-b((|g|')^q t^{-1})^{1/(q-1)}}$$

for all $q \in \mathbb{N} \setminus \{1\}$ and $e^{(q)}_{b',t}$ is defined analogously. Similarly,

$$e^{(n)}_{b',t}(h^{-1}g) \geq e^{(n)}_{b,t}(h)e^{(n)}_{b,t}(g)$$

for all $g, h \in G$ and $t > 0$.

Thirdly, if $t \geq 1$ and $g \in G$ then

$$\begin{aligned}
(G^{(m)}_{b',t} * G^{(n)}_{b',t})(g) &= V(t)^{-1/m}V(t)^{-1/n}\int_G dh\, e^{(m)}_{b',t}(h)e^{(n)}_{b',t}(h^{-1}g) \\
&\geq V(t)^{-1/m}V(t)^{-1/n}\int_G dh\, e^{(m)}_{b',t}(h)e^{(n)}_{b,t}(h)e^{(n)}_{b,t}(g) \\
&\geq V(t)^{-1/n}e^{(n)}_{b,t}(g)V(t)^{-1/m}\int_{\{h\in G:|h|'\leq t\}} dh\, e^{(m)}_{b',t}(h)e^{(n)}_{b,t}(h).
\end{aligned}$$

But if $|h|' \leq t$ then $e^{(n)}_{b,t}(h) \geq e^{(m)}_{b,t}(h)$. Moreover, $t^{1/m} \leq t$ since $t \geq 1$. Therefore

$$\begin{aligned}
&(G^{(m)}_{b',t} * G^{(n)}_{b',t})(g) \\
&\quad \geq V(t)^{-1/n}e^{(n)}_{b,t}(g)V(t)^{-1/m}\int_{\{h\in G:|h|'\leq t\}} dh\, e^{(m)}_{b',t}(h)e^{(m)}_{b,t}(h) \\
&\quad \geq V(t)^{-1/n}e^{(n)}_{b,t}(g)V(t)^{-1/m}\int_{\{h\in G:|h|'\leq t^{1/m}\}} dh\, e^{-2b((|h|')^m t^{-1})^{1/(m-1)}} \\
&\quad \geq cV(t)^{-1/n}e^{(n)}_{b,t}(g),
\end{aligned}$$

where

$$c = \inf_{s>0} V(s)^{-1/m} \int\limits_{\{h\in G:|h|'\le s^{1/m}\}} dh\, e^{-2b((|h|')^m s^{-1})^{1/(m-1)}}.$$

An elementary estimate shows that $c > 0$. Similarly

$$\begin{aligned}(G^{(m)}_{b',t} * G^{(n)}_{b',t})(g) &= V(t)^{-1/m} V(t)^{-1/n} \int\limits_G dh\, e^{(m)}_{b',t}(gh^{-1}) e^{(n)}_{b',t}(h)\\ &\ge V(t)^{-1/m} V(t)^{-1/n} \int\limits_G dh\, e^{(m)}_{b,t}(g) e^{(m)}_{b,t}(h) e^{(n)}_{b',t}(h)\\ &\ge cV(t)^{-1/n} e^{(m)}_{b,t}(g).\end{aligned}$$

Since $V(1) = 1$, by normalization, it follows that

$$E^{(m,n)}_{b,t} \le c^{-1} G^{(m)}_{b',t} * G^{(n)}_{b',t}$$

for all $t \ge 1$.

Finally, if $t \le 1$ then

$$\begin{aligned}&(G^{(m)}_{b',t} * G^{(n)}_{b',t})(g)\\ &\quad= V(t)^{-1/m} V(t)^{-1/n} \int\limits_G dh\, e^{(m)}_{b',t}(gh^{-1}) e^{(n)}_{b',t}(h)\\ &\quad\ge V(t)^{-1/m} V(t)^{-1/n} \int\limits_{\{h\in G:|h|'\le t^{1/m}\}} dh\, e^{(m)}_{b,t}(g) e^{(m)}_{b,t}(h) e^{(n)}_{b',t}(h)\\ &\quad\ge V(t)^{-1/m} e^{(m)}_{b,t}(g) V(t)^{-1/n} \int\limits_{\{h\in G:|h|'\le t^{1/m}\}} dh\, e^{-b} e^{(n)}_{b',t}(h)\\ &\quad\ge e^{-b} V(t)^{-1/m} e^{(m)}_{b,t}(g) V(t)^{-1/n} \int\limits_{\{h\in G:|h|'\le t^{1/n}\}} dh\, e^{(n)}_{b',t}(h)\\ &\quad\ge c_1 e^{-b} V(t)^{-1/m} e^{(m)}_{b,t}(g),\end{aligned}$$

where

$$c_1 = \inf_{s\le 1} V(s)^{-1/n} \int\limits_{\{h\in G:|h|'\le s^{1/n}\}} dh\, e^{(n)}_{b',s}(h) > 0.$$

Obviously $e^{(n)}_{b,t}(g) \le 1 \le e^b e^{(m)}_{b,t}(g)$ for all $g \in G$ with $|g|' \le t$. Alternatively, if $|g|' \ge t$ then $e^{(n)}_{b,t}(g) \le e^{(m)}_{b,t}(g)$. So

$$E^{(m,n)}_{b,t} \le c_1^{-1} e^{2b} G^{(m)}_{b',t} * G^{(n)}_{b',t}$$

for all $t \le 1$. This completes the proof of the estimate (11).

Next fix $b > 0$. Since $e^{\varrho|g|'} \le e^{\varrho|h'|} e^{\varrho|h^{-1}g|'}$ for all $\varrho > 0$ and $g, h \in G$ it follows that

$$\begin{aligned}
e^{\varrho|g|'}&(G^{(m)}_{b,t} * G^{(n)}_{b,t})(g) \\
&\le \int_G dh\, G^{(m)}_{b/2,t}(h) e^{(m)}_{b/2,t}(h) e^{\varrho|h|'} G^{(n)}_{b/2,t}(h^{-1}g) e^{(n)}_{b/2,t}(h^{-1}g) e^{\varrho|h^{-1}g|'} \\
&\le e^{\omega(\varrho^m+\varrho^n)t} \int_G dh\, G^{(m)}_{b/2,t}(h) G^{(n)}_{b/2,t}(h^{-1}g) \\
&\le c(V(t)^{-1/m} \wedge V(t)^{-1/n}) e^{\omega(\varrho^m+\varrho^n)t}
\end{aligned}$$

for all $t > 0$, $g \in G$ and $\varrho > 0$, where

$$c = \max(\sup_{s>0} \|G^{(m)}_{b/2,s}\|_1, \sup_{s>0} \|G^{(m)}_{b/2,s}\|_1) < \infty,$$
$$\omega = \max(m^{-1}(2b^{-1}(1-m^{-1}))^{m-1}, n^{-1}(2b^{-1}(1-n^{-1}))^{n-1})$$

and the $e^{(n)}_{b/2,t}$ are as before. So

$$G^{(m)}_{b,t} * G^{(n)}_{b,t} \le c N^{(m,n)}_{\omega,t}$$

for all $t > 0$. Since $(G^{(n)}_{b,t} * G^{(m)}_{b,t})(g) = (G^{(m)}_{b,t} * G^{(n)}_{b,t})(g^{-1})$ and $N^{(m,n)}_{\omega,t}(g) = N^{(m,n)}_{\omega,t}(g^{-1})$ this completes the proof of Statement I.

Since

$$G^{(m)}_{b,t}(g) = V(t)^{-1/m} \inf_{\varrho>0} e^{-\varrho|g|' + \omega\varrho^m t} \tag{12}$$

for all $t > 0$ and $g \in G$, where $\omega = m^{-1}(b^{-1}(1-m^{-1}))^{m-1}$ the estimates of Statement II follow from those of Statement I.

The estimate of Statement III follows from the equality (12), with m replaced by n, together with the bounds of Statement I.

Finally, if $b, \varepsilon > 0$ then

$$\begin{aligned}
e^{\varrho|g|'}&(G^{(m)}_{b,t} * G^{(m)}_{b,s})(g) \\
&\le \int_G dh\, G^{(m)}_{\varepsilon,t}(h) e^{(m)}_{b-\varepsilon,t}(h) e^{\varrho|h|'} G^{(m)}_{\varepsilon,s}(h^{-1}g) e^{(m)}_{b-\varepsilon,s}(h^{-1}g) e^{\varrho|h^{-1}g|'} \\
&\le e^{\omega\varrho^m(t+s)} \int_G dh\, G^{(m)}_{\varepsilon,t}(h) G^{(m)}_{\varepsilon,s}(h^{-1}g) \\
&\le c(V(t)^{-1/m} \wedge V(s)^{-1/m}) e^{\omega\varrho^m(t+s)}
\end{aligned}$$

for all $t, s > 0$, $g \in G$ and $\varrho > 0$, where $c = \sup_{u>0} \|G^{(m)}_{\varepsilon,u}\|_1 < \infty$ and $\omega = m^{-1}((b-\varepsilon)^{-1}(1-m^{-1}))^{m-1}$. But there is a $c' > 0$ such that $V(t) \vee V(s) \ge c' V(t+s)$ uniformly for all $t, s > 0$. So

$$\begin{aligned}
(G^{(m)}_{b,t} * G^{(m)}_{b,s})(g) &\le c(c')^{-1/m} \inf_{\varrho>0} e^{-\varrho|g|'} V(t+s)^{-1/m} e^{\omega\varrho^m(t+s)} \\
&= c(c')^{-1/m} G^{(m)}_{b-\varepsilon,t+s}(g)
\end{aligned}$$

for all $g \in G$. This proves Statement IV. ■

This completes the proof of Theorem 2.1.II. ■

The next theorem establishes that K_t converges in a strong sense to the kernel $K_t^{(\underline{m})}$ of $H_{\underline{m}}$ as $t \to \infty$, but we subsequently argue that one cannot usually expect simple Gaussian bounds for K.

THEOREM 2.12. *Suppose G is a connected nilpotent Lie group and $k \geq 2$. Let K and $K^{(\underline{m})}$ denote the kernels associated with H and $H_{\underline{m}}$. Set $\nu = (m_{k-1} - m_k)/m_k$. Then for all $\alpha \in J(d')$ there exist $b, c > 0$ such that*

$$|(A^\alpha K_t)(g) - (A^\alpha K_t^{(\underline{m})})(g)| \leq ct^{-\nu} t^{-|\alpha|/\underline{m}} (G_{b,t}^{(m)} * G_{b,t}^{(\underline{m})})(g)$$

for all $g \in G$ and $t \geq 1$.

Proof. First consider the case $|\alpha| < \underline{m}$. Let U_ϱ denote the multiplication operators used in the foregoing discussion of the Davies perturbation. Since

$$e^{\varrho|\psi(g)-\psi(e)|} |(A^\alpha K_t)(g) - (A^\alpha K_t^{(\underline{m})})(g)| \leq \|U_\varrho(A^\alpha S_t - A^\alpha S_t^{(\underline{m})})U_\varrho^{-1}\|_{1\to\infty}$$

where $S^{(\underline{m})}$ is the semigroup generated by $H_{\underline{m}}$ it suffices to prove that

$$\|U_\varrho(A^\alpha S_t - A^\alpha S^{(\underline{m})})U_\varrho^{-1}\|_{1\to\infty} \leq ct^{-\nu} t^{-|\alpha|/\underline{m}} V(t)^{-1/\underline{m}} e^{\omega(\varrho^m + \varrho^{\underline{m}})t}$$

for some $c, \omega > 0$ and all $t \geq 1$ and all $\varrho \in \mathbb{R}$. Then the bounds in terms of $G_{b,t}^{(m)} * G_{b,t}^{(\underline{m})}$ follow from Proposition 2.11. The foregoing estimates can, however, be derived by use of the Duhamel formula

$$U_\varrho(A^\alpha S_t - A^\alpha S_t^{(\underline{m})})U_\varrho^{-1} = \int_0^t ds\, U_\varrho A^\alpha S_{t-s}^{(\underline{m})}(H - H_{\underline{m}})S_s U_\varrho^{-1}$$

and the earlier kernel bounds.

The difference $H - H_{\underline{m}}$ is a linear combination of monomials A^β with $m_{k-1} \leq |\beta| \leq m$. But one has estimates

$$\int_0^t ds\, \|U_\varrho A^\alpha S_{t-s}^{(\underline{m})} A^\beta S_s U_\varrho^{-1}\|_{1\to\infty} \leq ct^{1-(|\alpha|+|\beta|)/\underline{m}} V(t)^{-1/\underline{m}} e^{\omega(\varrho^m + \varrho^{\underline{m}})t} \tag{13}$$

for $t \geq 1$ and all $\alpha, \beta \in J(d')$ with $|\alpha| < \underline{m}$. These are established in two steps. First, if $s \in [0, t/2]$ then

$$\begin{aligned}\|U_\varrho A^\alpha S_{t-s}^{(\underline{m})} A^\beta S_s U_\varrho^{-1}\|_{1\to\infty} &\leq \|U_\varrho A^\alpha S_{t-s}^{(\underline{m})} A^\beta U_\varrho^{-1}\|_{1\to\infty} \|S_s^\varrho\|_{1\to 1} \\ &\leq \|U_\varrho A^\alpha S_{(t-s)/2}^{(\underline{m})} U_\varrho^{-1}\|_{2\to\infty} \|U_\varrho S_{(t-s)/2}^{(\underline{m})} A^\beta U_\varrho^{-1}\|_{1\to 2} \|S_s^\varrho\|_{1\to 1}.\end{aligned}$$

Each term in the product can be bounded by integration of the kernel bounds given in Theorem 2.1.II. One finds bounds

$$\begin{aligned}\|U_\varrho A^\alpha S_{t-s}^{(\underline{m})} A^\beta S_s U_\varrho^{-1}\|_{1\to\infty} &\leq c(t-s)^{-(|\alpha|+|\beta|)/\underline{m}} V(t-s)^{-1/\underline{m}} e^{\omega(\varrho^m + \varrho^{\underline{m}})t} \\ &\leq c' t^{-(|\alpha|+|\beta|)/\underline{m}} V(t)^{-1/\underline{m}} e^{\omega(\varrho^m + \varrho^{\underline{m}})t}\end{aligned}$$

for all $t \geq 1$ where the latter bound uses $t - s \geq t/2$. Integration over $[0, t/2]$ then gives a bound of the same form as the right hand side of (13). Secondly, for $s \in [t/2, t]$ one makes the alternative estimate

$$\|U_\varrho A^\alpha S^{(\underline{m})}_{t-s} A^\beta S_s U_\varrho^{-1}\|_{1\to\infty} \leq \|U_\varrho A^\alpha S^{(\underline{m})}_{t-s} U_\varrho^{-1}\|_{\infty\to\infty} \|U_\varrho A^\beta S_s U_\varrho^{-1}\|_{1\to\infty}.$$

Integration of the kernel bounds now gives

$$\begin{aligned}\|U_\varrho A^\alpha S^{(\underline{m})}_{t-s} A^\beta S_s U_\varrho^{-1}\|_{1\to\infty} &\leq c(t-s)^{-|\alpha|/\underline{m}} s^{-|\beta|/\underline{m}} V(s)^{-1/\underline{m}} e^{\omega(\varrho^m + \varrho^{\underline{m}})t} \\ &\leq c'(t-s)^{-|\alpha|/\underline{m}} t^{-|\beta|/\underline{m}} V(t)^{-1/\underline{m}} e^{\omega(\varrho^m + \varrho^{\underline{m}})t}.\end{aligned}$$

Since $|\alpha|/\underline{m} < 1$ this bound is integrable for $s \in [t/2, t]$ and on integration one again obtains the same form as the right hand side of (13).

Since the expression for $H - H_{\underline{m}}$ only contains terms with $|\beta| \geq m_{k-1}$ it follows that

$$\int_0^t ds\, \|U_\varrho A^\alpha S^{(\underline{m})}_{t-s}(H - H_{\underline{m}}) S_s U_\varrho^{-1}\|_{1\to\infty} \leq c t^{-(m_{k-1} - \underline{m})/\underline{m}} t^{-|\alpha|/\underline{m}} e^{\omega(\varrho^m + \varrho^{\underline{m}})t}$$

for $t \geq 1$ and the proof for $|\alpha| < \underline{m}$ is complete.

The proof for $|\alpha| \geq \underline{m}$ requires a somewhat more complicated argument. One now starts from the Duhamel formula

$$\begin{aligned}S_t\varphi - S^{(\underline{m})}_t\varphi = &\int_0^{t/2} ds\, S^{(\underline{m})}_{(t-s)/2}(S^{(\underline{m})}_{(t-s)/2}(H_{\underline{m}} - H))S_s\varphi \\ &+ \int_{t/2}^t ds\, S^{(\underline{m})}_{t-s}((H_{\underline{m}} - H)S_s)\varphi.\end{aligned}$$

Note that by duality the operator $S^{(\underline{m})}_{(t-s)/2}(H_{\underline{m}} - H)$ extends to a bounded operator whose norm has a possible singularity at $s = t$. But there is no singularity at $s = 0$. Similarly $(H_{\underline{m}} - H)S_s$ has a possible singularity at $s = 0$ but there is no singularity at $s = t$. Next if one expands $H_{\underline{m}} - H = \sum_{m_{k-1}\leq|\beta|\leq m} c_\beta A^\beta$ and if $K^{(\underline{m})\beta}$ denotes the kernel of the operator $S^{(\underline{m})}_t A^\beta$ then the Duhamel formula gives

$$\begin{aligned}K_t(g) - K^{(\underline{m})}_t(g) = &\sum_{m_{k-1}\leq|\beta|\leq m} c_\beta \int_0^{t/2} ds\, (K^{(\underline{m})}_{(t-s)/2} * K^{(\underline{m})\beta}_{(t-s)/2} * K_s)(g) \\ &+ \sum_{m_{k-1}\leq|\beta|\leq m} c_\beta \int_{t/2}^t ds \int_G dh\, K^{(\underline{m})}_{t-s}(h)(L(h)A^\beta K_s)(g)\end{aligned}$$

for all $t > 0$ and $g \in G$. Note that $K^{(\underline{m})\beta}$ satisfies Gaussian bounds: there exist $b, c > 0$ such that $|K^{(\underline{m})\beta}_t(g)| \leq c t^{-|\beta|/\underline{m}} G^{(\underline{m})}_{b,t}(g)$ uniformly for all $|\beta| \leq$

m, $t > 0$ and $g \in G$. Hence if $\alpha \in J(d')$ then

$$(14)\qquad (A^\alpha K_t)(g) - (A^\alpha K_t^{(\underline{m})})(g) = \sum_{m_{k-1} \le |\beta| \le m} c_\beta \int_0^{t/2} ds\,((A^\alpha K^{(\underline{m})}_{(t-s)/2}) * K^{(\underline{m})\beta}_{(t-s)/2} * K_s)(g) + \sum_{m_{k-1} \le |\beta| \le m} c_\beta \int_{t/2}^{t} ds \int_G dh\, K^{(\underline{m})}_{t-s}(h)(A^\alpha L(h) A^\beta K_s)(g)$$

for all $t > 0$ and $g \in G$. We estimate the two terms separately. Using the kernel estimates of Theorem 2.1.II and Proposition 2.11 for the contribution over the interval $[0, t/2]$ gives

$$\int_0^{t/2} ds\,|((A^\alpha K^{(\underline{m})}_{(t-s)/2}) * K^{(\underline{m})\beta}_{(t-s)/2} * K_s)(g)|$$
$$\le c \int_0^{t/2} ds\,(t-s)^{-(|\alpha|+|\beta|)/\underline{m}} (G^{(\underline{m})}_{b,(t-s)/2} * G^{(\underline{m})}_{b,(t-s)/2} * (G^{(\underline{m})}_{b,s} * G^{(m)}_{b,s}))(g)$$
$$\le c_1 t^{-(|\alpha|+|\beta|)/\underline{m}} \int_0^{t/2} ds\,(G^{(\underline{m})}_{b/2,t-s} * (G^{(\underline{m})}_{b,s} * G^{(m)}_{b,s}))(g)$$

for all $t \ge 2$ and $g \in G$. But then it follows by repeated use of Proposition 2.11 that

$$\begin{aligned} G^{(\underline{m})}_{b/2,t-s} * (G^{(\underline{m})}_{b,s} * G^{(m)}_{b,s}) &\le c_2 G^{(m)}_{b_1,t-s} * G^{(\underline{m})}_{b_1,t-s} * G^{(\underline{m})}_{b,s} * G^{(m)}_{b,s} \\ &\le c_3 G^{(m)}_{b_1,t} * G^{(\underline{m})}_{b_2,t} * G^{(m)}_{b,s} \le c_4 G^{(\underline{m})}_{b_3,t} * G^{(m)}_{b_3,t} * G^{(m)}_{b,s} \\ &\le c_5 G^{(\underline{m})}_{b_3,t} * G^{(m)}_{b_4,t+s} \le c_5 G^{(\underline{m})}_{b_3,t} * G^{(m)}_{b_5,t} \end{aligned}$$

uniformly for all $t \ge 2$ and $s \in \langle 0, t/2]$. Hence

$$(15)\qquad \int_0^{t/2} ds\,|((A^\alpha K^{(\underline{m})}_{(t-s)/2}) * K^{(\underline{m})\beta}_{(t-s)/2} * K_s)(g)| \le c_5 t^{1-(|\alpha|+|\beta|)/\underline{m}} (G^{(\underline{m})}_{b_3,t} * G^{(m)}_{b_5,t})(g)$$

for all $t \ge 2$ and $g \in G$.

To bound the contribution over the subinterval $[t/2, t]$ we proceed similarly, although there is one new problem with the left translations. It follows from the proof of Lemma 4.3 of [ElR3] that there is a $c > 0$ and for all $\gamma \in J(d')$ with $|\alpha| \le |\gamma| \le r|\alpha|$ a function $f_\gamma : G \to \mathbb{R}$ such that

$$(16)\qquad L(h^{-1}) A^\alpha L(h) = \sum_{|\alpha| \le |\gamma| \le r|\alpha|} f_\gamma(h) A^\gamma$$

and $|f_\gamma(h)| \le c(|h|')^{|\gamma|-|\alpha|}$ for all γ and $h \in G$. (Since G is nilpotent the series expression given in [ElR3] terminates after a finite number, at most r, of terms.) Therefore

$$\begin{aligned}
&\int_{t/2}^{t} ds \int_G dh\, |K^{(\underline{m})}_{t-s}(h)(A^\alpha L(h) A^\beta K_s)(g)| \\
&\quad\le \sum_{|\alpha|\le|\gamma|\le r|\alpha|} \int_{t/2}^{t} ds \int_G dh\, |K^{(\underline{m})}_{t-s}(h)| \cdot |f_\gamma(h)| \cdot |(A^\gamma A^\beta K_s)(h^{-1}g)| \\
&\quad\le c' \sum_{|\alpha|\le|\gamma|\le r|\alpha|} \int_{t/2}^{t} ds \int_G dh\, |K^{(\underline{m})}_{t-s}(h)| \\
&\qquad\qquad \times (|h|')^{|\gamma|-|\alpha|} s^{-(|\beta|+|\gamma|)/\underline{m}} (G^{(\underline{m})}_{b,s} * G^{(m)}_{b,s})(h^{-1}g) \\
&\quad\le c'' t^{-(|\alpha|+|\beta|)/\underline{m}} \sum_{|\alpha|\le|\gamma|\le r|\alpha|} \int_{t/2}^{t} ds \int_G dh\, |K^{(\underline{m})}_{t-s}(h)| \\
&\qquad\qquad \times (|h|' s^{-1/\underline{m}})^{|\gamma|-|\alpha|} (G^{(\underline{m})}_{b,s} * G^{(m)}_{b,s})(h^{-1}g)
\end{aligned}$$

uniformly for all $g \in G$ and $t > 0$. But for $s \in [t/2, t]$ one has $s^{-1/\underline{m}} \le (t-s)^{-1/\underline{m}}$ and an elementary estimate gives

$$|K^{(\underline{m})}_{t-s}(h)|(|h|' s^{-1/\underline{m}})^{|\gamma|-|\alpha|} \le c G^{(\underline{m})}_{2b,t-s}(h)(|h|'(t-s)^{-1/\underline{m}})^{|\gamma|-|\alpha|} \le c' G^{(\underline{m})}_{b,t-s}(h)$$

uniformly for all $t \ge 2$, $s \in [t/2, t]$ and γ with $|\alpha| \le |\gamma| \le r|\alpha|$. Thus

$$\begin{aligned}
(17)\qquad &\int_{t/2}^{t} ds\, |((A^\alpha K^{(\underline{m})}_{t-s}) * (A^\beta K_s))(g)| \\
&\quad\le c t^{-(|\alpha|+|\beta|)/\underline{m}} \int_{t/2}^{t} ds\, (G^{(\underline{m})}_{b,t-s} * (G^{(\underline{m})}_{b,s} * G^{(m)}_{b,s}))(g) \\
&\quad\le c' t^{1-(|\alpha|+|\beta|)/\underline{m}} (G^{(\underline{m})}_{b',t} * G^{(m)}_{b,t})(g)
\end{aligned}$$

uniformly for all $t \ge 2$ because $G^{(m)}_{b,s}$ can be bounded by a multiple of $G^{(m)}_{b,t}$ for $s \in [t/2, t]$.

Combination of (14), (15), (17) and Proposition 2.11.I then gives the desired bounds. ■

Corollary 2.13. *If $k \ge 2$ then there is a $c > 0$ such that*

$$\|K_t - K^{(\underline{m})}_t\|_\infty \le c t^{-\nu} V(t)^{-1/\underline{m}} \quad \text{and} \quad \|K_t - K^{(\underline{m})}_t\|_1 \le c t^{-\nu}$$

for all $t \ge 1$, where $\nu = (m_{k-1} - m_k)/m_k$.

Proof. The first statement is an immediate consequence of the estimates of Theorem 2.12. The second follows straightforwardly by integration of the estimates. ∎

REMARK. The following example establishes that the exponent ν in these asymptotic estimates is optimal. Let $G = \mathbb{R}^d$ and $H_{m_j} = \Delta^{m_j/2}$, where $\Delta = -\sum_{i=1}^d \partial_i^2$. Then

$$K_t(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} d\xi\, e^{ix\cdot\xi} e^{-t(|\xi|^{m_1}+\ldots+|\xi|^{m_k})}$$

and

$$K_t^{(\underline{m})}(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} d\xi\, e^{ix\cdot\xi} e^{-t|\xi|^{\underline{m}}}$$

for $x \in \mathbb{R}^d$. Thus one finds that

$$\begin{aligned}
\|K_t - K_t^{(\underline{m})}\|_\infty &= |K_t(0) - K_t^{(\underline{m})}(0)| \\
&= (2\pi)^{-d} \int d\xi\, e^{-t|\xi|^{\underline{m}}}(1 - e^{-t(|\xi|^{m_1}+\ldots+|\xi|^{m_{k-1}})}) \\
&\geq (2\pi)^{-d} \int d\xi\, e^{-t|\xi|^{\underline{m}}}(1 - e^{-t|\xi|^{m_{k-1}}}) \\
&= (2\pi)^{-d} t^{-d/\underline{m}} \int_{\mathbb{R}^d} d\eta\, e^{-|\eta|^{\underline{m}}}(1 - e^{-t^{-\nu}|\eta|^{m_{k-1}}}) \\
&\geq (2\pi)^{-d} t^{-d/\underline{m}} \int_{\{\eta:|\eta|^{m_{k-1}}\leq\varepsilon\}} d\eta\, e^{-|\eta|^{\underline{m}}}(2^{-1}t^{-\nu}|\eta|^{m_{k-1}})
\end{aligned}$$

for $t \geq 1$, where $\varepsilon > 0$ is chosen small enough so that $1 - e^{-r} \geq 2^{-1}r$ holds for all $r \in [0, \varepsilon]$. Therefore one has an estimate $\|K_t - K_t^{\underline{m}}\|_\infty \geq c' t^{-\nu} t^{-d/\underline{m}}$ for $t \geq 1$. So the constant ν is optimal in this case. ∎

Note that for self-adjoint operators the kernel is positive at the identity.

COROLLARY 2.14. *If H and $H_{\underline{m}}$ are self-adjoint then there is a $c > 0$ such that*

$$K_t(e) \geq c(V(t)^{-1/m} \wedge V(t)^{-1/\underline{m}})$$

for all $t > 0$.

Proof. By [ElR5], Corollary 2.4, one has an estimate $K_t^{(\underline{m})}(e) \geq cV(t)^{-1/\underline{m}}$ for all $t > 0$. Combining this with the first statement of Corollary 2.13, it follows that there exist $c > 0$ and $T > 0$ such that

$$K_t(e) \geq cV(t)^{-1/\underline{m}}$$

for all $t \geq T$.

Alternatively, K_t satisfies mth order Gaussian bounds for $t \leq T$. Then by Corollary 2.2 of [ElR5] one obtains an estimate

$$K_t(e) \geq c' V(t)^{-1/m}$$

for $t \leq T$. ∎

Although Corollary 2.13 indicates that K_t approaches $K_t^{(\underline{m})}$ asymptotically it cannot be expected to be bounded by a Gaussian of order $\underline{m}$ uniformly for all t. We make this statement precise.

PROPOSITION 2.15. *Suppose $n \in \mathbb{N} \setminus \{1\}$. The following conditions are equivalent.*

I. *There exist $b, c > 0$ such that $|K_t| \le c G_{b,t}^{(n)}$ for all $t > 0$.*
II. *$n = m = \underline{m}$ or G is compact and $n \ge m$.*

Proof. "II⇒I". If $n = m = \underline{m}$ then Condition I follows from [ERS1], Theorem 3.5. Alternatively, assume G is compact. Then Condition I follows for $t \le 1$ from Theorem 2.12 and Proposition 2.11.II. Next let $b, c > 0$ be as in Theorem 2.12 for $|\alpha| = 0$. Then

$$|K_t(g)| \le c V(1)^{-1/\underline{m}} \le c V(1)^{-1/\underline{m}} |G|^{1/\underline{m}} e^{b x^{n/(n-1)}} G_{b,t}^{(n)}(g)$$

for all $t \ge 1$ and $g \in G$, where $x = \max\{|g|' : g \in G\}$ and G is the Haar measure of G.

"I⇒II". For $t > 0$ set $L_t = K_{t/2} * \overline{\check{K}_{t/2}}$, where $\check{K}_s(g) = K_s(g^{-1})$. Then it follows as in Step 2 of the proof of Theorem 1.1 of [Dun] that there exists a $c > 0$ such that $K_t(e) \ge c V(t)^{-1/n}$ for all $t > 0$. On the other hand, it is a consequence of Theorem 2.1.II and Proposition 2.11 that there is a $c' > 0$ such that $L_t(e) \le c'(V(t)^{-1/m} \wedge V(t)^{-1/\underline{m}})$ for all $t > 0$. These bounds are compatible for small t if, and only if, $n \ge m$. Moreover, they are compatible for large t if, and only if, $D/\underline{m} \le D/m$ and this gives the two possibilities of Condition II. ∎

Note that the only compact nilpotent Lie groups are products of tori.

3. General groups. Let G be a Lie group of polynomial growth and H a strongly subcoercive operator of step r. In the previous section we established that if G is nilpotent and its Lie algebra has rank r, or less, then H satisfies the strong Gårding inequality (2). In particular H is accretive. But for this nilpotency of G, or some more stringent assumption on the step of subcoercivity, is essential. The conclusion can fail even for compact G if the step is small.

Let $G = SO(2)$, the compact three-dimensional group of rotations. Thus $\mathfrak{g}$ has a vector space basis, i.e., an algebraic basis of rank 1, of elements a_1, a_2, a_3 satisfying $[a_1, a_2] = a_3$, $[a_2, a_3] = a_1$ and $[a_3, a_1] = a_2$. Then $\widetilde{G} = G(3,1) = \mathbb{R}^3$, by definition. Now consider the self-adjoint operator

$$H = -A_1^2 - A_2^2 - A_3^2 + i\lambda A_3 = -A_1^2 - A_2^2 - A_3^2 + i\lambda [A_1, A_2]$$

where $\lambda \in \mathbb{R}$. It follows from the second expression for H that it is subcoercive of step 1 and homogeneous of order 2. The spectrum of H consists, however, of a sequence of eigenvalues $l(l+1) - m\lambda$ with $l \in \mathbb{N}_0$ and $m \in \mathbb{Z}$

with $|m| \leq l$. Thus if $\lambda > 2$ then H has negative eigenvalues and certainly cannot satisfy a strong Gårding inequality. In fact the multiplicity of the negative spectrum can be made arbitrarily large by choosing λ sufficiently large.

This example has other interesting features. If $\lambda = 2$ then $H \geq 0$ but it has two zero eigenvalues corresponding to $l = 0 = m$ and $l = 1 = m$. The first of these eigenvalues has a constant eigenfunction but the second has a non-constant eigenfunction φ_1. Since $(\varphi_1, H\varphi_1) = 0$ but $\|A_i\varphi_1\|_2 \neq 0$ for at least one $i \in \{1,2,3\}$ the strong Gårding inequality must fail for H. Specifically, $\|A_iS_t\|_{2\to 2} = O(1 \vee t^{-1/2})$ for all $t > 0$ and $i \in \{1,2,3\}$. As $t \to \infty$ the norms are attained by the eigenfunction corresponding to $l = 1 = m$ and as $t \to 0$ their values are governed by the eigenfunctions with large $l = 2|m|$.

In the next section we consider Lie groups of polynomial growth which are a local direct product of a connected compact Lie group K and a connected nilpotent Lie group N. We argue that this restriction is natural by the results of [Dun] and [ERS2]. If H is a strongly subcoercive operator of step r with $r \in \mathbb{N}$ satisfying the Gårding inequality

$$\mathrm{Re}(\varphi, H\varphi) \geq \mu \sum_{|\alpha| = \underline{m}/2} \|A^\alpha \varphi\|_2^2$$

(which is weaker than the estimate (2)) and, moreover, if the kernel K of the semigroup generated by H satisfies Gaussian bounds

$$|K_t(g)| \leq c(G_{b,t}^{(m)} * G_{b,t}^{(\underline{m})})(g)$$

then it follows from the estimates

$$|(G_{b,t}^{(m)} * G_{b,t}^{(\underline{m})})(g)| \leq c' V(t)^{-1/\underline{m}} e^{-b'((|g|')^{\underline{m}} t^{-1})^{1/(m-1)}}$$

and the arguments in Section 2 of [Dun] that there exists a one-parameter family $(\eta_R)_{R\geq 1}$ of cutoff functions such that (3) is valid uniformly for all $|\alpha| \leq \underline{m}/2$ and $R \geq 1$. But if G is not a local direct product of a connected compact Lie group K and a connected nilpotent Lie group N then by Theorem 4.4 of [ERS2] these cutoff functions exist if, and only if, $\underline{m} \leq 2$.

In the discussion of product groups the strong Gårding inequality (2) for H is crucial. On a general group it implies that H is maximal accretive and consequently has a bounded H_∞-holomorphic calculus by [ADM], Theorem G. Therefore

$$|(S_t\varphi, HS_t\varphi)| \leq ct^{-1}\|\varphi\|_2^2$$

for all $t > 0$. Hence it follows from (2), with φ replaced by $S_t\varphi$, that

$$\max_{|\alpha|=m/2} \|A^\alpha S_t\|_{2\to 2} + \max_{|\alpha|=\underline{m}/2} \|A^\alpha S_t\|_{2\to 2} \leq c' t^{-1/2}$$

for all $t > 0$. Then by the usual $\varepsilon, \varepsilon^{-1}$ inequalities linking the powers of the A_i, i.e., the inequalities

$$\max_{|\alpha|=n_1} \|A^\alpha \varphi\|_2 \le \varepsilon^{n_2-n_1} \max_{|\beta|=n_2} \|A^\beta \varphi\|_2 + c\varepsilon^{-n_1} \|\varphi\|_2 \tag{18}$$

which are valid for $n_2 > n_1$, $\varepsilon > 0$ and for all $\varphi \in C_c^\infty(G)$, one deduces that

$$\max_{|\alpha|=n} \|A^\alpha S_t\|_{2\to 2} \le c(t^{-n/m} \wedge t^{-n/\underline{m}})$$

for a suitable $c > 0$, all $n \in \{1, 2, \ldots, m/2\}$ and all $t > 0$. These latter bounds are crucial in the discussion of product groups.

4. Local direct product groups. In this section we assume that $G = K \times_l N$ is a local direct product of a connected compact Lie group K and a connected nilpotent Lie group N with Lie algebra of rank r. Moreover, H is a strongly subcoercive operator of step r and order $m \ge 4$. The example in Section 3 of a second-order operator on a compact group demonstrates that one cannot expect to derive good asymptotic bounds on the corresponding semigroup kernel or to deduce boundedness of the Riesz transforms without further assumptions. Nevertheless one can characterize these properties in simpler terms. We first consider the Riesz transforms.

THEOREM 4.1. *Assume H is accretive. Then each of the conditions in the following two families*, *indexed by* $n \in \mathbb{N}$, *is equivalent.*

I$_n$. *There is a $\sigma_n > 0$ such that*

$$\max_{|\alpha|=n} \|A^\alpha S_t\|_{2\to 2} \le \sigma_n(t^{-n/m} \wedge t^{-n/\underline{m}})$$

uniformly for all $t > 0$.

II$_n$. *For all $\alpha \in J(d')$ with $|\alpha| = n$ and all $j \in \{1, \ldots, k\}$ one has $D(H^{n/m_j}) \subseteq D(A^\alpha)$ and there is a $c > 0$ such that*

$$\max_{|\alpha|=n} \|A^\alpha \varphi\|_2 \le c\|H^{n/m_j} \varphi\|_2$$

for all $\varphi \in D(H^{n/m_j})$.

Moreover, *if H satisfies the strong Gårding inequality* (2) *then all these conditions are satisfied.*

P r o o f. It follows from Condition II$_n$ that

$$\|A^\alpha S_t\|_{2\to 2} \le c\|H^{n/m_j} S_t\|_{2\to 2}$$

for all α with $|\alpha| = n$ and all $j \in \{1, \ldots, k\}$. But since H is maximal accretive it has a bounded H_∞-holomorphic functional calculus by [ADM], Theorem G. Consequently, $\|H^{n/m_j} S_t\|_{2\to 2} \le c' t^{-n/m_j}$ for all $t > 0$. Condition I$_n$ follows immediately. Next I$_n$ implies I$_1$ by two applications of the inequalities (18) with suitable choices of ε. The proof that I$_1$ implies II$_n$ is essentially a repetition of the arguments used to prove Proposition 4.1 in [ERS2]. Some extra argument is, however, required since H is not assumed to be self-adjoint.

First one proves the implication for a direct product $K \times N$ and then uses structure theory to lift the result to the local direct product. The latter argument is unchanged in the current context and one only needs to verify the former. But on the direct product one introduces a projection P onto the subspace of L_2 formed by functions constant over K. This is defined by averaging over the compact component, i.e., averaging over K. Then Condition II_n is satisfied on PL_2 by the results established for the nilpotent case in Section 2. On $(I-P)L_2$ one argues as in [ERS2], with the aid of I_1, to obtain spectral estimates

$$\|H^N(I-P)\varphi\|_2 \ge \mu^N \|(I-P)\varphi\|_2$$

for some $\mu > 0$, some $N \in \mathbb{N}$ and all $\varphi \in D(H^N)$. If H is self-adjoint this immediately implies that H restricted to $(I-P)L_2$ has spectrum in $[\mu, \infty\rangle$ and hence

$$\|H^n(I-P)\varphi\|_2 \ge \mu^n \|(I-P)\varphi\|_2$$

for all $n \in \mathbb{N}$ and all $\varphi \in D(H^n)$. The general case is, however, covered by the following spectral lemma for holomorphic semigroups on Banach space.

LEMMA 4.2. *Let H be the generator of a bounded holomorphic semigroup S. The following conditions are equivalent.*

I. *There exist $M \ge 1$ and $\omega > 0$ such that $\|S_t\| \le Me^{-\omega t}$ for all $t > 0$.*

II. *There exists $\mu > 0$ such that $\|H^n\varphi\| \ge \mu^n\|\varphi\|$ for all $\varphi \in D(H^n)$ and all $n \in \mathbb{N}$.*

III. *There exists $N \in \mathbb{N}$ and $\nu > 0$ such that $\|H^N\varphi\| \ge \nu^N\|\varphi\|$ for all $\varphi \in D(H^N)$.*

Proof. I⇒II. It follows by integration of S that H^{-1} is a bounded operator and

$$\|H^{-1}\| \le \int_0^\infty dt\, \|S_t\| \le M\omega^{-1}.$$

Hence $\|H^{-n}\| \le (M/\omega)^n$ and II is valid with $\mu = \omega/M$.

It is evident that II⇒III so it remains to prove that III⇒I.

First since S is uniformly bounded for all $n \in \mathbb{N}$ there is a $c_n > 0$ such that

$$\|H^N\varphi\| \le \varepsilon^n \|H^{N+n}\varphi\| + c_n\varepsilon^{-N}\|\varphi\| \tag{19}$$

for all $\varphi \in D(H^{N+n})$ and $\varepsilon > 0$ (see the proof of Lemma III.3.3 in [Rob]). Hence it follows from III that

$$\|H^{N+n}\varphi\| \ge (\nu^N - c_n\varepsilon^{-N})\varepsilon^{-n}\|\varphi\|$$

for all $\varphi \in D(H^{N+n})$ and all $\varepsilon > 0$. Therefore there is a $\kappa > 0$ such that

$$\|H^{N+n}\varphi\| \ge \kappa^N \|\varphi\|$$

for all $n \in \{0, \ldots, N-1\}$ and $\varphi \in D(H^{N+n})$. Another straightforward application of (19) leads to the further conclusion that there are $\sigma, r > 0$ such that

$$\|(\lambda I - H)^{N+n}\varphi\| \ge \sigma^N \|\varphi\|$$

for all $n \in \{0, \ldots, N-1\}$, $\varphi \in D(H^{N+n})$ and $\lambda \in \mathbb{C}$ with $|\lambda| < r$.

Secondly, let $\varrho(H)$ denote the resolvent set of H and $R(\lambda) = (\lambda I - H)^{-1}$ the resolvent for all $\lambda \in \varrho(H)$. If S is bounded holomorphic in the sector $\Delta(\theta)$ then $\mathbb{C} \setminus \overline{\Delta(\pi/2 - \theta)} \subseteq \varrho(H)$ and R is analytic in this set. But if $\lambda_0 \in \varrho(H)$ the Taylor series for R^N around this point can be rewritten in the form

$$\begin{aligned}(20)\quad R(\lambda)^N &= \sum_{n=0}^{\infty} \binom{N+n-1}{n} (\lambda_0 - \lambda)^n R(\lambda_0)^{N+n} \\ &= \sum_{m=1}^{\infty} \sum_{n=0}^{N-1} \binom{mN+n-1}{N-1} (\lambda_0 - \lambda)^{(m-1)N+n} R(\lambda_0)^{mN+n}.\end{aligned}$$

But if $\lambda_0 \in \langle -r, 0\rangle$ then $\lambda_0 \in \varrho(H)$ and the previous estimates show that $\|R(\lambda_0)^{mN+n}\| \le \|R(\lambda_0)^N\|^{m-1} \|R(\lambda_0)^{N+n}\| \le \sigma^{-mN}$. Therefore the series on the right hand side of (20) converges for $|\lambda - \lambda_0| < \sigma/2$ and defines an analytic extension R_N of R^N into the interior of the ball $B_{\sigma/2} = \{\lambda \in \mathbb{C} : |\lambda| < \sigma/2\}$.

Thirdly

$$\begin{aligned} S_t &= (2\pi i)^{-1} (N-1)! t^{-(N-1)} \int_{\Gamma} d\lambda\, e^{-\lambda t} (\lambda I - H)^{-N} \\ &= (2\pi i)^{-1} (N-1)! t^{-(N-1)} \int_{\Gamma} d\lambda\, e^{-\lambda t} R_N(\lambda)\end{aligned}$$

where Γ is a positively-oriented contour in $\varrho(H)$, enclosing $\Delta(\pi/2 - \theta)$, which runs from $\arg \lambda = -(\pi/2 - \theta) - \varepsilon$ to $\arg \lambda = (\pi/2 - \theta) + \varepsilon$, with $\varepsilon \in \langle 0, \theta\rangle$. This follows from the usual Cauchy representation for S through integration by parts. Since, by the foregoing, R_N has an analytic extension to the half-plane $\operatorname{Re} \lambda < 2^{-1}\sigma \sin\theta$ one can deform the contour Γ so that it lies totally in the half-plane $\operatorname{Re} \lambda \ge 4^{-1}\sigma \sin\theta$. It then follows from the integral representation that one has bounds $\|S_t\| \le M e^{-\omega t}$ for all $t \ge 1$ with $\omega = 4^{-1}\sigma \sin\theta$. As S is uniformly bounded these bounds extend to all $t > 0$ with an enlarged value for M, i.e., Condition I is satisfied. ∎

It now follows as in [ERS2] that Condition II_n is satisfied on $(I - P)L_2$. The result on L_2 is then pieced together from the results on the two components PL_2 and $(I - P)L_2$.

Finally if H satisfies (2) then Condition I_1 is satisfied by the discussion at the end of Section 3. ∎

The estimates of the second family of conditions in Theorem 4.1 can be rephrased as a direct statement of the boundedness of appropriate Riesz transforms if the group G is not compact. For example, combination of the equivalent Conditions $\mathrm{II}_{nm/2}$ and $\mathrm{II}_{n\underline{m}/2}$ yields bounds

$$\max_{n\underline{m}/2\leq|\alpha|\leq nm/2} \|A^\alpha H^{-n/2}\|_{2\to2} < \infty$$

for all $n \in \mathbb{N}$. Boundedness of the Riesz transforms is directly related to the existence of good asymptotic bounds on the semigroup kernel. The most straightforward statement to this effect is for self-adjoint H.

THEOREM 4.3. *Assume H is positive, symmetric. Then each of the conditions in the following two families, indexed by $n \in \mathbb{N}$, is equivalent to each of the conditions in the two families in Theorem* 4.1.

III_n. *There is a $\mu_n > 0$ such that*

$$(\varphi, H^n\varphi) \geq \mu_n\big(\max_{|\alpha|=nm/2} \|A^\alpha\varphi\|_2^2 + \max_{|\alpha|=n\underline{m}/2} \|A^\alpha\varphi\|_2^2\big)$$

for all $\varphi \in D(H^n)$.

IV_n. *There are $b, c > 0$ such that for each $\alpha \in J(d')$ with $|\alpha| = n$,*

$$\max_{|\alpha|=n} |(A^\alpha K_t)(g)| \leq c(t^{-n/m} \wedge t^{-n/\underline{m}})(G^{(m)}_{b,t} * G^{(\underline{m})}_{b,t})(g)$$

for all $g \in G$ and $t > 0$.

Moreover, if one of the equivalent conditions is satisfied then there are $b, c > 0$ such that

$$|K_t(g)| \leq c(G^{(m)}_{b,t} * G^{(\underline{m})}_{b,t})(g)$$

for all $g \in G$ and $t > 0$.

P r o o f. Condition $\mathrm{II}_{nm/2}$ with $j = 1$ implies that

$$(\varphi, H^n\varphi) \geq \mu \max_{|\alpha|=nm/2} \|A^\alpha\varphi\|_2^2$$

for all $\varphi \in D(H^n)$. Similarly, Condition $\mathrm{II}_{n\underline{m}/2}$ with $j = k$ implies that

$$(\varphi, H^n\varphi) \geq \mu \max_{|\alpha|=n\underline{m}/2} \|A^\alpha\varphi\|_2^2$$

for all $\varphi \in D(H^n)$. Since $\mathrm{II}_i \Leftrightarrow \mathrm{II}_j$ for all $i, j \in \mathbb{N}$ this means that $\mathrm{II}_n \Rightarrow \mathrm{III}_n$. But III_n implies I_1 because

$$\max_{|\alpha|=nm/2} \|A^\alpha S_t\|_{2\to2} \leq ct^{-n/2}$$

and then by use of (18) one deduces that

$$\max_{1\le i\le d'} \|A_i S_t\|_{2\to 2} \le c t^{-1/m}$$

for all $t > 0$. Similarly

$$\max_{1\le i\le d'} \|A_i S_t\|_{2\to 2} \le c t^{-1/\underline{m}}$$

for all $t > 0$. Thus Condition I_1 is valid. Next $\mathrm{IV}_n \Rightarrow \mathrm{I}_n$ by integration. Finally II_n together with the strong Gårding inequality III_1 implies IV_n by the arguments used in the nilpotent case. ■

It is not clear which condition on the coefficients of a non-symmetric operator H implies that H satisfies the strong Gårding inequality (2) on a local direct product group.

The next theorem states that as in Theorem 4.3 the strong Gårding inequality (2) implies Gaussian bounds for the kernel and all its derivatives.

THEOREM 4.4. *If H satisfies the strong Gårding inequality* (2) *then for all $\alpha \in J(d')$ there exist $b, c > 0$ such that*

$$|(A^\alpha K_t)(g)| \le c(t^{-|\alpha|/m} \wedge t^{-|\alpha|/\underline{m}})(G^{(m)}_{b,t} * G^{(\underline{m})}_{b,t})(g)$$

for all $g \in G$ and $t > 0$.

Proof. The proof is as in the nilpotent case. Since (2) is valid Condition II_n of Theorem 4.1 holds and this implies the Gaussian bounds. ■

One can also estimate the difference between the kernel K and its asymptotic limit $K^{(m)}$.

THEOREM 4.5. *Suppose both H and $H_{\underline{m}}$ satisfy the strong Gårding inequality* (2). *Let K and $K^{(\underline{m})}$ denote the kernels associated with H and $H_{\underline{m}}$. Suppose $k \ge 2$ and set $\nu = (m_{k-1} - m_k)/m_k$. Then for all $\alpha \in J(d')$ there exist $b, c > 0$ such that*

$$|(A^\alpha K_t)(g) - (A^\alpha K^{(\underline{m})}_t)(g)| \le c t^{-\nu} t^{-|\alpha|/\underline{m}} (G^{(m)}_{b,t} * G^{(\underline{m})}_{b,t})(g)$$

for all $g \in G$ and $t \ge 1$.

Proof. By Theorem 4.4 both the kernels K and $K^{(\underline{m})}$ have Gaussian bounds for all its derivatives. Then the proof of the theorem is almost a repetition of the proof of Theorem 2.12, but there is one difficulty with the decomposition (16), as there might be more terms involved. Precisely, in the current situation one has

$$L(h^{-1}) A^\alpha L(h) = \sum_{|\alpha|\le|\gamma|\le rs|\alpha|} f_\gamma(h) A^\gamma \tag{21}$$

instead of (16), where s is the rank of the algebraic basis.

We sketch the proof of (21). One only has to consider the case where $|\alpha| = 1$ and $|g|'$ is bounded away from 0, by Lemma 4.3 of [ElR3]. Let K

and N be the connected compact and nilpotent Lie groups such that G is the local direct product of K and N, i.e., $G = KN$, $K \cap N$ is discrete and K and N commute. If $i \in \{1, \ldots, d'\}$, the direction a_i has a unique decomposition $a_i = a_i^{(K)} + a_i^{(N)}$ with $a_i^{(K)} \in \mathfrak{k}$ and $a_i^{(N)} \in \mathfrak{n}$, where $\mathfrak{k}$ and $\mathfrak{n}$ are the Lie algebras of K and N. Then

$$L((kn)^{-1})A_i L(kn) = L(k^{-1})dL(a_i^{(K)})L(k) + L(n^{-1})dL(a_i^{(N)})L(n)$$

for all $k \in K$ and $n \in N$, since K and N commute. Now one can separately estimate each of the two terms on the right hand side, where the estimate on the second term reduces to an application of (16). We omit further details. ■

Let $G = K \times N$ be a direct product and assume H is positive, symmetric. Further let P be the projection onto the subspace of $L_2(G)$ formed by the functions which are constant over K. The key observation in the derivation of the kernel bounds is the spectral estimate

$$\|S_t(I - P)\|_{2\to 2} \leq M e^{-\omega t} \tag{22}$$

for all $t > 0$. Now if the Haar measure on K is normalized such that $|K| = 1$ then the semigroup S restricted to PL_2 has a kernel $\widehat{K}$ with

$$\widehat{K}_t((k, n)) = K_t^{(N)}(n)$$

where $K^{(N)}$ is the kernel of S acting on $L_2(N)$. If (22) is valid, e.g., if the equivalent conditions of Theorem 4.1 are satisfied, one immediately has

$$\begin{aligned} \|K_t - \widehat{K}_t\|_\infty &= \|S_t(I - P)\|_{1\to\infty} \\ &\leq \|S_{t/4}\|_{2\to\infty}^2 \|S_{t/2}(I - P)\|_{2\to 2} \leq cV(t)^{-1/\underline{m}} e^{-\omega t} \end{aligned}$$

for suitable $c, \omega > 0$ and all $t \geq 1$. Therefore the asymptotic form of K can be estimated by the asymptotic form of $\widehat{K}$. But the latter is determined by the nilpotent component and its form has been discussed in Section 2. Finally since K and $\widehat{K}$ are both bounded by a convolution of Gaussians, of order m and $\underline{m}$, the difference $K - \widehat{K}$ has a similar bound. Combining this observation with the uniform bound on the difference one deduces that there are $c', \omega' > 0$ such that

$$\|K_t - \widehat{K}_t\|_1 \leq c' e^{-\omega' t}$$

for all $t \geq 1$.

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