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A LIMIT INVOLVING FUNCTIONS IN $W_0^{1,p}(\Omega)$

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Abstract. We point out the following fact: if $\Omega\subset\mathbb{R}^n$ is a bounded open set, $\delta>0$, and $p>1$, then

$$\lim_{\varepsilon\to0^+}\inf_{u\in V_\varepsilon}\int_\Omega|\nabla u(x)|^p\,dx=\infty,$$

where $V_\varepsilon=\{u\in W_0^{1,p}(\Omega):\mathrm{meas}(\{x\in\Omega:|u(x)|<\delta\})<\varepsilon\}$.

Here and in the sequel, $\Omega\subset\mathbb{R}^n$ is a (non-empty) bounded open set, m denotes the Lebesgue measure in $\mathbb{R}^n$, $\delta>0$, $p>1$, and $W_0^{1,p}(\Omega)$ is the usual Sobolev space, equipped with the norm $\|u\|=(\int_\Omega|\nabla u(x)|^p\,dx)^{1/p}$.

The aim of this paper is to prove the following result which could be useful in certain cases:

THEOREM 1. *For each $\varepsilon>0$, put*

$$V_\varepsilon=\{u\in W_0^{1,p}(\Omega):m(\{x\in\Omega:|u(x)|<\delta\})<\varepsilon\}.$$

Then

$$\lim_{\varepsilon\to0^+}\inf_{u\in V_\varepsilon}\int_\Omega|\nabla u(x)|^p\,dx=\infty.$$

Before giving the proof of Theorem 1, we establish the following proposition:

PROPOSITION 1. *For each $u\in W_0^{1,p}(\Omega)$,*

$$m(\{x\in\Omega:|u(x)|<\delta\})>0.$$

Proof. For simplicity, let us introduce some notation. We first put

$$\Gamma=\{x\in\Omega:|u(x)|<\delta\}.$$

We think of Ω as a subset of $\mathbb{R}\times\mathbb{R}^{n-1}$. If $x\in\mathbb{R}^n$, we set $x=(t,\xi)$, where $t\in\mathbb{R}$ and $\xi\in\mathbb{R}^{n-1}$. We also denote by A (resp. B) the projection of Ω on $\mathbb{R}$ (resp. $\mathbb{R}^{n-1}$), and by m_1 (resp. m_{n-1}) the Lebesgue measure on $\mathbb{R}$ (resp.

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$\mathbb{R}^{n-1}$). So, A and B are (non-empty) open sets, and hence $m_1(A) > 0$ and $m_{n-1}(B) > 0$. Finally, for a generic set $S \subseteq \Omega$ and for each $\xi \in B$, put

$$S_\xi = \{t \in A : (t, \xi) \in S\}.$$

By well-known results ([1], [2]), we can assume that, for almost every $\xi \in B$, the function $u(\cdot, \xi)$ belongs to $W_0^{1,p}(\Omega_\xi)$, and so it is almost everywhere equal to a function which is continuous in $\overline{\Omega_\xi}$ and zero on $\partial\Omega_\xi$. Consequently, we have $m_1(\Gamma_\xi) > 0$ a.e. in B. Now, if χ_Γ denotes the characteristic function of Γ, then Fubini's theorem yields

$$m(\Gamma) = \int_{A\times B} \chi_\Gamma(t, \xi)\, dt\, d\xi = \int_B \Big(\int_{\Gamma_\xi} dt \Big)\, d\xi = \int_B m_1(\Gamma_\xi)\, d\xi > 0,$$

as claimed. ∎

Proof of Theorem 1. Clearly, the function $\varepsilon \mapsto \inf_{u\in V_\varepsilon} \int_\Omega |\nabla u(x)|^p\, dx$ is non-increasing. Consequently,

$$\lim_{\varepsilon\to 0^+} \inf_{u\in V_\varepsilon} \int_\Omega |\nabla u(x)|^p\, dx = \sup_{\varepsilon>0} \inf_{u\in V_\varepsilon} \int_\Omega |\nabla u(x)|^p\, dx.$$

Arguing by contradiction, assume that there is $M > 0$ such that

$$\inf_{u\in V_\varepsilon} \int_\Omega |\nabla u(x)|^p dx < M$$

for all $\varepsilon > 0$. Consider the function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(t) = \begin{cases} \delta - |t| & \text{if } |t| < \delta, \\ 0 & \text{if } |t| \geq \delta. \end{cases}$$

Consider also the functional $\Psi : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by putting

$$\Psi(u) = \int_\Omega g(u(x))\, dx$$

for all $u \in W_0^{1,p}(\Omega)$. Using the Rellich–Kondrashov theorem, one sees that Ψ is sequentially weakly continuous in $W_0^{1,p}(\Omega)$. Now, for each $h \in \mathbb{N}$, choose $u_h \in V_{1/(h\delta)}$ in such a way that

$$\int_\Omega |\nabla u_h(x)|^p\, dx < M.$$

So, the sequence $\{u_h\}$ is bounded in $W_0^{1,p}(\Omega)$. Consequently, since $p > 1$, there is a subsequence $\{u_{h_k}\}$ weakly converging to some $u_0 \in W_0^{1,p}(\Omega)$. For each $k \in \mathbb{N}$, we have

$$\Psi(u_{h_k}) = \int_{\{x\in\Omega: |u_{h_k}(x)|<\delta\}} (\delta - |u_{h_k}(x)|)\, dx < \frac{1}{h_k\delta}\delta = \frac{1}{h_k}.$$

Passing to the limit as $k \to \infty$, we then get $\Psi(u_0) = 0$. This implies that $m(\{x \in \Omega : |u_0(x)| < \delta\}) = 0$, contrary to Proposition 1. ■

For $p = 1$, we have the following result:

THEOREM 2. *Let* $n = 1$. *For each* $\varepsilon > 0$, *put*

$$U_\varepsilon = \{u \in W_0^{1,1}(\Omega) : m(\{x \in \Omega : |u(x)| < \delta\}) < \varepsilon\}.$$

If k *denotes the number (possibly infinite) of connected components of* Ω, *then*

$$\lim_{\varepsilon \to 0^+} \inf_{u \in U_\varepsilon} \int_\Omega |u'(x)|\, dx = 2k\delta.$$

Proof. First, assume that k is finite. Let $]a_i, b_i[$ $(i = 1, \ldots, k)$ denote the connected components of Ω. Suppose that $\varepsilon \le \min_{1 \le i \le k}(b_i - a_i)$. Let $v \in U_\varepsilon$. We can assume that v is absolutely continuous in each interval $[a_i, b_i]$. Fix i. Since $v(a_i) = v(b_i) = 0$, due to the choice of ε, there is $x_i \in\,]a_i, b_i[$ such that $|v(x_i)| = \delta$. Assume, for instance, that $v(x_i) = \delta$. Then

$$\delta = \int_{a_i}^{x_i} v'(x)\, dx \le \int_{a_i}^{x_i} |v'(x)|\, dx$$

and

$$\delta = -\int_{x_i}^{b_i} v'(x)\, dx \le \int_{x_i}^{b_i} |v'(x)|\, dx.$$

Hence,

$$2\delta \le \int_{a_i}^{b_i} |v'(x)|\, dx.$$

With obvious changes, one gets this inequality also if $v(x_i) = -\delta$. Consequently,

$$2k\delta \le \sum_{i=1}^{k} \int_{a_i}^{b_i} |v'(x)|\, dx = \int_\Omega |v'(x)|\, dx.$$

We then infer that

$$2k\delta \le \inf_{u \in U_\varepsilon} \int_\Omega |u'(x)|\, dx. \tag{1}$$

Now, consider the function $w : \Omega \to \mathbb{R}$ defined by

$$w(x) = \begin{cases} 4k\delta(x - a_i)/\varepsilon & \text{if } x \in\,]a_i, a_i + \varepsilon/(4k)], \\ \delta & \text{if } x \in\,]a_i + \varepsilon/(4k), b_i - \varepsilon/(4k)[, \\ 4k\delta(b_i - x)/\varepsilon & \text{if } x \in [b_i - \varepsilon/(4k), b_i[. \end{cases}$$

Clearly, $w \in U_\varepsilon$. Moreover, a simple calculation gives $\int_\Omega |w'(x)|\,dx = 2k\delta$. This and (1) then show that

$$\inf_{u\in U_\varepsilon} \int_\Omega |u'(x)|\,dx = 2k\delta.$$

Therefore, our conclusion is proved when k is finite.

Now, assume that Ω has infinitely many connected components. Let $r \in \mathbb{N}$. Let $]\alpha_i, \beta_i[$ $(i = 1, \dots, r)$ be r distinct connected components of Ω. Fix $\varepsilon \le \min_{1\le i\le r}(\beta_i - \alpha_i)$, and let $v \in U_\varepsilon$. Then, from the first part of the proof, we know that

$$2r\delta \le \sum_{i=1}^{r} \int_{\alpha_i}^{\beta_i} |v'(x)|\,dx \le \int_\Omega |v'(x)|\,dx.$$

Hence,

$$2r\delta \le \inf_{u\in U_\varepsilon} \int_\Omega |u'(x)|\,dx.$$

This, of course, implies that

$$\lim_{\varepsilon\to 0^+} \inf_{u\in U_\varepsilon} \int_\Omega |u'(x)|\,dx = \infty,$$

and the proof is complete. ■

REFERENCES

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