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ONE-PARAMETER FAMILIES OF BRAKE ORBITS IN DYNAMICAL SYSTEMS

BY

LENNARD F. BAKKER (PROVO, UT)

Abstract. We give a clear and systematic exposition of one-parameter families of brake orbits in dynamical systems on product vector bundles (where the fiber has the same dimension as the base manifold). A generalized definition of a brake orbit is given, and the relationship between brake orbits and periodic orbits is discussed. The brake equation, which implicitly encodes information about the brake orbits of a dynamical system, is defined. Using the brake equation, a one-parameter family of brake orbits is defined as well as two notions of nondegeneracy by which a given brake orbit embeds into a one-parameter family of brake orbits. The duality between the two notions of nondegeneracy for a brake orbit in a one-parameter family is described. Finally, four ways in which a given periodic brake orbit generates a one-parameter family of periodic brake orbits are detailed.

1. Introduction. First coined by Ruiz in 1977 (see [22] and [26]), the term "brake orbit" describes a particular kind of nonequilibrium solution of a dynamical system. In the context of a vector field on a vector bundle (where the dimension of the fiber is the same as that of the base manifold), a brake orbit is a nonequilibrium solution of the vector field whose fiber part vanishes at least twice. Brake orbits have found application in Hamiltonian dynamical systems (that satisfy a certain reversibility condition) as a tool productive in proving the existence of periodic orbits of prescribed energy. For example, see [1], [4], [6], [9], [11], [15], [18], [19], [20], [22], [23], [15], [13], [14], and [26]. However, not every brake orbit is a special kind of periodic orbit.

The rigorous development and exposition of a theory of one-parameter families of brake orbits in distinction to the well-known theory of one-parameter families of periodic orbits is therefore appropriate. Indeed, "...the final goal of dynamics embraces the characterization of *all types of movements* and of their interrelation" (italics added, p. 682 of [5]). That which is presented here is a refinement of the theory of one-parameter families of brake orbits as described in [3]. Although much of this theory is implicitly known by researchers in the field (at least within the context of Hamil-

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tonian dynamical systems), it has not previously been stated clearly and systematically in the literature.

2. Brake orbits and the brake equation. Consider the product vector bundle $M = Q \times \mathbb{R}^n$, where Q is a smooth manifold of dimension n. By smooth we mean C^∞. We let q denote a point in Q, and let p denote a point in the fiber $\{q\} \times \mathbb{R}^n$. A point in M is denoted by $z = (q, p)$. The vector bundle M comes equipped with three maps. The *canonical projection* $\pi : M \to Q$ is defined by $\pi(q, p) = q$. The *canonical injection* (or *zero-section map*) $\sigma : Q \to M$ is defined by $\sigma(q) = (q, 0)$. The *fiber projection* $\pi' : M \to \mathbb{R}^n$ is defined by $\pi'(q, p) = p$.

Let X be a smooth vector field on M and let F_t be the flow it generates. We assume that the flow is complete. The *orbit* of the flow F_t through an initial condition z is the directed set $\gamma(z) = \{F_t(z) : t \in \mathbb{R}\}$, where the direction is determined by the evolution of F_t as the time t progresses from $-\infty$ to ∞. The zero-section of M is the embedded submanifold $\sigma(Q)$. The set of all zero-section equilibria of a vector field X on M is $Z_{\mathrm{e}}(X) = \{q \in Q : X(\sigma(q)) = 0\}$.

The notion of a brake describes the relationship between the zero-section $\sigma(Q)$ and the flow F_t. An orbit $\gamma(z_0)$ is said to *brake* at time t if

$$\pi' F_t(z_0) = 0.$$

Geometrically, this says that the orbit $\gamma(z)$ intersects $\sigma(Q)$. With each orbit $\gamma(z_0)$ we associate two sets, the *braking point set* $\mathfrak{BP}(z_0) = \{F_t(z_0) : \pi' F_t(z_0) = 0,\ t \in \mathbb{R}\}$, and the *braking time set* $\mathfrak{BT}(z_0) = \{t \in \mathbb{R} : \pi' F_t(z_0) = 0\}$. For a set S, we denote by $|S|$ the cardinality of S.

Definition 2.1. An orbit $\gamma(z_0)$ of F_t is called a *brake orbit* if $X(z_0) \neq 0$ and $|\mathfrak{BT}(z_0)| \geq 2$.

For a brake orbit $\gamma(z_0)$ there is, by a time translation, a point q_0 in Q such that $\gamma(\sigma(q_0)) = \gamma(z_0)$. Because $X(z_0) \neq 0$, the point q_0 does not belong to $Z_{\mathrm{e}}(X)$. A nonzero time T_0 in $\mathfrak{BT}(\sigma(q_0))$ is called an *interbraking time* for $\gamma(\sigma(q_0))$. Because $\gamma(\sigma(q_0))$ brakes at time T_0, the braking point $F_{T_0}\sigma(q_0)$ belongs to $\sigma(Q)$. The pair (T_0, q_0) is a solution of the *brake equation*

$$\pi' F_t \sigma(q) = 0.$$

This implicit equation encodes information about the brake orbits of the flow F_t. Certainly, a necessary condition for the existence of brake orbits is that $Z_{\mathrm{e}}(X) \neq Q$.

Solutions of the brake equation are not in a one-to-one correspondence with the brake orbits of the flow F_t. If (T_0, q_0) is a solution of the brake equation and $T \in \mathfrak{BT}(\sigma(q_0))$ is not equal to T_0, then the pairs $(-T, \pi F_T \sigma(q_0))$ and $(T_0 - T, \pi F_T \sigma(q_0))$ are also solutions of the brake equation. We say that two solutions (t, q) and $(\widetilde{t}, \widetilde{q})$ of the brake equation are *equivalent* if and only if $\gamma(\sigma(q)) = \gamma(\sigma(\widetilde{q}))$. This is an equivalence relation on the solutions of the brake equation. Consequently, there is a one-to-one correspondence between the equivalence classes of solutions of the brake equation and the brake orbits of F_t.

A given brake orbit may or may not be periodic. A brake orbit $\gamma(\sigma(q_0))$ is called a *periodic brake orbit* if there is a $t \neq 0$ such that $F_t\sigma(q_0) = \sigma(q_0)$. A periodic brake orbit is a special kind of periodic orbit. Given a periodic brake orbit $\gamma(\sigma(q_0))$ (which is a nonequilibrium orbit by definition), the smallest positive time T for which $F_T\sigma(q_0) = \sigma(q_0)$ is called the *prime period*. On the other hand, there may or may not be a smallest positive interbraking time for a brake orbit (periodic or otherwise). That is, given a brake orbit $\gamma(\sigma(q_0))$, the set of positive interbraking times $\mathfrak{BT}^+(\sigma(q_0))$ may be empty or $\inf \mathfrak{BT}^+(\sigma(q_0))$ may be zero.

DEFINITION 2.2. A brake orbit $\gamma(z_0)$ is called *finite* if $\mathfrak{BP}(z_0)$ is finite.

PROPOSITION 2.3. *If $\gamma(\sigma(q_0))$ is a finite brake orbit for which $\mathfrak{BT}^+(\sigma(q_0))$ is nonempty, then $\gamma(\sigma(q_0))$ has a smallest positive interbraking time.*

Proof. Suppose that there is not a smallest positive interbraking time for $\gamma(\sigma(q_0))$. Then, as $\mathfrak{BT}^+(\sigma(q_0))$ is nonempty, there is a positive sequence $t_j \to 0$ for which $\pi' F_{t_j}\sigma(q_0) = 0$. So $\{t_j\} \subset \mathfrak{BT}(\sigma(q_0))$ and $\{F_{t_j}\sigma(q_0)\} \subset \mathfrak{BP}(\sigma(q_0))$. As $\gamma(\sigma(q_0))$ is a finite brake orbit, there is a $\sigma(q_1) \in \mathfrak{BP}(\sigma(q_0))$ and a subsequence t_{j_k} such that $F_{t_{j_k}}\sigma(q_1) = \sigma(q_1)$. Hence $\gamma(\sigma(q_1))$ is a periodic brake orbit with periods t_{j_k}. Thus, each t_{j_k} must be an integer multiple of the prime period of $\gamma(\sigma(q_1))$. But this is impossible as $t_{j_k} \to 0$. ■

PROPOSITION 2.4. *If $\gamma(\sigma(q_0))$ is a finite brake orbit and $|\mathfrak{BT}(\sigma(q_0))| = \infty$, then $\gamma(\sigma(q_0))$ is a periodic brake orbit which has a smallest positive interbraking time.*

Proof. By the hypotheses, $|\mathfrak{BP}(\sigma(q_0))| < |\mathfrak{BT}(\sigma(q_0))| = \infty$. So there are $T_1, T_2 \in \mathfrak{BT}(\sigma(q_0))$ with $T_1 \neq T_2$ for which $F_{T_1}\sigma(q_0) = F_{T_2}\sigma(q_0)$. Thus, $F_{T_2 - T_1}\sigma(q_0) = \sigma(q_0)$ and hence $\gamma(\sigma(q_0))$ is a periodic brake orbit. As we can take $T_2 > T_1$, it follows that $T_2 - T_1 \in \mathfrak{BT}^+(\sigma(q_0))$, and so by Proposition 2.2, $\gamma(\sigma(q_0))$ has a smallest positive interbraking time. ■

The use of Proposition 2.4 to determine if a given brake orbit is a periodic brake orbit is limited to the knowledge one has of the flow F_t. On the other hand, there is a global condition on the vector field X which implies that every brake orbit (finite or otherwise) is a periodic brake orbit. The *momentum reversing involution* $R : M \to M$ is defined by $R(q, p) = (q, -p)$. We say that X is *R-reversible* if $R_*X = -X$ where $R_*X = TR \circ X \circ R^{-1}$.

PROPOSITION 2.5. *Suppose that X is R-reversible. Then any brake orbit $\gamma(\sigma(q_0))$ of X is a periodic brake orbit. Moreover, if $\gamma(\sigma(q_0))$ is a finite brake orbit, then $|\mathfrak{BP}(\sigma(q_0))| = 2$ and the prime period of $\gamma(\sigma(q_0))$ is $2T_0$, where T_0 is the smallest positive interbraking time of $\gamma(\sigma(q_0))$.*

Proof. The R-reversibility of X is equivalent to R and F_t satisfying $RF_{-t} = F_tR$ (see [3]). Any point in $\sigma(Q)$ is a fixed point of R, that is, $R(\sigma(q)) = \sigma(q)$. Let T_0 be an interbraking time for a brake orbit $\gamma(\sigma(q_0))$. Then

$$F_{2T_0}\sigma(q_0) = F_{T_0}RF_{T_0}\sigma(q_0) = RF_{-T_0}F_{T_0}\sigma(q_0) = \sigma(q_0),$$

and hence $\gamma(\sigma(q_0))$ is a periodic brake orbit. (This is a well-known fact within the context of an R-reversible Hamiltonian system [10], [16].)

Now suppose that $\gamma(\sigma(q_0))$ is a finite brake orbit. By the periodicity of $\gamma(\sigma(q_0))$, $\mathfrak{BT}^+(\sigma(q_0))$ is not empty. By Proposition 2.3, we can take T_0 to be the smallest positive interbraking time of $\gamma(\sigma(q_0))$. By the calulcation of the previous paragraph, $2T_0$ is a period of $\gamma(\sigma(q_0))$.

We show that T_0 is not the prime period. On the contrary, suppose it is. Then $F_{T_0}\sigma(q_0) = \sigma(q_0)$. Applying $F_{-T_0/2}$ to both sides we get $F_{T_0/2}\sigma(q_0) = F_{-T_0/2}\sigma(q_0)$. Then

$$RF_{T_0/2}\sigma(q_0) = RF_{-T_0/2}\sigma(q_0) = F_{T_0/2}R\sigma(q_0) = F_{T_0/2}\sigma(q_0).$$

This says that $F_{T_0/2}\sigma(q_0) \in \sigma(Q)$. Hence $T_0/2$ is a positive interbraking time that is smaller than T_0.

Now we show that $2T_0$ is the prime period. On the contrary, suppose it is not. Then by periodicity, $2T_0/k$, for an integer $k \geq 2$, is the prime period. We eliminated the case $k = 2$ above. So $k \geq 3$. But then $2T_0/k$ would be a positive interbraking time smaller than T_0.

Finally, we show that $|\mathfrak{BP}(\sigma(q_0))| = 2$. Suppose that $\sigma(q_0)$ is the only braking point of $\gamma(\sigma(q_0))$. Then $F_{T_0}\sigma(q_0) = \sigma(q_0)$, and arguing as above, we reach a contradiction. So $|\mathfrak{BP}(\sigma(q_0))| \geq 2$. Now suppose that there is a third braking point $F_{T_1}\sigma(q_0)$ besides $\sigma(q_0)$ and $F_{T_0}\sigma(q_0)$. By periodicity and the fact that T_0 is the smallest positive interbraking time, we can take $T_0 < T_1 < 2T_0$. Then $T_2 = 2T_0 - T_1$ satisfies $0 < T_2 < T_0$. Also, as

$$F_{T_2}F_{T_1}\sigma(q_0) = F_{2T_0-T_1+T_1}\sigma(q_0) = \sigma(q_0),$$

T_2 belongs to $\mathfrak{BT}(F_{T_1}\sigma(q_0))$. By the argument that implies $2T_0$ is a period of $\gamma(\sigma(q_0))$, we deduce that $2T_2$ is a period of $\gamma(F_{T_1}\sigma(q_0)) = \gamma(\sigma(q_0))$. But $2T_2 < 2T_0$, where $2T_0$ is the prime period of $\gamma(\sigma(q_0))$, a contradiction. Hence $|\mathfrak{BP}(\sigma(q_0))| \leq 2$. ■

Within the class of reversible Hamiltonian systems, the term "brake orbit", as coined by Ruiz, corresponds to our notion of a brake orbit which is finite (with exactly two braking points) and periodic. However, in the class of linear Hamiltonian systems, there are examples of periodic brake orbits for which the number of braking points is larger than 2 but finite (see [11]) as well as examples of nonperiodic brake orbits for which the number of braking points is exactly two (see [3]). In these examples, the corresponding vector fields are not R-reversible.

By their nature, brake orbits can have a relationship with orbits homoclinic and heteroclinic to zero-section equilibria. Indeed, one may think of an orbit homoclinic to a zero-section equilibrium or an orbit heteroclinic to two zero-section equilibria as a "brake orbit with an infinite interbraking time". It is this idea that underlies some of the existence results for homoclinic and heteroclinic connections in Hamiltonian systems (see [21] and [2], for example).

3. One-parameter families of brake orbits. With the brake equation, we define the notion of a one-parameter family of brake orbits for a generic parameter. Let η be a real parameter belonging to an interval (η^-, η^+). A function $\tau : (\eta^-, \eta^+) \to \mathbb{R} \setminus \{0\}$ is called an *interbraking time function* if it is continuous. A function $\vartheta : (\eta^-, \eta^+) \to Q \setminus Z_{\mathrm{e}}(X)$ is called an *initial brake function* if it is a local homeomorphism.

Definition 3.1. A *one-parameter family of brake orbits* is a curve $\eta \mapsto (\tau(\eta), \vartheta(\eta))$ for η in an interval (η^-, η^+) where τ is an interbraking time function and ϑ is an initial brake function such that $\pi' F_{\tau(\eta)} \sigma \vartheta(\eta) = 0$ for all η in (η^-, η^+).

If the interbraking time and initial brake functions of a one-parameter family of brake orbits are smooth, we say that the one-parameter family of brake orbits is *smooth*.

The parameter is not intrinsic to a one-parameter family of brake orbits: we can always reparameterize the family. Let ν be a real parameter belonging to the interval (ν^-, ν^+). Let $h : (\eta^-, \eta^+) \to (\nu^-, \nu^+)$ be a homeomorphism. Define $\overline{\tau} = \tau h^{-1}$ and $\overline{\vartheta} = \vartheta h^{-1}$. Then $\nu \mapsto (\overline{\tau}(\nu), \overline{\vartheta}(\nu))$, $\nu \in (\nu^-, \nu^+)$, is a *reparameterization* of $\eta \mapsto (\tau(\eta), \vartheta(\eta))$, $\eta \in (\eta^-, \eta^+)$.

Rather than reparameterizing the whole of a one-parameter family of brake orbits, we may reparameterize just a part. Let U be an open subinterval of (η^-, η^+). Restricting τ and ϑ to U we obtain a *subfamily* $\eta \mapsto (\tau(\eta), \vartheta(\eta))$, $\eta \in U$. Let $\nu \in (\nu^-, \nu^+)$ and let $h : U \to (\nu^-, \nu^+)$ be a homeomorphism. With $\overline{\tau} = \tau h^{-1}$ and $\overline{\vartheta} = \vartheta h^{-1}$, the one-parameter family $\nu \mapsto (\overline{\tau}(\nu), \overline{\vartheta}(\nu))$, $\nu \in (\nu^-, \nu^+)$, is a *reparameterization of a subfamily* of $\eta \mapsto (\tau(\eta), \vartheta(\tau))$.

The requirement that the initial brake function be a local homeomorphism is to allow for different solutions of the brake equation to correspond to different brake orbits. For example, if (T_0, q_0) is a solution of the brake equation, we want to avoid calling $\eta \mapsto (T_0, q_0)$ a one-parameter family of brake orbits. The local homeomorphism property of the initial brake function does not, however, prevent it from self-intersections. A self-intersection means that the one-parameter family of brake orbits visits the same brake orbit twice, and allows for the possibility that it is "periodic" in the parameter.

DEFINITION 3.2. A one-parameter family of brake orbits $\eta \mapsto (\tau(\eta), \vartheta(\eta))$, $\eta \in (\eta^-, \eta^+)$ is called *cyclic* if it has a reparameterization $\nu \mapsto (\overline{\tau}(\nu), \overline{\vartheta}(\nu))$, $\nu \in \mathbb{R}$, that is periodic in ν.

4. Nondegeneracy with respect to an interbraking time.

A natural choice for the parameter in a one-parameter family of brake orbits is an interbraking time. Given a brake orbit $\gamma(\sigma(q_0))$ with an interbraking time $T_0 \neq 0$, we can consider the problem of solving the brake equation $\pi' F_t \sigma(q) = 0$ for a one-parameter family of brake orbits $T \mapsto (T, \vartheta(T))$, where T belongs to an interval (T^-, T^+) containing T_0 (but not zero) and $\vartheta(T_0) = q_0$. Let $\mathbf{T}$ denote the tangent functor.

DEFINITION 4.1. A brake orbit $\gamma(\sigma(q_0))$ with an interbraking time of T_0 is called *nondegenerate with respect to* T_0 if the linear map

$$K(T_0) = \frac{\partial \pi' F_t \sigma(q)}{\partial q}\bigg|_{t=T_0,\, q=q_0} = \mathbf{T}\pi'(F_{T_0}\sigma(q_0))\mathbf{T}F_{T_0}(\sigma(q_0))\mathbf{T}\sigma(q_0)$$

from $\mathbf{T}_{q_0}Q$ to $\mathbb{R}^n$ is invertible; otherwise, the brake orbit $\gamma(\sigma(q_0))$ is called *degenerate with respect to* T_0.

Define $Y(z) = \mathbf{T}\pi'(z)X(z)$.

PROPOSITION 4.2. *If $\gamma(\sigma(q_0))$ is a brake orbit that is nondegenerate with respect to T_0, then $Y(\sigma(q)) \neq 0$ for some $q \in Q$.*

Proof. Suppose that $Y(\sigma(q)) = 0$ for all $q \in Q$. Then $\sigma(Q)$ is an invariant set for F_t. Thus $\pi' F_t \sigma(q) \equiv 0$. This implies that $K(T_0) \equiv 0$. Therefore, any brake orbit is degenerate with respect to every one of its interbraking times. ∎

DEFINITION 4.3. The vector field X is called *transverse to $\sigma(Q \setminus Z_{\mathrm{e}}(X))$ at $\sigma(q)$* if $Y(\sigma(q_0)) \neq 0$. It is called *transverse to $\sigma(Q \setminus Z_{\mathrm{e}}(X))$* if it is transverse to $\sigma(Q \setminus Z_{\mathrm{e}}(X))$ at $\sigma(q)$ for all $q \in Q \setminus Z_{\mathrm{e}}(X)$.

THEOREM 4.4. *Suppose that $\gamma(\sigma(q_0))$ is a brake orbit with an interbraking time of $T_0 \neq 0$. If X is transverse to $\sigma(Q \setminus Z_{\mathrm{e}}(X))$ at $F_{T_0}\sigma(q_0)$ and $\gamma(\sigma(q_0))$ is nondegenerate with respect to T_0, then there is a unique and smooth one-parameter family of brake orbits $T \mapsto (T, \vartheta(T))$, $T \in (T^-, T^+)$, such that $T_0 \in (T^-, T^+)$ and $\vartheta(T_0) = q_0$, and for each $T \in (T^-, T^+)$, ϑ is* *an immersion at* T, *the brake orbit* $\gamma(\vartheta(T))$ *is nondegenerate with respect to* T, *and* X *is transverse to* $\sigma(Q \setminus Z_e(X))$ *at* $F_T\sigma\vartheta(T)$.

REMARK. The verification of the transversality of X on $\sigma(Q \setminus Z_e(X))$ at $F_{T_0}\sigma(q_0)$ is problematic because the point $F_{T_0}\sigma(q_0)$ is not, in general, explicitly known. In practice, the stronger condition that X is transverse to $\sigma(Q \setminus Z_e(X))$ is assumed. This stronger condition is satisfied, for example, by the vector field of a mechanical, or kinetic plus (minus) potential, Hamiltonian system on M.

Proof (of Theorem 4.4.) Suppose that $\gamma(\sigma(q_0))$ is nondegenerate with respect to T_0. Then by the Implicit Function Theorem, there is an interval (T^-, T^+) containing T_0 and a unique and smooth function $\vartheta : (T^-, T^+) \to Q$ such that $\vartheta(T_0) = q_0$ and $\pi' F_T\sigma\vartheta(T) = 0$ for all T in (T^-, T^+). Moreover, we can take (T^-, T^+) so that it does not contain 0 and $\vartheta(T) \notin Z_e(X)$ for all $T \in (T^-, T^+)$. It follows from the proof of the Implicit Function Theorem that

$$K(T) = \left.\frac{\partial \pi' F_t\sigma(q)}{\partial q}\right|_{t=T, q=\vartheta(T)}$$

is invertible for $T \in (T^-, T^+)$. Hence, each brake orbit $\gamma(\sigma\vartheta(T))$ in the family is nondegenerate with respect to T.

We show that ϑ is an immersion. To get an equation that $d\vartheta/dT$ at $T = T_0$ satisfies, we differentiate $\pi' F_T\sigma(\vartheta(T)) = 0$ with respect to T and set $T = T_0$. Thus, after some manipulation, we obtain

$$K(T_0)\left.\frac{d\vartheta}{dT}\right|_{T=T_0} = -Y(F_T\sigma(q_0)).$$

By the hypotheses, $K(T_0)$ is invertible and $Y(F_{T_0}\sigma(q_0)) \neq 0$. Thus, we can take (T^-, T^+) so that $d\vartheta/dT \neq 0$ for all T in (T^-, T^+). So, $T \mapsto \vartheta(T)$ is an immersion, and hence it is a local homeomorphism [12].

By the continuity of X and the fact that X is transverse to $\sigma(Q \setminus Z_e(X))$ at $F_{T_0}\sigma(q_0)$, we can take (T^-, T^+) so that X is transverse to $\sigma(Q \setminus Z_e(X))$ at $F_T\sigma\vartheta(T)$ for all T in (T^-, T^+). ∎

A one-parameter family of brake orbits $T \mapsto (T, \vartheta(T))$, $T \in (T^-, T^+)$, may be a subfamily of a cyclic one-parameter family of brake orbits, but cannot be cyclic in and of itself. Suppose on the contrary that it is cyclic. Then there is a reparameterization $\nu \mapsto (h^{-1}(\nu), \vartheta(h^{-1}(\nu)))$, $\nu \in \mathbb{R}$, which is periodic in ν where $h : (T^-, T^+) \to \mathbb{R}$ is a homeomorphism. So there exists a ν_0 such that $h^{-1}(\nu + \nu_0) = h^{-1}(\nu)$ for all $\nu \in \mathbb{R}$. Then $h^{-1}(\nu_0) = h^{-1}(0)$, which implies that h is not a homeomorphism, a contradiction.

For an R-reversible vector field X we can say more about brake orbits, their nondegeneracy with respect to an interbraking time, and their inclusion into one-parameter families of brake orbits.

PROPOSITION 4.5. *If X is R-reversible, then X is transverse to $\sigma(Q \setminus Z_{\mathrm{e}}(X))$.*

Proof. Let $q \in Q \setminus Z_{\mathrm{e}}(X)$. By the R-reversibility of X, $\mathbf{T}RX(\sigma(q)) = -X(R(\sigma(q))) = -X(\sigma(q))$. The discrete symmetry R satisfies $\pi R = \pi$. Hence

$$\mathbf{T}\pi X(\sigma(q)) = \mathbf{T}(\pi R)X(\sigma(q)) = \mathbf{T}\pi\mathbf{T}RX(\sigma(q)) = -\mathbf{T}\pi X(\sigma(q)).$$

This implies that $\mathbf{T}\pi X(\sigma(q)) = 0$. As $q \notin Z_{\mathrm{e}}(X)$, we have $X(\sigma(q)) \neq 0$, and so it follows that $\mathbf{T}\pi' X(\sigma(q)) \neq 0$. ■

PROPOSITION 4.6. *Suppose that $\gamma(\sigma(q_0))$ is a brake orbit. If the vector field X is R-reversible, then $T_0 \in \mathfrak{BT}(\sigma(q_0))$ if and only if $-T_0 \in \mathfrak{BT}(\sigma(q_0))$, and moreover, $\gamma(\sigma(q_0))$ is nondegenerate with respect to T_0 if and only if $\gamma(\sigma(q_0))$ is nondegenerate with respect to $-T_0$.*

Proof. The R-reversibility of X implies that $RF_t = F_{-t}R$. As $R\sigma(q) = \sigma(q)$ for all $q \in \sigma(Q)$ and $\pi' R = -\pi'$, we have

$$\pi' F_t \sigma(q) = \pi' F_t R\sigma(q) = \pi' RF_{-t}\sigma(q) = -\pi' F_{-t}\sigma(q).$$

It follows that $\pi' F_{T_0}\sigma(q_0) = 0$ if and only if $\pi' F_{-T_0}\sigma(q_0) = 0$. Differentiating $\pi' F_t\sigma(q) = -\pi' F_{-t}\sigma(q)$ with respect to q and evaluating at $t = T_0$, $q = q_0$ we get $K(T_0) = -K(-T_0)$. Hence, $\gamma(\sigma(q_0))$ is nondegenerate with respect to T_0 if and only if $\gamma(\sigma(q_0))$ is nondegenerate with respect to $-T_0$. ■

THEOREM 4.7. *If X is R-reversible and $\gamma(\sigma(q_0))$ is a brake orbit that is nondegenerate with respect to T_0, then there exists a unique and smooth one-parameter family $T \mapsto (T, \vartheta(T))$, $T \in (T^-, T^+)$, of periodic brake orbits for which $\vartheta(T_0) = q_0$.*

Proof. By Proposition 4.5, X is tranverse to $\sigma(Q \setminus Z_{\mathrm{e}}(X))$ at $F_{T_0}\sigma(q_0)$. Thus by Theorem 4.4, there is a unique and smooth one-parameter family $T \mapsto (T, \vartheta(T))$, $T \in (T^-, T^+)$, of brake orbits for which $\vartheta(T_0) = q_0$. By Proposition 2.5, each of the brake orbits in the family is a periodic brake orbit. ■

5. Nondegeneracy with respect to a first integral. Another natural choice for the parameter in a one-parameter family of brake orbits is the value of a first integral. Recall that a smooth function $H : M \to \mathbb{R}$ is a *first integral* of X if $dH(X) = 0$. We let E denote the value of H. Given a first integral H and a brake orbit $\gamma(\sigma(q_0))$ with a first integral value of $E_0 = H(\sigma(q_0))$ and an interbraking time of $T_0 \neq 0$, we can consider the problem of solving the equations

$$H(\sigma(q)) - E = 0, \qquad \pi' F_t \sigma(q) = 0$$

simultaneously for a one-parameter family of brake orbits $E \mapsto (\tau(E), \vartheta(E))$, where E belongs to an interval (E^-, E^+) containing E_0, $\tau(E_0) = T_0$, and $\vartheta(E_0) = q_0$.

DEFINITION 5.1. Suppose that H is a first integral of X. A brake orbit $\gamma(\sigma(q_0))$ with a first integral value of $E_0 = H(\sigma(q_0))$ and an interbraking time of $T_0 \neq 0$ is called *nondegenerate with respect to* E_0 *given* T_0 if the linear map

$$L(E_0, T_0) = \frac{\partial((H(\sigma(q)) - E) \times \pi' F_t \sigma(q))}{\partial(t, q)}\bigg|_{t=T_0,\, q=q_0,\, E=E_0}$$

from $\mathbb{R} \times \mathbf{T}_{q_0} Q$ to $\mathbb{R} \times \mathbb{R}^n$ is invertible; otherwise the brake orbit $\gamma(\sigma(q_0))$ is called *degenerate with respect to* E_0 *given* T_0.

There is a more workable expression for the linear map $L(E_0, T_0)$. For $(w, v) \in \mathbb{R} \times T_{q_0} Q$,

$$L(E_0, T_0) : (w, v) \mapsto d(H\sigma)(q_0)v \times [Y(T_0)w + K(T_0)v],$$

where, by a slight abuse of notation, $Y(T_0) = \mathbf{T}\pi'(F_{T_0}\sigma(q_0))X(F_{T_0}\sigma(q_0))$.

PROPOSITION 5.2. *Suppose that H is a first integral for X. If $\gamma(\sigma(q_0))$ is a brake orbit that is nondegenerate with respect to $E_0 = H(\sigma(q_0))$ given an interbraking time T_0, then q_0 is a regular point of $H\sigma$ (that is, $d(H\sigma)(q_0) \neq 0$), and* $\operatorname{rank} K(T_0) \geq n - 1$.

Proof. If q_0 is a critical point of $H\sigma$, then $L(E_0, T_0)$ is not surjective. If the rank of $K(T_0)$ is smaller than $n-1$, then $\operatorname{span}\{Y(T_0)\} + \operatorname{range} K(T_0) \neq \mathbb{R}^n$ and so $L(E_0, T_0)$ is not surjective. ■

THEOREM 5.3. *Suppose that H is a first integral of X and suppose that $\gamma(\sigma(q_0))$ is a brake orbit with a first integral value of $E_0 = H\sigma(q_0)$ and an interbraking time of T_0. Then $\gamma(\sigma(q_0))$ is nondegenerate with respect to E_0 given T_0 if and only if*

(1) *X is transverse to $\sigma(Q \setminus Z_{\mathrm{e}}(X))$ at $F_{T_0}\sigma(q_0)$, that is, $Y(T_0) \neq 0$,*
(2) $\ker d(H\sigma)(q_0) \cap \ker K(T_0) = \{0\}$, *and*
(3) $K(T_0)(\ker d(H\sigma)(q_0)) \cap \operatorname{span}\{Y(T_0)\} = \{0\}$.

Proof. Suppose that (1)–(3) hold. Set $L(E_0, T_0)(w, v) = (0, 0)$. Then $v \in \ker d(H\sigma)(q_0)$ and $Y(T_0)w + K(T_0)v = 0$. By (3), $Y(T_0)w = 0$ and $K(T_0)v = 0$. By (1), $w = 0$, and by (2), $v = 0$. Therefore $L(E_0, T_0)$ is injective and hence invertible.

On the other hand, if X is not transverse to $\sigma(Q \setminus Z_{\mathrm{e}}(X))$ at $F_{T_0}\sigma(q_0)$, then $(w, v) \mapsto d(H\sigma)(q_0)v \times K(T_0)v$ is not injective. Also, if there is $0 \neq v \in \ker d(H\sigma)(q_0) \cap \ker K(T_0)$, then $L(E_0, T_0)(0, v) = (0, 0)$, which means that $L(E_0, T_0)$ is not injective. Lastly, suppose that there is a $0 \neq v \in K(T_0)(\ker d(H\sigma)(q_0)) \cap \operatorname{span}\{Y(T_0)\}$. We can take $v = Y(T_0)$. Let

$u \in \ker d(H\sigma)(q_0)$ be such that $v = K(T_0)u$. Then $L(E_0, T_0)(-1, u) = (d(H\sigma(q_0)u, -v + K(T_0)u) = (0, 0)$ and hence $L(E_0, T_0)$ is not injective. ∎

The two notions of nondegeneracy for a brake orbit are independent of each other. The invertibility of $K(T_0)$ does not imply the invertibility of $L(E_0, T_0)$ because condition (3) listed in Theorem 5.3 may fail to hold. On the other hand, the invertibility of $L(E_0, T_0)$ implies that the rank of $K(T_0)$ is only at least $n - 1$, and so $K(T_0)$ may not be invertible.

THEOREM 5.4. *Suppose that H is a first integral for X. If $\gamma(\sigma(q_0))$ is a brake orbit with an interbraking time of T_0 and a first integral value of E_0 that is nondegenerate with respect to E_0 given T_0, then there is a unique and smooth one-parameter family of brake orbits $E \mapsto (\tau(E), \vartheta(E))$, $E \in (E^-, E^+)$, such that $E_0 \in (E^-, E^+)$, $\tau(E_0) = T_0$ and $\vartheta(E_0) = q_0$, and for each $E \in (E^-, E^+)$, ϑ is an immersion at E, the brake orbit $\gamma(\vartheta(E))$ is nondegenerate with respect to E given $T = \tau(E)$, and ϑ transversally intersects $\mathrm{zms}_H(E) = \{q \in Q : H(\sigma(q)) = E\}$ at $\vartheta(E)$.*

REMARK. The $\mathrm{zms}_H(E)$ is the *zero-momentum surface of energy E* for H.

Proof (of Theorem 5.4). Suppose that $\gamma(\sigma(q_0))$ is nondegenerate with respect to E_0 given T_0. By the Implicit Function Theorem, there is an interval (E^-, E^+) containing E_0 and unique and smooth functions $T = \tau(E)$, $q = \vartheta(E)$ such that $\pi' F_{\tau(E)}\sigma\vartheta(E) = 0$ and $H(\sigma\vartheta(E)) - E = 0$ for all E in (E^-, E^+). Moreover, we can take (E^-, E^+) so that for all E in (E^-, E^+), $\tau(E) \neq 0$ and $\vartheta(E) \notin Z_{\mathrm{e}}(X)$. It follows from the proof of the Implicit Function Theorem that

$$L(E, \tau(E)) = \frac{\partial((H(\sigma(q)) - E) \times \pi' F_t \sigma(q))}{\partial(t, q)}\bigg|_{t=\tau(E),\, q=\vartheta(E)}$$

is invertible for all $E \in (E^-, E^+)$. Hence each brake orbit $\gamma(\sigma\vartheta(E))$ in the family is nondegenerate with respect to E given $\tau(E)$.

We show that ϑ is an immersion. To get an expression involving $d\vartheta/dE$ we differentiate $H(\sigma\vartheta(E)) = E$ with respect to E. Doing so, we obtain

$$d(H\sigma)(\vartheta(E))\frac{d\vartheta(E)}{dE} = 1.$$

Thus, $d\vartheta(E)/dE$ is nonzero for all $E \in (E^-, E^+)$. So $E \mapsto \vartheta(E)$ is an immersion, and hence it is a local homeomorphism [12].

For each $E \in (E^-, E^+)$, $\vartheta(E)$ is a regular point of $H\sigma$ by Proposition 5.2. So there is a neigborhood of $\vartheta(E)$ in $\mathrm{zms}_H(E)$ which is a submanifold with $T_{\vartheta(E)}\,\mathrm{zms}_H(E) = \ker d(H\sigma)(\vartheta(E))$. As $d(H\sigma)(\vartheta(E))d\vartheta(E)/dE = 1$, we have $\mathrm{span}\{d\vartheta(E)/dE\} + T_{\vartheta(E)}\,\mathrm{zms}_H(E) = T_{\vartheta(E)}Q$. That is, for each $E \in (E^-, E^+)$, ϑ transversally intersects $\mathrm{zms}_H(E)$ at $\vartheta(E)$. ∎

A one-parameter family of brake orbits $E \mapsto (\tau(E), \vartheta(E))$, $E \in (E^-, E^+)$ may be a subfamily of a cyclic one-parameter family of brake orbits, but cannot be a cyclic family in and of itself. Suppose to the contrary that it is cyclic. Then there is a reparameterization $\nu \mapsto (\tau h^{-1}(\nu), \vartheta(h^{-1}(\nu))$, $\nu \in \mathbb{R}$, which is periodic in ν where $h : (E^-, E^+) \to \mathbb{R}$ is a homeomorphism. So there is a ν_0 such that $\vartheta(h^{-1}(\nu + \nu_0)) = \vartheta(h^{-1}(\nu))$ for all $\nu \in \mathbb{R}$. In particular, $\vartheta(h^{-1}(\nu_0)) = \vartheta(h^{-1}(0))$. Then $H\sigma\vartheta(h^{-1}(\nu_0)) = H\sigma\vartheta(h^{-1}(0))$. As $h^{-1}(\nu)$ is the energy, $h^{-1}(\nu_0) = h^{-1}(0)$, which is a contradiction to h being a homeomorphism.

For an R-reversible vector field X that has a first integral H we can say more about brake orbits, their nondegeneracy with respect to its first integral value given an interbraking time, and their inclusion into one-parameter families of brake orbits.

PROPOSITION 5.5. *If X is R-reversible, has a first integral H, and $\gamma(\sigma(q_0))$ is a brake orbit, then $\gamma(\sigma(q_0))$ is nondegenerate with respect to $E_0 = H(\sigma(q_0))$ given T_0 if and only if $\gamma(\sigma(q_0))$ is nondegenerate with respect to E_0 given $-T_0$.*

Proof. Arguing as we did in Proposition 4.6, we have $\pi' F_t\sigma(q) = -\pi' F_{-t}\sigma(q)$. Differentiating this equation with respect to t and evaluating at $t = T_0$ and $q = q_0$ we get $Y(T_0) = Y(-T_0)$. Differentiating $\pi' F_t\sigma(q) = -\pi' F_{-t}\sigma(q)$ with respect to q and evaluating at $t = T_0$, $q = q_0$ we get $K(T_0) = -K(-T_0)$. Consequently, $\ker K(T_0) = \ker K(-T_0)$ and $K(T_0)(\ker d(H\sigma)(q_0)) = K(-T_0)(\ker d(H\sigma)(q_0))$. We therefore have

$$\ker d(H\sigma)(q_0) \cap \ker K(T_0) = \ker d(H\sigma)(q_0) \cap \ker K(-T_0))$$

and

$$\begin{aligned} K(T_0)(\ker d(H\sigma)(q_0)) &\cap \operatorname{span}\{Y(T_0)\} \\ &= K(-T_0)(\ker d(H\sigma)(q_0)) \cap \operatorname{span}\{Y(-T_0)\}. \end{aligned}$$

It now follows from Theorem 5.3 that $\gamma(\sigma(q_0))$ is nondegenerate with respect to E_0 given T_0 if and only if $\gamma(\sigma(q_0))$ is nondegenerate with respect to E_0 given $-T_0$. ∎

THEOREM 5.6. *If X is R-reversible, H is a first integral of X, and $\gamma(\sigma(q_0))$ is a brake orbit that is nondegenerate with respect to $E_0 = H(\sigma(q_0))$ given T_0, then there is a unique and smooth one-parameter family $E \mapsto (\tau(E), \vartheta(E))$, $E \in (E^-, E^+)$, of periodic brake orbits for which $\tau(E_0) = T_0$ and $\vartheta(E_0) = q_0$.*

Proof. By Theorem 5.4, there is a unique and smooth one-parameter family $E \mapsto (\tau(E), \vartheta(E))$, $E \in (E^-, E^+)$, of brake orbits for which $\tau(E_0) = T_0$ and $\vartheta(E_0) = q_0$. By Proposition 2.5, each of these brake orbits is periodic. ∎

6. First integral-interbraking time duality. In the presence of a first integral, there is a relationship between the first integral values and the interbraking times of the members of a one-parameter family of brake orbits. Suppose that H is a first integral of X, and suppose that $\eta \mapsto (\tau(\eta), \vartheta(\eta))$, $\eta \in (\eta^-, \eta^+)$, is a smooth one-parameter family of brake orbits. Define the *first integral function* by $\xi(\eta) = H\sigma\vartheta(\eta)$.

DEFINITION 6.1. The first integral function $\xi(\eta)$ is called *strongly monotone at* η_0 if $d\xi/d\eta|_{\eta=\eta_0} \neq 0$. The interbraking time function $\tau(\eta)$ is called *strongly monotone at* η_0 if $d\tau/d\eta|_{\eta=\eta_0} \neq 0$.

If $\tau(\eta)$ is strongly monotone at η_0, then we can reparameterize a subfamily of $\eta \mapsto (\tau(\eta), \vartheta(\eta))$ by the interbraking time $T = \tau(\eta)$. This does not necessarily imply that $\gamma(\sigma\vartheta(\eta_0))$ is nondegenerate with respect to $T_0 = \tau(\eta_0)$. Similarly, if $\xi(\eta)$ is strongly monotone at η_0, then we can reparameterize a subfamily of $\eta \mapsto (\tau(\eta), \vartheta(\eta))$ by the first integral value $E = H\sigma\vartheta(\eta)$. This does not necessarily imply that $\gamma(\sigma\vartheta(\eta_0))$ is nondegenerate with respect to $E_0 = H\sigma\vartheta(\eta_0)$ given $T_0 = \tau(\eta_0)$.

LEMMA 6.2. *Suppose H is a first integral of X and $\eta \mapsto (\tau(\eta), \vartheta(\eta))$, $\eta \in (\eta^-, \eta^+)$, is a smooth one-parameter family of brake orbits. Let $\xi(\eta) = H\sigma\vartheta(\eta)$. For $\eta_0 \in (\eta^-, \eta^+)$, set $q_0 = \vartheta(\eta_0)$, $T_0 = \tau(\eta_0)$, and $E_0 = H\sigma\vartheta(\eta_0)$. Then*

(a) $\left.\dfrac{d\vartheta}{d\eta}\right|_{\eta=\eta_0} \in \ker d(H\sigma)(q_0)$ *if and only if* $\left.\dfrac{d\xi}{d\eta}\right|_{\eta=\eta_0} = 0$, *and*

(b) $\left.\dfrac{d\vartheta}{d\eta}\right|_{\eta=\eta_0} \in \ker K(T_0)$ *if and only if* $Y(T_0)\left.\dfrac{d\tau}{d\eta}\right|_{\eta=\eta_0} = 0$.

Proof. Differentiation of the equations $\xi(\eta) = H\sigma\vartheta(\eta)$ and $\pi' F_{\tau(\eta)}\sigma\vartheta(\eta) = 0$ with respect to η and setting $\eta = \eta_0$ gives the respective equations

$$d(H\sigma)(q_0)\left.\frac{d\vartheta}{d\eta}\right|_{\eta=\eta_0} = \left.\frac{d\xi}{d\eta}\right|_{\eta=\eta_0} \quad \text{and} \quad Y(T_0)\left.\frac{d\tau}{d\eta}\right|_{\eta=\eta_0} + K(T_0)\left.\frac{d\vartheta}{d\eta}\right|_{\eta=\eta_0} = 0,$$

from which the lemma follows. ∎

THEOREM 6.3 (Duality). *Suppose that H is a first integral of X and that $\eta \mapsto (\tau(\eta), \vartheta(\eta))$, $\eta \in (\eta^-, \eta^+)$, is a smooth one-parameter family of brake orbits. Let $\eta_0 \in (\eta^-, \eta^+)$, and set $q_0 = \vartheta(\eta_0)$, $T_0 = \tau(\eta_0)$, and $E_0 = \xi(\eta_0)$. Then $\gamma(\sigma(q_0))$ is nondegenerate with respect to T_0 and $\xi(\eta)$ is strongly monotone at η_0 if and only if $\gamma(\sigma(q_0))$ is nondegenerate with respect to E_0 given T_0 and $\tau(\eta)$ is strongly monotone at η_0.*

Proof. Suppose that $\gamma(\sigma(q_0))$ is nondegenerate with respect to T_0 and that $\xi(\eta)$ is strongly monotone at η_0. By part (a) of Lemma 6.2, ϑ is an immersion at η_0 and $d\vartheta/d\eta|_{\eta=\eta_0} \notin \ker d(H\sigma)(q_0)$. By part (b) of that lemma,

$Y(T_0) \neq 0$ and $\tau(\eta)$ is strongly monotone at η_0. The invertibility of $K(T_0)$ and the equation $Y(T_0)d\tau/d\eta|_{\eta=\eta_0} + K(T_0)d\vartheta/d\eta|_{\eta=\eta_0} = 0$ imply that

$$K(T_0)\left(\operatorname{span}\left\{\frac{d\vartheta}{d\eta}\bigg|_{\eta=\eta_0}\right\}\right) = \operatorname{span}\left\{Y(T_0)\frac{d\tau}{d\eta}\bigg|_{\eta=\eta_0}\right\}.$$

Hence, $K(T_0)(\ker d(H\sigma)(q_0)) \cap \operatorname{span}\{Y(T_0)\} = \{0\}$. The invertibility of $K(T_0)$ implies that $\ker d(H\sigma)(q_0) \cap \ker K(T_0) = \{0\}$. Therefore, by Theorem 5.3, $\gamma(\sigma(q_0))$ is nondegenerate with respect to E_0 given T_0.

Suppose that $\gamma(\sigma(q_0))$ is nondegenerate with respect to E_0 given T_0 and $\tau(\eta)$ is strongly monotone at η_0. By Theorem 5.3, $Y(T_0) \neq 0$. By Theorem 5.4, there is a unique and smooth one-parameter family $E \mapsto (\overline{\tau}(E), \overline{\vartheta}(E))$, $E \in (E^-, E^+)$, such that $q_0 = \overline{\vartheta}(E_0)$, $T_0 = \overline{\tau}(E_0)$. By uniqueness, $\overline{\tau}(E) = \tau(\eta)$ whenever $E = \xi(\eta)$ for all E in a neighborhood of E_0. Hence

$$\frac{d\overline{\tau}}{d\eta}\bigg|_{E=E_0}\frac{d\xi}{d\eta}\bigg|_{\eta=\eta_0} = \frac{d\tau}{d\eta}\bigg|_{\eta=\eta_0}.$$

Since $\tau(\eta)$ is strongly monotone at η, $\xi(\eta)$ is strongly monotone at η_0. In the case of $n = 1$, it follows from part (b) of Lemma 6.2 that $K(T_0)$ is invertible. In the case of $n \geq 2$, the vector $Y(T_0)$ belongs to the range of $K(T_0)$ since $Y(T_0)d\tau/d\eta|_{\eta=\eta_0} + K(T_0)d\vartheta/d\eta|_{\eta=\eta_0} = 0$. By Proposition 5.2, the rank of $K(T_0)$ is at least $n-1$. If the rank of $K(T_0)$ is $n-1$, then $L(E_0, T_0)$ is not surjective since $Y(T_0)$ belongs to the range of $K(T_0)$. Therefore, $\gamma(\sigma(q_0))$ is nondegenerate with respect to T_0. ■

In a one-parameter family of periodic brake orbits parameterized by a first integral, duality gives a sufficient condition for the strict monotonicity of a period as a function of the first integral. The interbraking time function of a one-parameter family $E \mapsto (\tau(E), \vartheta(E))$, $E \in (E^-, E^+)$, is called a *period function* if $F_{\tau(E)}\sigma\vartheta(E) = \sigma\vartheta(E)$ for all $E \in (E^-, E^+)$.

COROLLARY 6.4. *Suppose that H is a first integral of X and that $E \mapsto (\tau(E), \vartheta(E))$, $E \in (E^-, E^+)$, is a smooth one-parameter family of periodic brake orbits such that τ is a period function. Let $E_0 \in (E^-, E^+)$. If $\gamma(\sigma\vartheta(E_0))$ is nondegenerate with respect to $\tau(E_0)$, then near E_0, $\tau(E)$ is a strictly monotone function of E.*

Proof. Take $\eta = E$ in Theorem 6.3. ■

In certain R-reversible Hamiltonian dynamical systems, [7] and [13] give sufficient conditions for the strict monotonicity of the period function of a one-parameter family of periodic brake orbits parameterized by the Hamiltonian (or energy). These seem to be different from the sufficient condition given in Corollary 6.4. However, the relationship between a critical point of the period function and possible bifurcations is in accordance with [8].

7. One-parameter families of periodic brake orbits. For an R-reversible vector field X, Theorems 4.7 and 5.6 give two ways of generating a one-parameter family of periodic brake orbits from a given periodic brake orbit in terms of the two notions of nondegeneracy for a brake orbit. The analogous notions of nondegeneracy for a periodic orbit when applied to a periodic brake orbit of an R-reversible vector field X that has a first integral H also generate one-parameter families of periodic brake orbits. This then gives four ways to generate one-parameter families of periodic brake orbits from a given periodic brake orbit.

A nonconstant orbit $\gamma(z_0)$ of the flow F_t of X is periodic if there is a $T_0 > 0$ for which $F_{T_0}(z_0) = z_0$. A *one-parameter family of periodic orbits* is a continuous curve $\eta \mapsto (\tau(\eta), z(\eta))$, $\eta \in (\eta^-, \eta^+)$, for which $F_{\tau(\eta)}z(\eta) = z(\eta)$ for all $\eta \in (\eta^-, \eta^+)$ and $z(\eta)$ is a local homeomorphism.

LEMMA 7.1. *If X is R-reversible and $\eta \to (\tau(\eta), z(\eta))$, $\eta \in (\eta^-, \eta^+)$, is a one-parameter family of periodic orbits, then $\eta \mapsto (\tau(\eta), Rz(\eta))$, $\eta \in (\eta^-, \eta^+)$, is also a one-parameter family of periodic orbits.*

Proof. From the hypotheses, $RF_t = F_{-t}R$ and $F_{\tau(\eta)}z(\eta) = z(\eta)$ for all $\eta \in (\eta^-, \eta^+)$. Then

$$F_{\tau(\eta)}Rz(\eta) = F_{\tau(\eta)}RF_{\tau(\eta)}z(\eta) = F_{\tau(\eta)}F_{-\tau(\eta)}Rz(\eta) = Rz(\eta)$$

for all $\eta \in (\eta^-, \eta^+)$. Thus, $\eta \mapsto (\tau(\eta), Rz(\eta))$, $\eta \in (\eta^-, \eta^+)$, is a one-parameter family of periodic orbits. ∎

A periodic orbit $\gamma(z_0)$ with a period of T_0 is called *elementary with respect to T_0* if the rank of

$$A(T_0) = \frac{\partial(F_t(z) - z))}{\partial z}\bigg|_{t=T_0,\, z=z_0}$$

is $2n - 1$. Otherwise, it is called *nonelementary with respect to T_0*.

THEOREM 7.2. *Suppose that H is a first integral of an R-reversible vector field X, and that $\gamma(\sigma(q_0))$ is a periodic brake orbit with period T_0. If $\sigma(q_0)$ is a regular point of H and $\gamma(\sigma(q_0))$ is elementary with respect to T_0, then there exists a unique and smooth one-parameter family of periodic brake orbits $T \mapsto (T, \vartheta(T))$, $T \in (T^-, T^+)$, where $\vartheta(T_0) = q_0$.*

Proof. By Proposition 4.5, X is transverse to $\sigma(Q \setminus Z_{\mathrm{e}}(X))$ at $\sigma(q_0)$. We can therefore choose coordinates $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ on M so that $X(\sigma(q_0))$ is transverse to the hyperplane $\{p_n = 0\}$. For $d(H\sigma)(q_0) \neq 0$ and $\gamma(\sigma(q_0))$ elementary with respect to T_0, it is a classical result (p. 147 of [24]) that there exists a smooth and unique function $T \mapsto z(T)$ defined on an interval (T^-, T^+) containing T_0 for which $z(T_0) = \sigma(q_0)$, and $z(T) \in \{p_n = 0\}$ and $F_T z(T) = z(T)$ for all $T \in (T^-, T^+)$. Differentiating $F_T z(T) = z(T)$

with respect to T and evaluating at $T = T_0$ yields

$$A(T_0)\frac{dz}{dT}\bigg|_{T=T_0} = -X(\sigma(q_0)),$$

from which it follows that $dz/dT|_{T=T_0} \neq 0$. Hence $T \mapsto z(T)$ is an immersion and hence a local homeomorphism in a neighborhood of T_0. Therefore, by taking a smaller interval containing T_0 if necessary, $T \mapsto (T, z(T))$, $T \in (T^-, T^+)$, is a one-parameter family of periodic orbits. By Lemma 7.1, $T \to (T, Rz(T))$, $T \in (T^-, T^+)$, is also a one-parameter family of periodic orbits. Now $Rz(T_0) = R\sigma(q_0) = \sigma(q_0) = z(T_0)$. Also, $Rz(T) \in \{p_n = 0\}$ for all $T \in (T^-, T^+)$. By the uniqueness of $z(T)$ we must have $Rz(T) = z(T)$ for all $T \in (T^-, T^+)$. This implies that $z(T) = \sigma\vartheta(T)$ where $\vartheta(T) = \pi z(T)$. By taking a smaller neighborhood of T_0, we can assume that $\vartheta(T) \notin Z_{\mathrm{e}}(X)$ for all $T \in (T^-, T^+)$. So $z(T) \in \sigma(Q \setminus Z_{\mathrm{e}}(X))$, which implies that $\gamma(z(T))$ is a periodic brake orbit. Also, as $z(T)$ is smooth and a local homeomorphism, so is $\vartheta(T)$. Consequently, as $\gamma(z(T)) = \gamma(\sigma\vartheta(T))$, $T \mapsto (T, \vartheta(T))$, $T \in (T^-, T^+)$, is a unique and smooth one-parameter family of periodic brake orbits for which $\vartheta(T_0) = q_0$. ∎

A periodic orbit $\gamma(z_0)$ with first integral value E_0 and period T_0 is called *elementary with respect to E_0 given T_0* if the rank of

$$B(E_0, T_0) = \frac{\partial((F_t(z) - z) \times (H(z) - E))}{\partial(t, z)}\bigg|_{t=T_0,\, E=E_0,\, z=z_0}$$

is $2n$. Otherwise, it is called *nonelementary with respect to E_0 given T_0*.

THEOREM 7.3. *Suppose that H is a first integral of an R-reversible vector field X, and that $\gamma(\sigma(q_0))$ is a periodic brake orbit with first integral value $E_0 = H(\sigma(q_0))$ and period T_0. If $\sigma(q_0)$ is a regular point of H and $\gamma(\sigma(q_0))$ is elementary with respect to E_0 given T_0, then there exists a unique and smooth one-parameter family of periodic brake orbits $E \mapsto (\tau(E), \vartheta(E))$, $E \in (E^-, E^+)$, for which $\tau(E_0) = T_0$ and $\vartheta(E_0) = q_0$.*

Proof. Arguing as in the proof of Theorem 7.2, we find that $X(\sigma(q_0))$ is transverse to $\{p_n = 0\}$. As $\sigma(q_0)$ is a regular point of H and $\gamma(\sigma(q_0))$ is elementary with respect to E_0 given T_0, it is a classical result (p. 147 of [24], p. 136 of [17]) that there exists a unique and smooth function $E \mapsto (\tau(E), z(E))$ defined on an interval (E^-, E^+) containing E_0 for which $\tau(E_0) = T_0$, $z(E_0) = \sigma(q_0)$, and $z(E) \in \{p_n = 0\}$, $H(z(E)) = E$, $F_{\tau(E)}z(E) = z(E)$ for all $E \in (E^-, E^+)$. Differentiating $H(z(E)) = E$ with respect to E and evaluating at $E = E_0$, we get

$$dH(\sigma(q_0))\frac{dz}{dE}\bigg|_{E=E_0} = 1.$$

This implies that $z(E)$ is an immersion at E_0, and hence in a neighborhood of E_0, $z(E)$ is a local homeomorphism. Arguing as in the last part of the proof of Theorem 7.2, we find that $E \mapsto (\tau(E), \vartheta(E))$, $E \in (E^-, E^+)$, where $\vartheta(E) = \pi z(E)$, is a unique and smooth one-parameter family of periodic brake orbits for which $\tau(E_0) = T_0$ and $\vartheta(E_0) = q_0$. ■

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Department of Mathematics
Brigham Young University
292 Talmage Math/Computer Building
PO Box 26539
Provo, UT 84602-6539, U.S.A.
E-mail: bakkerl@member.ams.org

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