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ADDITIVE PROPERTIES AND UNIFORMLY COMPLETELY RAMSEY SETS

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Abstract. We prove some properties of uniformly completely Ramsey null sets (for example, every hereditarily Menger set is uniformly completely Ramsey null).

1. Introduction. The notion of UCR_0 sets was considered in [Da] where it was proved that every UCR_0 set has the Marczewski s_0 property. The main problem concerning these sets is whether one can prove the existence of such a set of size continuum without any extra axioms (see [Da], Question 1). We are still unable to give a complete answer to this problem. However, in Section 4 we will show that every hereditarily Menger set belongs to the class of UCR_0 sets.

2. Notation. $\exists_n^\infty$ and $\forall_n^\infty$ stand for "there exists infinitely many n" and "for all but finitely many n" respectively. We use $\omega^{\omega\uparrow}$ to denote the family of all strictly increasing functions from ω^ω. In $\omega^{\omega\uparrow}$ we define the order $\prec$ in the standard way:

$$x \prec y \Leftrightarrow \exists_{n<\omega}\forall_{k>n} x(k) \le y(k).$$

Using the characteristic function, we can view $[\omega]^\omega$ as a subset of 2^ω. So we will look at 2^ω as the union $[\omega]^\omega \cup [\omega]^{<\omega}$. Sometimes we identify $[\omega]^\omega$ with the space $\omega^{\omega\uparrow}$ via the standard homeomorphism.

If $U \in [\omega]^\omega$, $F \in [\omega]^{<\omega}$ and $\max(F) < \min(U)$ then $[F, U]$ denotes $\{A \in [\omega]^\omega : F \subseteq A \subseteq F \cup U\}$. We call such a set an *Ellentuck set*.

3. Definitions. Let us define the main notions of this article.

A set $X \subseteq [\omega]^\omega$ is *Ramsey* iff there exists $A \in [\omega]^\omega$ such that either $[A]^\omega \subseteq X$ or $[A]^\omega \cap X = \emptyset$.

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We say that a set $X \subseteq [\omega]^\omega$ is *Ramsey null* (or for short X is CR_0) iff for every Ellentuck set $[F,V]$ there exists an Ellentuck set $[F,U] \subseteq [F,V]$ such that $[F,U] \cap X = \emptyset$.

A set $X \subseteq 2^\omega$ is *uniformly completely Ramsey null* iff for every continuous function $F : 2^\omega \to 2^\omega$ and every $Y \subseteq X$, $F^{-1}(Y)$ is Ramsey. We then write $X \in \mathrm{UCR}_0$.

We say that a sequence of functions $f_k : X \to \mathbb{R}$ *converges quasinormally* to f ($f_k \xrightarrow{\mathrm{QN}} 0$) if there is a sequence $\varepsilon_n \to 0$ such that for each x there is k_0 such that $|f(x) - f_k(x)| < \varepsilon_k$ for all $k > k_0$.

A subset $X \subset 2^\omega$ is a QN *set* if for each sequence of continuous functions $f_k : X \to \mathbb{R}$, $(f_k \to 0) \Rightarrow (f_k \xrightarrow{\mathrm{QN}} 0)$; and X is a wQN *set* if for each sequence of continuous functions $f_k : X \to \mathbb{R}$ with $f_k \to 0$ there is a subsequence k_l such that $f_{k_l} \xrightarrow{\mathrm{QN}} 0$. The last two notions were introduced in [BRR].

We say that $X \subseteq 2^\omega$ has the *Menger property* iff every continuous image $f(X)$ of X in ω^ω is a *nondominating family*, which means that there exists $g \in \omega^\omega$ such that $\forall_{x\in X}\forall_n \exists_{m>n} g(m) > f(x)(m)$. We say that X is a *hereditarily Menger set* iff every subspace of X has the Menger property. We say that $X \subseteq 2^\omega$ has the *Hurewicz property* iff every continuous image of X in ω^ω is a bounded family. It is evident that if X has the Hurewicz property then it has the Menger property.

A tree $S \subseteq \omega^{<\omega\uparrow}$ is *superperfect* iff $\forall_{t\in S}\exists_{s\supseteq t}\exists^\infty_{n<\omega} s^\frown\langle n\rangle \in S$. If $T \subseteq \omega^{<\omega\uparrow}$ is a tree then we define $[T] = \{x \in \omega^{\omega\uparrow} : \forall_n x|n \in T\}$; moreover, $\mathrm{stem}(T)$ is the unique $s \in T$ with $\forall_{t\in T} s \subseteq t \vee t \subseteq s$ and $|\{n \in \omega : s^\frown\langle n\rangle \in T\}| \geq 2$.

A tree $S \subseteq \omega^{<\omega\uparrow}$ is called a *Laver tree* iff $\forall_{s\in S}$ if $\mathrm{stem}(S) \subseteq s$ then $\exists^\infty_n s^\frown\langle n\rangle \in S$.

We say that $X \subseteq \omega^{\omega\uparrow}$ is an m_0 *set* iff for every superperfect tree $T \subseteq \omega^{<\omega\uparrow}$ one can find a superperfect tree $S \subseteq T$ such that $[S] \cap X = \emptyset$; and X is an l_0 *set* iff for every Laver tree $T \subseteq \omega^{<\omega\uparrow}$ one can find a Laver tree $S \subseteq T$ such that $[S] \cap X = \emptyset$.

4. Results. We start this section with the following simple but useful characterization of UCR_0 sets:

THEOREM 1. *Let $X \subseteq 2^\omega$. Then X is* UCR_0 *iff for every continuous function $F : 2^\omega \to 2^\omega$ there exists $A \in [\omega]^\omega$ such that*

$$|F(P(A)) \cap X| \leq \omega.$$

Proof. $\Rightarrow$ Let $X \subseteq 2^\omega$ be UCR_0 and let $F : 2^\omega \to 2^\omega$ be a continuous function. By the definition of UCR_0 one can find $A \in [\omega]^\omega$ such that $[A]^\omega \subseteq F^{-1}(X) \vee [A]^\omega \cap F^{-1}(X) = \emptyset$. Consider the following two cases:

CASE 1: $[A]^\omega \subseteq F^{-1}(X)$. By [Da], Theorem 3, X is (s_0). Thus there is no uncountable analytic subset of X. As $F([A]^\omega)$ is an analytic set this implies that $|F([A]^\omega)| \le \omega$. So we have shown that $|F(P(A))| \le \omega$ and finally $|F(P(A)) \cap X| \le \omega$.

CASE 2: $[A]^\omega \cap F^{-1}(X) = \emptyset$. First note that $F(P(A)) \subseteq F([A]^\omega) \cup F([\omega]^{<\omega})$ and $X \cap F([A]^\omega) = \emptyset$. This implies that

$$X \cap F(P(A)) \subseteq X \cap (F([A]^\omega) \cup F([\omega]^{<\omega})) \subseteq F([\omega]^{<\omega}).$$

Thus $|X \cap F(P(A))| \le \omega$.

$\Leftarrow$ Suppose that $F : 2^\omega \to 2^\omega$ is a continuous function and $Y \subseteq X$. By assumption, there exists $A \in [\omega]^\omega$ such that $|F(P(A)) \cap X| \le \omega$. Note first that $Y \cap F(P(A))$ is a Borel set, since it is countable. Then, by the classical Galvin–Prikry Theorem (see [Ke], Theorem 19.11) applied to the set $Y \cap F[P(A)]$ and the space $P(A)$ which is homeomorphic to 2^ω, there exists $B \in [A]^\omega$ such that either

$$F([B]^\omega) \subseteq Y \cap F(P(A)) \quad \text{or} \quad F([B]^\omega) \cap Y \cap F(P(A)) = \emptyset.$$

If $F([B]^\omega) \subseteq Y \cap F(P(A))$ then we are done. If $F([B]^\omega) \cap Y \cap F(P(A)) = \emptyset$ then $F([B]^\omega) \cap Y = \emptyset$, and the assertion is also proved in this case. ■

In addition to Theorem 1 we record the following simple but useful observation:

OBSERVATION 1. *Suppose that $X \in \mathrm{UCR}_0$, $A \in [\omega]^\omega$ and $F : P(A) \to 2^\omega$ is a continuous function. Then there exists $B \in [A]^\omega$ such that $|F(P(B)) \cap X| \le \omega$.*

Proof. Fix any bijection $g : \omega \to A$. For $Z \subseteq \omega$ define $G(Z) := g(Z)$. It is clear that $G : 2^\omega \to 2^A$. It is also easy to see that G is a homeomorphism. Applying Theorem 1 to the function $F \circ G$ shows that there exists $C \in [\omega]^\omega$ such that $|(F \circ G)(P(C)) \cap X| \le \omega$. But $G(P(C)) = P(B)$, where $B = g(C)$ and $B \in [A]^\omega$, so we have $|F(P(B)) \cap X| \le \omega$. ■

THEOREM 2. *Let $X \subseteq 2^\omega$. Then X is UCR_0 iff for every continuous function $h : [\omega]^{<\omega} \to 2^\omega$ there exists $B \in [\omega]^\omega$ such that $|\overline{h([B]^{<\omega})} \cap X| \le \omega$.*

Proof. $\Rightarrow$ Take any continuous function $h : [\omega]^{<\omega} \to 2^\omega$. One can find a G_δ set, say G, and a continuous function $h^* : G \to 2^\omega$ such that $[\omega]^{<\omega} \subseteq G$ and $h^*|[\omega]^{<\omega} = h$.

We will frequently use the following well-known lemma:

LEMMA 1. *Given a G_δ set $H' \supseteq [\omega]^{<\omega}$, $H' \subseteq 2^\omega$ one can find $A \in [\omega]^\omega$ such that $P(A) \subseteq H'$.*

Applying this lemma to G yields a set $A \in [\omega]^\omega$ such that $P(A) \subseteq G$. Applying Observation 1 to the set A and to the function $h^* : P(A) \to 2^\omega$ we obtain $B \in [A]^\omega$ such that $|h^*(P(B)) \cap X| \le \omega$. Obviously, $P(B)$ is compact.

Thus $h^*(P(B))$ is closed and of course

$$h^*(P(B)) \supseteq h([B]^{<\omega}).$$

Hence

$$\overline{h([B]^{<\omega})} \subseteq h^*(P(B)).$$

From this it easily follows that $|\overline{h([B]^{<\omega})} \cap X| \le \omega$.

$\Leftarrow$ Let $F : 2^\omega \to 2^\omega$ be continuous. By Theorem 1 it is sufficient to find $B \in [\omega]^\omega$ such that $F(P(B)) \cap X$ is countable. Take the restriction $F|[\omega]^{<\omega}$ for h. Then there exists $B \in [\omega]^\omega$ such that

$$|\overline{h([B]^{<\omega})} \cap X| \le \omega.$$

However, $h([B]^{<\omega})$ is dense in $F(P(B))$, so $\overline{h([B]^{<\omega})} \supseteq F(P(B))$. Thus $F(P(B)) \cap X$ is countable. ∎

In the sequel we will show that every hereditarily Menger set is UCR_0. We start with the following lemma:

LEMMA 2. *Let $F : 2^\omega \to 2^\omega$ be a continuous function and $B : 2^\omega \to 2^\omega$ a Borel function. Then there exists $A \in [\omega]^\omega$ such that the restriction of B to $F([A]^\omega) \setminus F([\omega]^{<\omega})$ is continuous.*

Proof. We use the following classical result (see [Ke], Exercise 19.19):

LEMMA 3. *If $D : 2^\omega \to 2^\omega$ is a Borel function then there exists $A \in [\omega]^\omega$ such that $D|[A]^\omega$ is continuous.*

From this lemma, there exists $A \in [\omega]^\omega$ such that $(B \circ F)|[A]^\omega$ is continuous on $[A]^\omega$. We now show that this A works. Fix any closed set $K \subseteq 2^\omega$. Then $F^{-1}(B^{-1}(K)) \cap [A]^\omega$ is closed in $[A]^\omega$. Pick a closed $L \subseteq P(A)$ such that

$$L \cap [A]^\omega = F^{-1}(B^{-1}(K)) \cap [A]^\omega.$$

Let us verify that

$$F(L) \cap (F([A]^\omega) \setminus F([\omega]^{<\omega})) = B^{-1}(K) \cap (F([A]^\omega) \setminus F([\omega]^{<\omega})),$$

which will prove that B is continuous after restriction to $F([A]^\omega) \setminus F([\omega]^{<\omega})$.

Let $a \in F(L) \cap (F([A]^\omega) \setminus F([\omega]^{<\omega})$. Then $F(l) = a$ for some $l \in L$. Note that $B(a) = B(F(l))$ and $l \notin [\omega]^{<\omega}$, since $a = F(l) \notin F([\omega]^{<\omega})$. Thus $l \in L \setminus [\omega]^{<\omega} \subseteq [A]^\omega$. But $l \in L \cap [A]^\omega \subseteq F^{-1}(B^{-1}(K))$ so $B(a) = B(F(l)) \in K$.

Conversely, if $a \in B^{-1}(K) \cap F([A]^\omega) \setminus F([\omega]^{<\omega})$ then there exists $l \in [A]^\omega$ such that $F(l) = a$. Since clearly $B(a) \in K$ we see that $B(F(l)) = B(a) \in K$. Observe that

$$l \in F^{-1}(B^{-1}(K)) \cap [A]^\omega \subseteq L,$$

which implies $a = F(l) \in F(L)$.

This proves that $B|F([A]^\omega) \setminus F([\omega]^{<\omega})$ is continuous. ∎

THEOREM 3. *If* $X \subseteq 2^\omega$ *is a hereditarily Menger set then* X *is* UCR_0.

Proof. Suppose $F : 2^\omega \to 2^\omega$ is continuous. First we define a Borel function $B : 2^\omega \to 2^\omega$ by

$$B(x) = \begin{cases} \Omega(F^{-1}(x)) & \text{if } F^{-1}(x) \neq \emptyset \wedge F^{-1}(x) \subseteq [\omega]^\omega, \\ \underline{0} & \text{if } F^{-1}(x) = \emptyset \vee F^{-1}(x) \not\subseteq [\omega]^\omega, \end{cases}$$

where $\forall_k\ \underline{0}(k) = 0$ and $\Omega(K)(k)$ denotes $\max\{x(k) : x \in K\}$ for every nonempty compact $K \subseteq [\omega]^\omega$ (recall that we treat K as a subset of $\omega^{\omega\uparrow}$).

Since the graph of F is compact, the definition of B shows that B is Borel. Also note that $D \prec B(F(D))$ provided $F(D) \notin F[[\omega]^{<\omega}]$.

Apply Lemma 2 with the functions F and B to find $A \in [\omega]^\omega$ such that $B|Z$ is continuous, where $Z = F([A]^\omega) \setminus F([\omega]^{<\omega})$.

Since $X \cap Z$ has the Menger property, we conclude that $B[X \cap Z]$ is a nondominating family in $[\omega]^\omega$ (where $[\omega]^\omega$ is treated as $\omega^{\omega\uparrow}$). Fix $f \in \omega^{\omega\uparrow}$ such that $f \in [A]^\omega$ and

$$(\dagger) \qquad \forall_{g \in B(X \cap Z)} f \not\prec g.$$

We will show that $F([f]^\omega) \cap X \subseteq F([\omega]^{<\omega})$.

Assume that for some $D \in [f]^\omega$,

$$F(D) \in X \setminus F([\omega]^{<\omega}).$$

Since we know that $D \in [f]^\omega$ we conclude that $f \prec D$. Moreover, $D \prec B(F(D))$, so $f \prec B(F(D))$. Hence

$$F(D) \in Z = F([A]^\omega) \setminus F([\omega]^{<\omega})$$

and $F(D) \in X$, so

$$B(F(D)) \in B(X \cap Z),$$

which contradicts ($\dagger$). We have thus proved that

$$F([f]^\omega) \cap X \subseteq F([\omega]^{<\omega}),$$

which ends the proof of Theorem 3. ∎

For the next conclusion we will introduce the notion of $D^\ddagger$ set (see [PR]). We say a subset X of 2^ω is $D^\ddagger$ iff every Borel image of X in ω^ω is a nondominating family.

CONCLUSION 1. *Every* $D^\ddagger$ *set is* UCR_0.

CONCLUSION 2. $\mathrm{non}(\mathrm{UCR}_0) \geq \mathrm{d}$.

THEOREM 4. *Every* QN *set is* UCR_0.

Proof. By Theorem 3 it is sufficient to show that every QN set has the hereditary Hurewicz property.

Let $X \subseteq 2^\omega$ be a QN set, $Y \subseteq X$ and let $f : Y \to \omega^\omega$ be continuous. Note that we can extend the domain of f to a G_δ subset of X. Thus the proof will be completed if we show that every continuous function f defined on a G_δ subset of X with values in ω^ω is bounded.

Since every QN set is a σ set (see [Rec]) we see that every G_δ subset of X is also an F_σ subset. From the results of [BRR] it follows that every F_σ subset of X is a QN set and every QN set has the Hurewicz property. Therefore f is bounded. ■

It is natural to formulate the following problem:

Problem 1. *Is every* wQN *set* UCR_0?

Note that every wQN set $X \subseteq [\omega]^\omega$ is bounded in the space $\omega^{\omega\uparrow}$ (recall from the preliminary section that we identify $[\omega]^\omega$ with the space $\omega^{\omega\uparrow}$ via the standard homeomorphism). It follows that every such set is Ramsey null. However, it is not clear whether every wQN set $X \subseteq 2^\omega$ is Ramsey null. We can also state a weak form of Problem 1:

Problem 2. *Is every* wQN *set* $X \subseteq 2^\omega$ *Ramsey null?*

It is known (see [Br]) that not every Ramsey null set is an m_0 set, and not every m_0 set is Ramsey null. However, we will prove that every UCR_0 set is both an m_0 set and an l_0 set.

Theorem 5. *Every* UCR_0 *set is an* m_0 *set.*

Proof. Suppose $X \subseteq [\omega]^\omega \subseteq 2^\omega$ is UCR_0. Let $T \subseteq \omega^{<\omega\uparrow}$ be a superperfect tree. For every $s \in T$ we fix $t_s \supseteq s$, $t_s \in T$ such that $\exists_n^\infty t_s{}^\frown\langle n\rangle \in T$. Fix $k_0^{(s)} < k_1^{(s)} < k_2^{(s)} < \ldots$ such that

$$\forall_{i\in\omega} t_s{}^\frown\langle k_i^{(s)}\rangle \in T.$$

We define by induction the function $F : \omega^{<\omega\uparrow} \to \omega^{<\omega\uparrow}$ in the following way:

1. $F(\emptyset) = s$, where s is any fixed member of $\mathrm{stem}(T)$.
2. If we have already defined $F(s)$ for $|s| = n$, then for $i > \max\mathrm{ran}(s)$ we put

$$F(s^\frown\langle i\rangle) = t_{F(s)}{}^\frown k^{(F(s))}_{i-\max\mathrm{ran}\,F(s)-1}.$$

It is clear that F is strictly monotonic, which means that if $s \subset t$ then $F(s) \subset F(t)$.

Observation 2. *The function F extends to a continuous* $F^* : 2^\omega \to 2^\omega$.

To see this, simply define

$$F^*(x) = \begin{cases} F(x) & \text{iff } x \in \omega^{<\omega\uparrow}, \\ \bigcup_{n<\omega} F(x|n) & \text{iff } x \in \omega^{\omega\uparrow}. \end{cases}$$

Since X is UCR_0, we can find (by Theorem 1) a set $A \in [\omega]^\omega$ such that

$$|F^*(P(A)) \cap X| \le \omega.$$

It is easy to see that $F^*(P(A)) \cap [\omega]^\omega$ is equal to $[S_A]$ for some superperfect tree $S_A \subseteq T$. But $|[S_A] \cap X| \leq \omega$ so we can find a superperfect tree $S \subseteq S_A$ such that $[S] \cap X = \emptyset$. The proof of Theorem 5 is therefore complete. ∎

Conclusion 3. *If X is hereditarily Menger then X is an m_0 set.*

Note that the same argument as in Theorem 5 yields the following result:

Theorem 6. *Every* UCR_0 *set is an l_0 set.*

Theorem 7. *Let $F : 2^\omega \to 2^\omega$ be a continuous function and $X \subseteq 2^\omega$ a* UCR_0 *set. Assume also that to every $x \in X$ we have assigned a set $Z_x \subseteq F^{-1}(\{x\})$ which is also* UCR_0. *Then*

$$Z = \bigcup_{x \in X} Z_x$$

is a UCR_0 *set.*

Proof. Let $G : 2^\omega \to 2^\omega$ be continuous. By Theorem 1 the proof of our theorem will be completed if we show that $\exists_{B \in [\omega]^\omega} |G(P(B)) \cap Z| \leq \omega$.

Since X is a UCR_0 set, we conclude from Theorem 1 that there exists $A \in [\omega]^\omega$ such that

$$|F(G(P(A))) \cap X| \leq \omega.$$

Then

$$W = \bigcup_{x \in F(G(P(A))) \cap X} Z_x \in \mathrm{UCR}_0,$$

hence (again from Theorem 1) there exists $B \in [A]^\omega$ such that

$$|G(P(B)) \cap W| \leq \omega.$$

It follows that

$$G(P(B)) \cap Z = G(P(B)) \cap \bigcup_{x \in X} Z_x = G(P(B)) \cap \bigcup_{x \in F(G(P(A))) \cap X} Z_x = G(P(B)) \cap W.$$

Hence $|G(P(B)) \cap Z| \leq \omega$, which shows that $Z \in \mathrm{UCR}_0$. ∎

As an easy consequence we obtain the following corollary:

Corollary 1. *Let $X \subseteq 2^\omega$ and let $Y \subseteq 2^\omega$ be a* UCR_0 *set. Then the set $X \times Y$ (contained in the space $2^\omega \times 2^\omega$ homeomorphic to 2^ω) is also* UCR_0.

Conclusion 4. *Assuming MA there exists a* UCR_0 *set $X \subseteq 2^\omega$ and a continuous function $F : 2^\omega \to 2^\omega$ such that $F(X) = 2^\omega$.*

Proof. Take a generalized Luzin set $L \subseteq 2^\omega$ such that $L + L = 2^\omega$. From [Da], Theorem 12, we know that under MA every generalized Luzin set is UCR_0. Put $X = L \times L$ and define $F : 2^\omega \times 2^\omega \to 2^\omega$ by $F(x,y) = x + y$. Clearly, these X and F work. ∎

It is well known (see [Da], Theorem 9) that every Sierpiński and every Luzin set is UCR_0. We will prove the following intriguing fact:

THEOREM 8. *Let $L \subseteq 2^\omega$ be a Luzin set and $S \subseteq 2^\omega$ a Sierpiński set. Assume that $F : 2^\omega \times 2^\omega \to 2^\omega$ is a continuous function such that for every $y \in 2^\omega$, $F^{-1}(\{y\})$ is of measure zero. Then $F(L \times S)$ is* UCR_0.

Proof. Fix a continuous $G : 2^\omega \to 2^\omega$.

LEMMA 4. *There exists $A \in [\omega]^\omega$ such that $F^{-1}(G(P(A)))$ has measure zero.*

Proof. One can easily find a G_δ set, say H, such that the (countable) set $G([\omega]^{<\omega})$ is included in H and $F^{-1}(H)$ has measure zero. Applying Lemma 1 to the G_δ set $G^{-1}(H)$ yields $A \in [\omega]^\omega$ such that $P(A) \subseteq G^{-1}(H)$. Then $G(P(A)) \subseteq H$ and so $F^{-1}(G(P(A))) \subseteq F^{-1}(H)$. Thus $F^{-1}(G(P(A)))$ has measure zero.

In the next part of our proof of Theorem 8 we use the following interesting fact observed by J. Pawlikowski (private communication):

LEMMA 5. *Let $A \subseteq 2^\omega \times 2^\omega$ be a co-null G_δ set. Then there exists a co-meager set $B \subseteq 2^\omega$ and co-null set $C \subseteq 2^\omega$ such that $B \times C \subseteq A$.*

We leave it to the reader to verify this lemma.

It is easy to see that the set $(2^\omega \times 2^\omega) \setminus F^{-1}(G(P(A)))$ satisfies the assumption of Lemma 5. Indeed, from our previous results we know that $F^{-1}(G(P(A)))$ has measure zero. Also $F^{-1}(G(P(A)))$ is closed (because $P(A) \subseteq 2^\omega$ is compact). Consequently, let $B \subseteq 2^\omega$ be a co-meager set and $C \subseteq 2^\omega$ be a co-null set such that

$$B \times C \subseteq (2^\omega \times 2^\omega) \setminus F^{-1}(G(P(A))).$$

This can be written as

$$(B \times C) \cap F^{-1}(G(P(A))) = \emptyset. \tag{1}$$

Then we have

$$(L \times S) \setminus (B \times C) \subseteq [(L \setminus B) \times S] \cup [L \times (S \setminus C)],$$

where $L_1 = L \setminus B$ and $S_1 = S \setminus C$ are countable. Thus

$$\begin{aligned} F(L \times S) &= F((L \times S) \setminus (B \times C)) \cup F(B \times C) \\ &\subseteq F(L_1 \times S) \cup F(L \times S_1) \cup F(B \times C). \end{aligned}$$

From (1) we know that $F(B \times C) \cap G(P(A)) = \emptyset$. Thus

$$F(L \times S) \cap G(P(A)) \subseteq [F(L_1 \times S) \cup F(L \times S_1)] \cap G(P(A)).$$

However, $F(L_1 \times S) \cup F(L \times S_1)$ has the UCR_0 property. Indeed, $L_1 \times S$ as a countable sum of Sierpiński sets is also a Sierpiński set, so $F(L_1 \times S)$ is UCR_0. Analogously, $F(L \times S_1)$ is also UCR_0.

Choose $B \in [A]^\omega$ such that

$$|G(P(B)) \cap [F(L_1 \times S) \cup F(L \times S_1)]| \le \omega.$$

Since

$$F(L \times S) \cap G(P(B)) \subseteq [F(L_1 \times S) \cup F(L \times S_1)] \cap G(P(B)),$$

we finally obtain

$$|F(L \times S) \cap G(P(B))| \le \omega,$$

which shows that $F(L \times S)$ is UCR_0. ■

As an immediate consequence of Theorem 8 we obtain:

COROLLARY 2. *If $L \subseteq 2^\omega$ is a Luzin set and $S \subseteq 2^\omega$ is a Sierpiński set then the algebraic sum*

$$L + S = \{x + y : x \in L,\ y \in S\}$$

has the UCR_0 *property.*

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