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# INFINITE ERGODIC INDEX $\mathbb{Z}^d$ -ACTIONS IN INFINITE MEASURE

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**Abstract.** We construct infinite measure preserving and nonsingular rank one $\mathbb{Z}^d$-actions. The first example is ergodic infinite measure preserving but with nonergodic, infinite conservative index, basis transformations; in this case we exhibit sets of increasing finite and infinite measure which are properly exhaustive and weakly wandering. The next examples are staircase rank one infinite measure preserving $\mathbb{Z}^d$-actions; for these we show that the individual basis transformations have conservative ergodic Cartesian products of all orders, hence infinite ergodic index. We generalize this example to obtain a stronger condition called power weakly mixing. The last examples are nonsingular $\mathbb{Z}^d$-actions for each Krieger ratio set type with individual basis transformations with similar properties.

**1. Introduction.** In this paper we construct families of ergodic infinite measure preserving and nonsingular free actions of $\mathbb{Z}^d$ on the real line. The method is by the natural generalization of the "cutting and stacking" constructions for integer actions. This method has been used in Park–Robinson [PR] and Adams [A] to construct ergodic finite measure preserving $\mathbb{Z}^2$-actions with various properties, but we do not know of its use for infinite measure preserving $\mathbb{Z}^2$-actions. Recently there has been much interest in constructing examples of ergodic actions of groups other than the integers; cf. [Sch] and the references therein.

To simplify the exposition we first exhibit the examples for the case when $d = 2$; the changes needed for general $d$ are in general straightforward. The first examples we construct are the analogues in $\mathbb{Z}^2$ of the well-known ergodic infinite measure preserving transformation of Hajian and Kakutani [HK2]. In this case we study the weakly wandering sets for these actions, and introduce the notion of properly exhaustive sets, a notion that becomes important in $\mathbb{Z}^2$-actions. We exhibit properly exhaustive weakly wandering sets of finite increasing measure and of infinite measure.

However, it is easy to see that for the ergodic $\mathbb{Z}^2$-actions mentioned above, the basis transformations (individual horizontal and vertical integer

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actions) are not ergodic, though we show that all their Cartesian products are conservative. In [AS], Adams and Silva constructed rank one mixing finite measure preserving $\mathbb{Z}^d$-actions, $d \geq 2$. In Section 4 we modify the staircase $\mathbb{Z}^d$ constructions of [AS] to obtain infinite measure preserving $\mathbb{Z}^d$-actions. For these actions the basis transformations are indeed ergodic and also have continuous $L^\infty$ spectrum, hence are weak mixing; in fact, we show that all their Cartesian products are ergodic, i.e., have infinite ergodic index. (In infinite measure, ergodicity of the $k$-fold Cartesian product does not imply ergodicity of the $(k+1)$st Cartesian product [KP].)

The difficulty in the infinite measure preserving case is that there is currently no formulation of a pointwise ergodic theorem for $\mathbb{Z}^d$-actions in infinite measure, as the counter-example of Brunel and Krengel [Kre], p. 217 prevents the obvious formulation. Also, in infinite measure, the weakly wandering sets preclude a useful notion of mixing (the notions of mixing for infinite measure in the literature do not imply ergodicity, and in the existing examples ergodicity has to be shown separately).

Next, we modify the construction of the infinite staircase actions to obtain a new action called a multistep action, where the earlier proof applies and shows that the action is power weakly mixing, a condition stronger than having every nontrivial element of the action of infinite ergodic index.

The last section constructs, for each $0 \leq \lambda \leq 1$, conservative ergodic free nonsingular type $\mathrm{III}_\lambda$ $\mathbb{Z}^d$-actions. For the case of $0 < \lambda \leq 1$ we prove that the basis transformations have infinite ergodic index. For the case $\lambda = 0$ we show that the basis transformations are weakly mixing.

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**2. Preliminaries.** We let $X$ denote a finite or infinite interval, $\mathfrak{B}$ the Borel $\sigma$-algebra in $X$, and $\mu$ Lebesgue measure. A $\mathbb{Z}^d$-*action* is a measurable map $T : \mathbb{Z}^d \times X \to X$ such that if $e$ is the identity in $\mathbb{Z}^d$ then for a.a. $x \in X$, $T^e(x) = x$, and for all $p, q \in \mathbb{Z}^d$, $T^p(T^q(x)) = T^{p+q}(x)$ a.e. We write

$T^p(x)$ instead of $T(p,x)$. An action of $\mathbb{Z}^d$ is determined by $d$ commuting *basis transformations* $T^{(1,0,\dots,0)},\dots,T^{(0,\dots,0,1)}$. The action is *free* if $\mu\{x : T^p(x)=x$ for some $p\neq e\}=0$. All our actions will be free by definition or construction.

The action of $T$ on $(X,\mathfrak{B},\mu)$ is *measure preserving* if for every $p\in\mathbb{Z}^d$ and all $A\in\mathfrak{B}$, $\mu(T^pA)=\mu(A)$. The action is *nonsingular* if for every $p\in\mathbb{Z}^d$ and all $A\in\mathfrak{B}$, $\mu(T^pA)>0$ if and only if $\mu(A)>0$. Further, $T$ is *ergodic* if for all measurable sets $A$, if $T^pA=A$ for all $p\in\mathbb{Z}^d$ then $\mu(A)=0$ or $\mu(A^{\mathrm{c}})=0$. It is *properly ergodic* if it is ergodic and no orbit of a single point a.e. covers the whole space $X$. As our measures are nonatomic, our ergodic actions are properly ergodic.

A set $W\in\mathfrak{B}$ with $\mu(W)>0$ is *wandering* under the action $T$ if for all $p,q\in\mathbb{Z}^d$ with $p\neq q$, we have $\mu(T^pW\cap T^qW)=0$. An action is *conservative* if it has no wandering sets. A set $W$ is *weakly wandering* on a sequence $\{p_i\}$ of elements of $\mathbb{Z}^d$ if for all $m,n\in\mathbb{Z}$ with $m\neq n$, we have $\mu(T^{p_m}W\cap T^{p_n}W)=0$. A set $W$ which is weakly wandering on a sequence $\{p_i\}$ of elements of $\mathbb{Z}^d$ is *exhaustive* if

$$\mu\Big(X-\bigcup_{i=0}^{\infty}T^{p_i}W\Big)=0.$$

We say that the set $W$ is *properly exhaustive* if the sequence $\{p_i\}$ is not generated by a single element, i.e., there is no $p\in\mathbb{Z}^d$ such that $p_i=n_ip$ for some sequence $n_i\in\mathbb{Z}$. We will frequently write e.w.w. for exhaustive weakly wandering.

If the action has some element $p$ that is ergodic as a $\mathbb{Z}$-action, by [JK] there will be an e.w.w. set for the action of $p$, and trivially for the $\mathbb{Z}^2$-action; however, this set will not be properly exhaustive for the $\mathbb{Z}^2$-action. For the examples below we construct properly e.w.w. sequences.

If $T$ is a nonsingular action, for any $x\in X$ and any $p\in\mathbb{Z}^d$, we let

$$\omega_p(x)=\left(\frac{d(\mu\circ T^p)}{d\mu}\right)(x).$$

The notion of ratio set was introduced by Krieger [Kri], who proved its basic properties. The *ratio set* of an action $T$, denoted by $r(T)$, is the set

$$\begin{aligned}r(T)=\{t\in[0,\infty):\forall\varepsilon>0,\ \forall A \text{ with } \mu(A)>0,\\ \exists p\in\mathbb{Z}^d \text{ such that } \mu(A\cap T^{-p}A\cap\{x:\omega_p(x)\in N_\varepsilon(t)\})>0\},\end{aligned}$$

where $N_\varepsilon(t)=\{s\geq 0:|s-t|<\varepsilon\}$. Krieger showed (cf. [Kri], [HO]) that the ratio set of an ergodic action is invariant under change to an equivalent measure, and $r(T)\setminus\{0\}$ must be a multiplicative subgroup of $\mathbb{R}^+$. This allows four possibilities:

1. $r(T) = \{1\}$,
2. $r(T) = \{0, 1\}$,
3. $r(T) = \{0\} \cup \{\lambda^k : 0 < \lambda < 1,\ k \in \mathbb{Z}\}$,
4. $r(T) = \mathbb{R}^+$.

The first possibility is called *type* II and these are actions that admit an equivalent sigma-finite invariant measure; if the invariant measure is infinite it is type $\mathrm{II}_\infty$, otherwise type $\mathrm{II}_1$. The others are types $\mathrm{III}_0$, $\mathrm{III}_\lambda$ and $\mathrm{III}_1$, respectively.

Given a nonsingular transformation $T$, an $L^\infty$ *eigenvalue* is a complex number $\lambda$ such that for some nonnull function $f$ in $L^\infty$, $f(Tx) = \lambda f(x)$ a.e. Since the $L^\infty$ norms of $f$ and $f \circ T$ are equal, eigenvalues must have modulus 1. If $T$ is ergodic then $|f|$ must be constant a.e. Further, $T$ is said to be *weakly mixing* if for every finite measure preserving ergodic transformation $(Y, \nu, S)$, $(X \times Y, \mu \times \nu, T \times S)$ is ergodic. These notions for the case of nonsingular transformations were studied in [ALW], where it is shown that $T$ is weakly mixing if and only if $T$ is ergodic and its only $L^\infty$ eigenvalue is 1. We say that a transformation $T$ has $L^\infty$ *continuous spectrum* if it is ergodic and its only $L^\infty$ eigenvalue is 1.

The following lemma is well known for finite measure preserving transformations, but we include a proof for the general case.

LEMMA 2.1. *Let $T$ be a nonsingular transformation. If $T$ has continuous $L^\infty$ spectrum, then for all $n \in \mathbb{N}$, $T^n$ has continuous $L^\infty$ spectrum.*

Proof. Suppose that there exists a function $f \in L^\infty$ such that $f \circ T^n = \lambda f$, where $|f| = 1$ and $|\lambda| = 1$. Set $F = f \cdot f \circ T \cdot \ldots \cdot f \circ T^{n-1}$. Since $F \circ T = \lambda F$, it follows that $\lambda = 1$. It remains to prove that $T^n$ is ergodic. Suppose the contrary. Then it is easy to see that there is a measurable subset $A$ so that $X$ is the disjoint union of $A, TA, \ldots, T^{r-1}A$ for some $r < n$. We set $H = \sum_{k=0}^{r-1} \alpha^k \chi_{T^k A}$, where $\alpha = e^{2\pi i/r}$ and $\chi_{T^k A}$ is the characteristic function of $T^k A$. Since $H \circ T = \alpha H$, we have $\alpha = 1$, a contradiction. ∎

If the basis transformations of a $\mathbb{Z}^d$-action are weakly mixing then by Lemma 2.1 they are totally ergodic and by the same proof as in [AS] any $d$-dimensional subgroup of $\mathbb{Z}^d$ acts ergodically.

If $T \times T$ is ergodic then it is clear that $T$ must have continuous $L^\infty$ spectrum. However, in the infinite measure preserving and nonsingular cases the converse is not true [ALW], [AFS]. A nonsingular transformation $T$ is said to have *infinite ergodic index* if for all $k$, the Cartesian product of $k$ copies of $T$ is ergodic; it follows that all products are also conservative. Kakutani and Parry [KP] constructed the first examples of infinite measure preserving transformations with the $k$th Cartesian product ergodic but the $(k+1)$st not ergodic, and of infinite ergodic index. *Infinite conservative* *index* is defined in an analogous way. After the first version of this paper was written (which contained the proof of Theorem 4.3 but not of Theorem 4.9), a stronger condition was introduced in [DGMS]. An action $T$ of a group $G$ is said to be *power weakly mixing* if for all $g_1, \ldots, g_r \in G \setminus \{e\}$, $T^{g_1} \times \ldots \times T^{g_r}$ is ergodic. Clearly, any power weakly mixing transformation has infinite ergodic index. An infinite measure preserving transformation that is power weakly mixing is constructed in [DGMS]. Recently, it has been shown that there exists an integer action that has infinite ergodic index but is not power weakly mixing [AFS2].

A nonsingular transformation $T$ is said to be *partially rigid* if there exists an $\eta > 0$, an increasing sequence $r_n$, and a constant $R > 0$ such that for all sets $A$ of finite measure, $\liminf_{n\to\infty} \mu(T^{r_n}A \cap A) \geq \eta\mu(A)$ and $\omega_{T^{r_n}}(x) < R$ a.e. In [AFS], it was shown that if $T$ and $S$ are partially rigid under the same sequence $r_n$ then $T \times S$ is partially rigid under $r_n$, and that partially rigid transformations are conservative. As remarked in [AFS], it follows that if $T$ is partially rigid then it has infinite conservative index.

We use the following notation for certain squares in the integer lattice:

$$\mathcal{SQ}(h) = \{(a,b) : a, b \in \mathbb{Z},\ 0 \leq a < h \text{ and } 0 \leq b < h\}.$$

Given a nonnegative integer $h$, a *grid* $G$ of *length* $h$ is a collection of $h^2$ disjoint intervals in $\mathbb{R}^+$ indexed by $\mathcal{SQ}(h)$-elements. (All intervals in this paper are assumed left closed and right open.) Thus a bijection $\mathrm{Loc}_G : G \to \mathcal{SQ}(h)$ is implicit. For an interval $I \in G$, we call $\mathrm{Loc}_G(I)$ the *location* of $I$, and define $G(i,j) = \mathrm{Loc}_G^{-1}(i,j)$. A grid $G$ partially defines transformations $T^{(1,0)}$ and $T^{(0,1)}$ in the following way. Given an interval $I \in G$ with location $(i,j)$, define $T^{(1,0)}$ on $I$ to be the (orientation preserving) affine map that sends $I$ to the interval with location $(i+1,j)$; if no such interval exists $T^{(1,0)}$ remains undefined. Similarly, let $T^{(0,1)}$ take an interval $I$ to the interval with location $(i,j+1)$; again if no interval exists $T^{(0,1)}$ remains undefined.

Let $G$ and $H$ be two grids of length $g$ and $h$ respectively. Given nonnegative integers $a$ and $b$ such that $\max\{a+g, b+g\} < h$, we say the subgrid $G'$ defined by $G'(i,j) = H(a+i, b+j)$, for $0 \leq i < g$ and $0 \leq j < g$, is a *copy* of $G$ in $H$ located at $(a,b)$, if $G'(i,j) \subset G(i,j)$ for $0 \leq i < g$ and $0 \leq j < g$, and

$$T^{(1,0)}(G'(i,j)) = G'(i+1,j), \qquad T^{(0,1)}(G'(i,j)) = G'(i,j+1).$$

We denote the location $(a,b)$ by $\mathrm{Loc}_H(G')$.

**3. A $\mathbb{Z}^2$ skyscraper.** In this section we define a simple family of actions which exhibits sequences on which $[0,1)$ is properly exhaustive and weakly wandering. This $\mathbb{Z}^2$-action is analogous to the Hajian–Kakutani skyscraper [HK2] since it sweeps out all the spacers in each grid before proceeding to the next grid.

Weakly wandering sets for integer actions were introduced in [HK1] by Hajian and Kakutani who showed, among other things, that ergodic infinite measure preserving transformations admit weakly wandering sets of positive measure. In [HK2], Hajian and Kakutani constructed an example of an ergodic infinite measure preserving transformation with an exhaustive weakly wandering set of finite measure. The sequence under which the set is exhaustive has interesting arithmetical properties and this has been studied e.g. in Eigen–Hajian–Kakutani [EHK]. In [JK], Jones and Krengel showed that every ergodic infinite measure preserving integer action admits a weakly wandering set that is exhaustive, though possibly of infinite measure. In [HI], Hajian and Ito showed that an arbitrary group of measurable nonsingular transformations admits an equivalent finite invariant measure if and only if it does not admit a weakly wandering set of positive measure. It remains an open question whether every ergodic infinite measure preserving $\mathbb{Z}^d$-action admits a properly exhaustive weakly wandering set.

**3.1.** *Construction.* To define the basis transformations $T^{(1,0)}$ and $T^{(0,1)}$ we first construct inductively a sequence of grids $G_n$ of length $h_n$. Let $h_0 = 1$ and $G_0 = \{[0,1)\}$. Given $G_n$, we set $h_{n+1} = 4h_n$ and divide each interval interal $I \in G$ into four equal parts: $I = \bigcup_{i=0}^{3} I_i$ enumerated from the left to the right. Now set $\mathrm{Loc}_{G_{n+1}}(I_0) = \mathrm{Loc}_{G_n}(I)$, $\mathrm{Loc}_{G_{n+1}}(I_1) = \mathrm{Loc}_{G_n}(I) + (0,h_n)$, $\mathrm{Loc}_{G_{n+1}}(I_2) = \mathrm{Loc}_{G_n}(I) + (h_n,0)$, and $\mathrm{Loc}_{G_{n+1}}(I_3) = \mathrm{Loc}_{G_n}(I) + (h_n,h_n)$. Finally, we consider the elements of $\mathcal{SQ}(h_{n+1})$ which do not yet have intervals assigned to them; to these we assign a *spacer*, a new interval chosen from $\mathbb{R}^+$ of the same length as the previous ones. We choose each spacer interval so that it is disjoint from all previously chosen spacers and from $[0,1)$, and that it abuts on the previously chosen spacer (or, if it is the first spacer, so that it abuts on the unit interval).

The construction is a process of "cutting and tiling", analogous to the "cutting and stacking" with which rank one $\mathbb{Z}$-actions are constructed. It is easy to see that the number of intervals in a grid $G_n$ is $4^{2n}$, and that the length of each interval is $1/4^n$. Thus the measure of the union of the intervals within that grid, which we denote by $G_n^*$, is $4^n$. Thus, as $n \to \infty$, $G_n^* \to X = \mathbb{R}^+$.

Next, we define our transformations $T^{(1,0)}$ and $T^{(0,1)}$ on a grid $G_n$ as explained earlier. One can check that $T^{(1,0)}$ and $T^{(0,1)}$ are defined everywhere as $n \to \infty$. In this section, as well as in Section 4, all grids consist of intervals of the same length.

THEOREM 3.1. *The $\mathbb{Z}^2$-action $T$ defined by the above construction is measure preserving and properly ergodic. The basis transformations $T^{(0,1)}$ and $T^{(1,0)}$ are not ergodic but are partially rigid under the same sequence*

*$r_n = h_n$, hence the Cartesian products of any finite number of basis transformations is conservative.*

Proof. It is clear that intervals are sent to intervals of the same length and since the intervals in the union of the grids generate, the action is measure preserving.

Now we show that for any two sets $A$ and $B$ of positive measure, there exists an element $g \in \mathbb{Z}^2$ such that $\mu(T^g A \cap B) > 0$. There exists a grid $G_n$ and intervals $I, J \in G_n$ such that $\mu(A \cap I) > 0.5\mu(I)$ and $\mu(B \cap J) > 0.5\mu(I)$. Let $g = \mathrm{Loc}_{G_n}(J) - \mathrm{Loc}_{G_n}(I)$. Clearly, $\mu(T^g A \cap B) > 0$. It follows that $T$ is ergodic. Since $\mu$ is nonatomic, $T$ is properly ergodic and conservative.

To show that $T^{(1,0)}$ is partially rigid, since $T^{(1,0)}$ is measure preserving, it is enough to show only the first condition (the same argument applies to $T^{(0,1)}$). Moreover, by [AFS], Lemma 1.2, it suffices to show the result on an algebra that approximates all sets of finite measure. Let $r_n = h_n$ for all $n > 0$. Let $A \in G_k$ be an interval for some $k > 0$, and let $\eta = 1/4$. Note that in the grid $G_{n+1}$ for $n > k$, $T^{(h_n,0)}G_n^{(0,0)} = G_n^{(1,0)}$ and for $(i,j) \in \mathcal{SQ}(4)$, $\mu\big(G_n^{(i,j)} \cap A\big) = \frac{1}{4}\mu(A)$. (The first equality is understood to mean

$$T^{(h_n,0)}(G_n^{(0,0)}(\mathrm{Loc}_{G_{n+1}}(G_n^{(0,0)}) + (i,j)) = G_n^{(1,0)}(\mathrm{Loc}_{G_{n+1}}(G_n^{(1,0)}) + (i,j))$$

for all $0 \le i, j \le h_n$; similar equalities later in the paper are interpreted in the same way.)

Therefore

$$\mu(T^{(h_n,0)}A \cap Aw\big) \ge \mu(T^{(h_n,0)}(G_k^{(0,0)} \cap A) \cap (G_k^{(1,0)} \cap A)) \ge \tfrac{1}{4}\mu(A).$$

To show the basis transformations are nonergodic, let $A = [0, 1/4)$ and $B = [3/4, 1)$. Let $n > 0$ be an integer. Choose the first $k$ such that $n < h_{k-1}$. For each $I \in G_k$, $I \subset A$, we have $T^{(n,0)}I \in G_k$ and

$$\mathrm{Loc}_{G_k}(T^{(n,0)}I) = (2a + n, 2b)$$

for some integers $a$ and $b$. Now if $J \subset B$, $J \in G_k$ then $\mathrm{Loc}_{G_k}(J) = (2c+1, 2d+1)$ for some $c$ and $d$. As $T^{(n,0)}I \in G_k$ it follows that $T^{(n,0)}I \cap J = \emptyset$. Also, $T^{(0,1)}$ is nonergodic by the symmetry of the construction. ∎

**3.2.** *The sequence* $\{w_i\}$. We use a similar technique to that in [HK2] to construct a sequence $\{w_i \in \mathbb{Z}^2 : i = 0, 1, \ldots\}$ on which the set $W = [0, 1)$ is properly exhaustive weakly wandering. Let $w_0 = (0, 0)$. Given $i > 0$, we consider its quartic expansion:

$$i = 4^0\varepsilon_0 + 4^1\varepsilon_1 + \ldots + 4^k\varepsilon_k,$$

where $\varepsilon_j = \varepsilon_j(i) \in \{0, 1, 2, 3\}$, for $j = 0, 1, \ldots, k$ and some $k$ depending on $i$.

Now we assign to each $\varepsilon_j$ a $\delta_j$ as follows:

$$\delta_j = \delta_j(\varepsilon_j) = \begin{cases} (0,0) & \text{if } \varepsilon_j = 0, \\ (2,0) & \text{if } \varepsilon_j = 1, \\ (0,2) & \text{if } \varepsilon_j = 2, \\ (2,2) & \text{if } \varepsilon_j = 3, \end{cases}$$

Finally, define the weakly wandering sequence $w_i$ by

$$w_i = 4^0\delta_0 + 4^1\delta_1 + \ldots + 4^k\delta_k.$$

THEOREM 3.2. *$W$ is weakly wandering and properly exhaustive along the sequence $\{w_i\}$.*

Proof. The proof is inductive on the hypothesis that, for $n > 0$, the following two conditions hold:

(1) the sets $\{T^{w_i}W : 0 \le i < 4^n\}$ are pairwise disjoint, and
(2) $G_n^* = \bigcup_{i=0}^{4^n-1} T^{w_i}W$.

This is clearly true for $n = 1$. We show that (1) and (2) hold for $n+1$. Actually,

$$\bigcup_{i=0}^{4^{n+1}-1} T^{w_i}W = \bigcup_{j=0}^{3} T^{4^n\delta_j}\Big(\bigcup_{i=0}^{4^n-1} T^{w_i}W\Big) = \bigcup_{j=0}^{3} T^{4^n\delta_j}G_n^* = G_{n+1}^*,$$

as $\mathcal{SQ}(h_{n+1}) = \bigcup_{j=0}^{3}(\mathcal{SQ}(2\cdot 4^n) + 4^4\delta_j)$. To show (1), recall that $\mu(W) = 1$, $\mu(G_{n+1}) = 4^{n+1}$, and $T$ is measure preserving. Then

$$\sum_{i=0}^{4^{n+1}-1} \mu(T^{w_i}W) = \sum_{i=0}^{4^{n+1}-1} 1 = \mu(G_{n+1}^*) = \mu\Big(\bigcup_{i=0}^{4^{n+1}-1} T^{w_i}W\Big). \ \blacksquare$$

**3.3.** *Sets of increasing and infinite measure.* In [C], Crabtree describes exhaustive weakly wandering sets on the example of Hajian and Kakutani [HK2] whose measures are greater than 1; in particular, he details the construction of both an increasing sequence of e.w.w. sets and an infinite measure e.w.w. set.

**3.3.1.** *Properly exhaustive weakly wandering sets of increasing measure.* For any integer $n$, we can take $W = G_n$. If we let

$$w_i = 4^n(\delta_0 4^0 + \delta_1 4^1 + \ldots + \delta_k 4^k),$$

this is a properly exhaustive w.w. sequence for $W$. Thus we have the increasing sequence $G_0, G_1, \ldots$ of e.w.w. sets; the proof that each is e.w.w. is identical to the proof of Theorem 3.2, with each dimension scaled up by $4^n$.

**3.3.2.** *An infinite measure properly exhaustive weakly wandering set.* The construction of an infinite measure weakly wandering set $W_\infty$ is inductive on $n$. We begin with $W_0 = [0, 1/2)$. Given $W_{n-1}$, let $W_n = W_{n-1} \cup T^{(h_n/2,0)} W_{n-1}$. Note that $W_n$ is well defined in $G_n$ and $\mu(W_n) = 2\mu(W_{n-1})$.

This construction makes translations by $(h_n/2, 0)$ and $(h_n/2, h_n/2)$ inadmissible in $\{w_i\}$ but admits translations by $(1, h_n/2)$ and $(1, 0)$. We define a sequence $v_i$ with binary coding of $i$:

$$i = 2^0\varepsilon_0 + 2^1\varepsilon_1 + \ldots + 2^k\varepsilon_k,$$

where $\varepsilon_j \in \{0, 1\}$ for $j = 0, 1, \ldots, k$ and

$$\delta_j = \begin{cases} (0,0) & \text{if } \varepsilon_j = 0, \\ (0,2) & \text{if } \varepsilon_j = 1. \end{cases}$$

Put $v_i = 4^0\delta_0 + 4^1\delta_1 + \ldots + 4^k\delta_k$; the weakly wandering sequence $w_i$ is given by

$$w_{2i} = v_i \quad \text{and} \quad w_{2i+1} = v_i + (1, 0).$$

**4. Infinite measure actions.** In this section we first modify the finite measure preserving staircase actions of [AS] to construct infinite measure, measure preserving $\mathbb{Z}^2$-actions for which the basis transformations have infinite ergodic index. It is possible to choose a sequence $\{c_n\}$ of cuts (as defined below) for the staircase action of [AS] so that the resulting space has infinite measure; however, in this case the sequence $\{c_n\}$ will be unbounded. Our methods do not apply if $\liminf c_n = \infty$. While one could modify the construction on a subsequence to obtain $\liminf c_n < \infty$, we in fact define a new family of staircase actions that has infinite measure but with a bounded sequence of cuts; adapting techniques from [AFS] we show that in this case the basis transformations have infinite ergodic index. In the second part of the section we extend this construction to the multistep actions which we show are power weakly mixing.

**4.1.** *Staircase actions.* Given a positive integer c, a grid $H$ is defined to be an *infinite staircase c-cut* of a grid $G$, of length $g$, if $G \subset H$ and $H$ is a grid of least size that contains $(c+1)^2$ copies of $G$ located at

$$(2ig + i(i-1)/2 + ij, 2jg + j(j-1)/2 + ij)$$

for $(i, j) \in \mathcal{SQ}(h)$. The copy at this location is denoted by $G_n^{(i,j)}$. The length of $H$ is $h = 2(c+1)g + c(c-1)/2 + c^2$.

As before, we define on the grid $G$ two commuting transformations, $T^{(1,0)}$ as the translation mapping $G(i, j)$ onto $G(i+1, j)$, for $0 \le i < h-1$ and $0 \le j < h$; and $T^{(0,1)}$ as the translation mapping $G(i, j)$ onto $G(i, j+1)$, for $0 \le j < h-1$ and $0 \le i < h$. Figure 1 shows an infinite staircase 3-cut.

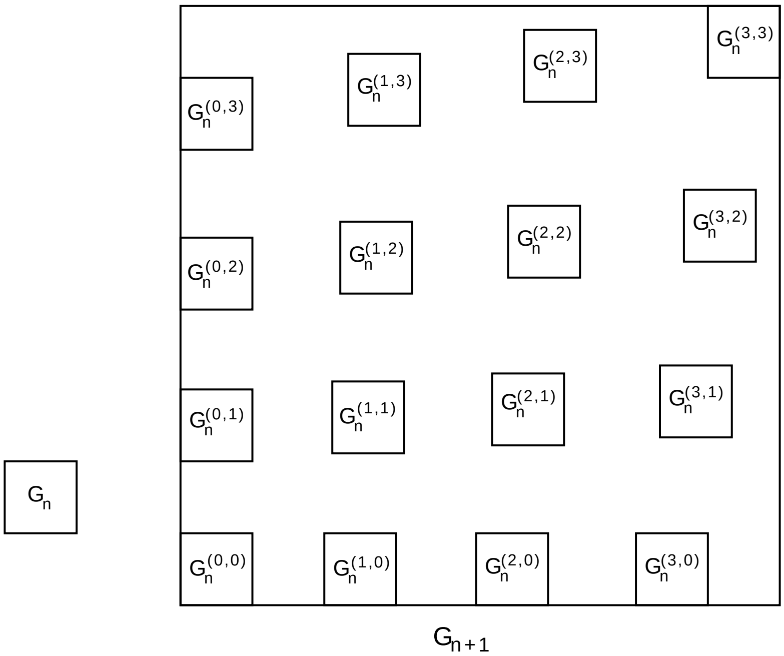


Fig. 1. An infinite staircase 3-cut

An *infinite staircase action* is defined by giving a sequence of positive numbers $\{c_n\}$ and a sequence of grids $\{G_n\}$ such that $G_0 = \{[0, 1)\}$ and $G_{n+1}$ is an infinite staircase $c_n$-cut of $G_n$.

Let $h_n$ denote the length of $G_n$. Then $h_0 = 1$ and

$$h_{n+1} = (2c_n + 1)h_n + c_n(c_n - 1)/2 + c_n^2. \tag{1}$$

It is clear that $T^{(1,0)}$ and $T^{(0,1)}$ so defined commute, and that the staircase $\mathbb{Z}^2$-action is measure preserving and ergodic.

PROPOSITION 4.1. *Let $T$ be an infinite measure staircase action with sequence $\{c_n\}$ of cuts. Then $T$ is defined on an infinite measure space.*

Proof. It suffices to consider the worst case $c_n = 1$ for all $n$. From (1) we deduce that $h_{n+1} > 4h_n > 4^{n+1}$. If $I \in G_n$ is an interval then $\mu(I) = 1/4^n$. There are $h_n^2$ intervals in $G_n$, so $\mu(G_n) = h_n^2/4^n > 4^n$. ■

If $I \in G_n$ is an interval and $0 < t \le h_n$ then let $\triangle(I, t)$, the *t-triangle under* $I$, denote the collection of all intervals $J \in G_n$ such that

$$\mathrm{Loc}_{G_n}(J) = \mathrm{Loc}_{G_n}(I) - (i, j)$$

where

$$0 \le i \le t, \quad 0 \le j \le t, \quad j \le i.$$

(Depending on the location of $I$, sometimes it may not look like a proper triangle.)

For concreteness, in the remainder of this section we will assume that $c_n = 3$ for all $n$, but one can verify that similar arguments work for $c_n \geq 2$.

LEMMA 4.2. *Let $T$ be the infinite measure staircase action with $c_i = 3$ for all $i \geq 0$. Given positive integers $n$ and $t$ there exists an integer $l = l(n,t) > 0$ such that if $I$ and $J$ are any two intervals in $G_n$ with $J \in \triangle(I,t)$ then*

$$\mu(T^{(l,0)} I \cap J) \geq \frac{1}{16^t}\mu(J).$$

Proof. For all $k \geq 0$, $G_{n+k+1}$ will contain 16 copies of $G_{n+k}$ where for $(i,j) \in \mathcal{SQ}(3+1)$, $\mu(G^{(i,j)}_{n+k}) = \frac{1}{16}\mu(G_{n+k})$. Observe that

$$\begin{aligned} T^{(2h_{n+k},0)} G^{(0,0)}_{n+k} &= G^{(1,0)}_{n+k}, \\ T^{(2h_{n+k},0)} G^{(1,0)}_{n+k} &= T^{(-1,0)} G^{(2,0)}_{n+k}, \\ T^{(2h_{n+k},0)} G^{(0,1)}_{n+k} &= T^{(-1,-1)} G^{(1,1)}_{n+k}. \end{aligned}$$

Using this idea, we set

$$l = \sum_{k=0}^{t-1} 2h_{n+k}.$$

Let $J \subset \triangle(I,t)$ and set $(x,y) = \mathrm{Loc}_{G_n}(I) - \mathrm{Loc}_{G_n}(J)$ (note that $0 \leq y \leq x$). Define intervals $I_k$ recursively for $0 \leq k \leq t$. Let $I_0 = I$. Then $\mu(I_0 \cap T^{(x,y)} J) = \mu(J)$. Assume that $I_k$ has been defined. If $k+1 \leq y$, let $I_{k+1} = T^{(2h_{n+2k},0)}(I_k \cap G^{(0,1)}_{n+2k})$. Then

$$\mu(I_{k+1} \cap T^{(x-(k+1),y-(k+1))} J) \geq \frac{1}{16^{k+1}}\mu(J).$$

If $y < k+1 \leq x$, let $I_{k+1} = T^{(2h_{n+2k},0)}(I_k \cap G^{(1,0)}_{n+2k})$. Then

$$\mu(I_{k+1} \cap T^{(x-(k+1),0)} J) \geq \frac{1}{16^{k+1}}\mu(J).$$

If $x < k+1 \leq t$, let $I_{k+1} = T^{(2h_{n+2k},0)}(I_k \cap G^{(0,0)}_{n+2k})$. Then

$$\mu(I_{k+1} \cap J) \geq \frac{1}{16^{k+1}}\mu(J).$$

Thus $I_t$ has been defined and $\mu(I_t \cap J) \geq \frac{1}{16^t}\mu(J)$. Also, $I_t \subset T^{(l,0)} I$. ∎

THEOREM 4.3. *Let $T$ be an infinite staircase action with sequence of cuts $c_n = 3$. Then the basis transformations $T^{(1,0)}$ and $T^{(0,1)}$ have infinite ergodic index.*

Proof. Let $k > 0$ and $S$ be the Cartesian product of $k$ copies of $T^{(1,0)}$. By symmetry it suffices to show that $S$ is ergodic. Let $A'$ and $B'$ be sets of positive measure in the product space and let $\mu_k$ denote product measure. Choose intervals $I_i$ and $J_i$, $i = 1, \ldots, k$, in some grid $G_m$ such that for $I = I_1 \times \ldots \times I_k$ and $J = J_1 \times \ldots \times J_k$,

$$\frac{\mu_k(A' \cap I)}{\mu_k(I)} > \frac{5}{6} \quad \text{and} \quad \frac{\mu_k(B' \cap J)}{\mu_k(J)} > \frac{5}{6}.$$

By taking a finer approximation in the grid $G_{m-1}$, and using the structure of the 16 copies of $G_{m-1}$ in $G_m$ we may assume that for each $i = 1, \ldots, k$, $J_i \in \triangle(I_i, t_i)$ for some $t_i$ (since any interval in $G_n^{(0,0)}$ is in the $t$-triangle of any interval in $G_n^{(3,1)}$ for some $t$). Let $A = A' \cap I$, $B = B' \cap J$, and $t = \max\{t_i : i = 1, \ldots, k\}$. Then $t \le h_m$. Choose $\delta = 1/16^t$. For any $n \ge m$ let

$$\Gamma_n = \Big\{1, \ldots, \prod_{i=m}^{n-1} (c_i + 1)^2\Big\}$$

and label the copies of $G_m$ in $G_n$ with integers from $\Gamma_n$. To find a finer approximation within $I$, choose a sufficiently large $n > m$ such that there is a set $I'$ of the form

$$I' = \bigcup_{\overline{u} \in U'} I_{\overline{u}} \quad \text{where } U' \subseteq \Gamma_n^k$$

so that $\mu_k(I' \triangle A) < \frac{1}{18}\delta^k \mu_k(I)$. Further, each $I_{\overline{u}}$ is of the form $I_{\overline{u}} = I_{u_1} \times \ldots \times I_{u_k}$ where $I_{u_i}$ is in $I_i$ and in the $u_i$ copy of $G_m$ in $G_n$. Similarly, there exists a subset $V' \subseteq \Gamma_n^k$ where $J' = \bigcup_{\overline{v} \in V'} J_{\overline{v}}$ so that $\mu_k(J' \triangle B) < \frac{1}{18}\delta^k \mu_k(J)$. Using the triangle inequality one obtains

$$\mu_k(I' \triangle I) < \tfrac{1}{3}\mu_k(I) \quad \text{and} \quad \mu_k(J' \triangle J) < \tfrac{1}{3}\mu_k(J).$$

Next we choose the "good" subintervals by letting

$$U'' = \big\{\overline{u} \in U' : \mu_k(I_{\overline{u}} \setminus A) < \tfrac{1}{3}\delta^k \mu_k(I_{\overline{u}})\big\}$$

and $I'' = \bigcup_{\overline{u} \in U''} I_{\overline{u}}$, and constructing $V''$ and $J''$ in a similar way. Now we have

$$\mu_k(I' \setminus I'') = \sum_{\overline{u} \in U' \setminus U''} \mu_k(I_{\overline{u}}) \le \sum_{\overline{u} \in U' \setminus U''} \frac{3}{\delta^k} \mu_k(I_{\overline{u}} \setminus A) \le \frac{3}{\delta^k} \mu_k(I' \setminus A).$$

Thus $\mu_k(I'' \triangle I') < \frac{1}{6}\mu_k(I)$, and

$$\mu_k(I'' \triangle I) < \tfrac{1}{6}\mu_k(I) + \tfrac{1}{3}\mu_k(I) = \tfrac{1}{2}\mu_k(I).$$

Likewise, $\mu_k(J'' \triangle J) < \frac{1}{2}\mu_k(J)$. Thus both $I''$ and $J''$ cover more than half of $I$ and $J$ respectively, and so there must exist an element $\overline{w} \in U'' \cap V''$. By Lemma 4.2 there is an integer $l = l(n, t)$ such that

$$\mu_k(S^l I_{\overline{w}} \cap J_{\overline{w}}) \ge \delta^k \mu_k(J_{\overline{w}}).$$

As $\overline{w}$ is in $U''$ and $V''$, it follows that

$$\begin{aligned}\mu_k(S^l A \cap B) &\geq \mu_k(S^l I_{\overline{w}} \cap J_{\overline{w}}) - \mu_k((S^l I_{\overline{w}} \cap J_{\overline{w}}) \setminus (S^l A \cap B)) \\ &\geq \delta^k \mu_k(J_w) - \frac{\delta^k}{3}\mu_k(I_w) - \frac{\delta^k}{3}\mu_k(J_2) > 0. \ \blacksquare\end{aligned}$$

The proof of the next result is similar to that of partial rigidity in Theorem 3.1.

THEOREM 4.4. *Let $T$ be an infinite staircase action with the sequence of cuts $c_n = 3$. Then the transformations $T^{(1,0)}$ and $T^{(0,1)}$ are partially rigid.*

REMARK 4.5. The previous proofs for infinite measure staircase $\mathbb{Z}^2$-actions can be generalized in a natural way to infinite measure staircase $\mathbb{Z}^d$-actions for $d > 2$. We leave this as an exercise for the reader.

**4.2.** *Multistep actions.* Here we modify the infinite staircase to construct a $\mathbb{Z}^2$-action that is power weakly mixing. As mentioned earlier, a power weakly mixing infinite transformation was constructed recently in [DGMS]. It remains open whether our infinite staircase actions are power weakly mixing, but we show how to modify the construction so that essentially the same proof of infinite ergodic index yields power weakly mixing for the new actions. For clarity of exposition we do this in two steps. First, we define step actions, then we generalize this to multistep actions and show how the same idea in the proof of Theorem 4.3 proves that multistep actions are power weakly mixing.

Given a positive integer $c$ and $(m, n) \in \mathbb{Z}^2$ where $m$ and $n$ are positive, a grid $H$ is an $(m, n)$-*step $c$-cut* of a grid $G$ of *length $g$* if $G \subset H$ and $H$ is a grid of least size that contains $(c+1)^2$ copies of $G$ located at

$$\begin{aligned}&((mi+nj)g+i(i-1)/2+ij, (ni+mj)g+j(j-1)/2+ij) && \text{for } m \neq n, \\ &((mi+nj)g+i(i-1)/2+ij, (ni+mj+cj)g+j(j-1)/2+ij) && \text{for } m = n\end{aligned}$$

for $(i, j) \in \mathcal{SQ}(c+1)$. We need the extra condition for the $m = n$ case or else $G^{(i,j)} = G^{(j,i)}$. The length of $H$ is

$$h = \begin{cases} ((m+n)c+1)g + c(c-1)/2 + c^2 & \text{for } m \neq n, \\ ((m+n+c)c+1)g + c(c-1)/2 + c^2 & \text{for } m = n. \end{cases}$$

Note that an $(m, n)$-step $c$-cut is identical to an $(n, m)$-step $c$-cut. Figure 2 shows a $(2, 1)$-step 2-cut.

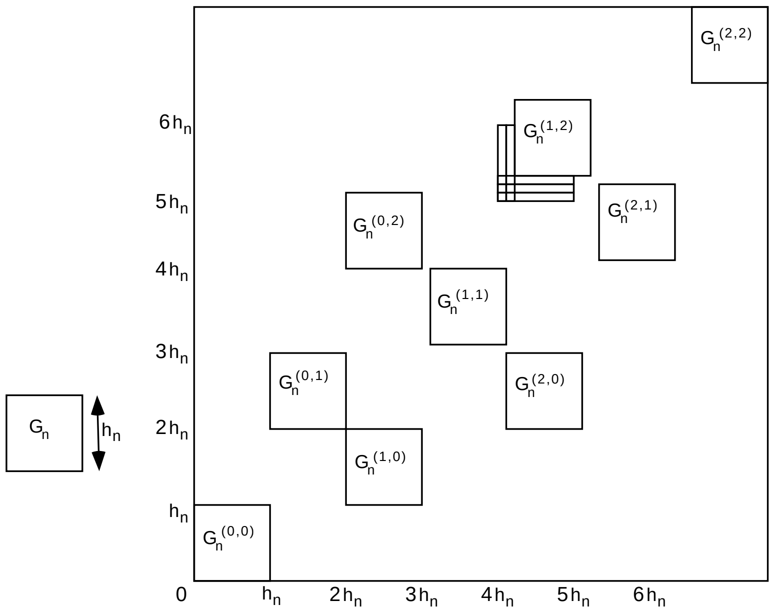


Fig. 2. A $(2,1)$-step 2-cut. The grid $G_n$ is shown next to $G_{n+1}$ and the indexed copies of $G_n$ are drawn. Note that $G^{(1,2)}$ is located at position $(4g+2, 5g+3)$, and we include the rows of intervals to show the offset.

A *step action* is defined by giving an initial grid $G_0$, a sequence $\{c_i\}$ of positive numbers called the cutting sequence, and a sequence $\{a_i\}$, $a_i = (m_i, n_i)$, $i \geq 0$, called the tiling sequence, where $m_i$ and $n_i$ are positive integers. Then a sequence $\{G_i\}$, $i \geq 0$, of grids is defined so that $G_0 = \{[0,1)\}$ and $G_{i+1}$ is an $a_i$-step $c_i$-cut of $G_i$. The length of each grid is $h_i$.

It is clear that $T^{(1,0)}$ and $T^{(0,1)}$ so defined commute, and that the $\mathbb{Z}^2$-step action is measure preserving, ergodic and defined on an infinite measure space. It is possible to choose a tiling sequence $(m_i, n_i)$ so that for each positive $(m,n)$, $T^{(m,n)}$ satisfies the corresponding equalites similar to those in the proof of Lemma 4.2, and then the proof of Theorem 4.3 can be adapted to show that for all $(m,n) \neq (0,0)$, $T^{(m,n)}$ has infinite ergodic index; however, we omit the details since our emphasis is on the multistep actions.

For the case of multistep actions, we will use the $(m,n)$-step 3-cuts of the step action to define a sequence of grids to prove a generalization of Lemma 4.2, which is Lemma 4.8 below.

Let $a = ((m_1, n_1), \ldots, (m_k, n_k)) \in \mathbb{Z}^{2k}$. Let $G_n$ be a grid of length $g_n$. We say that a grid of least size $H$ of length $h$ is an *$a$-multistep cut* of $G_n$ if $H$ is obtained as follows: first cut $G_n$ into $k$ copies, denoted by $G'_{n,1}, \ldots, G'_{n,k}$. For each $G'_{n,j}$ where $j = 1, \ldots, k$, cut $G'_{n,j}$ into 16 copies and arrange them in a grid $G_{n,j}$ so that $G_{n,j}$ is an $(m_j, n_j)$-step 3-cut of $G'_{n,j}$ and $G_{n,j}$ has length $h_{n,j}$. Denote the copies of $G'_{n,j}$ in $G_{n,j}$ by $G^{(x,y)}_{n,j}$ where $(x, y) \in \mathcal{SQ}(4)$. Now let $H$ be constructed by tiling the $G_{n,j}$'s so that $G_{n,j}$ is located at $(\sum_{i=1}^{j} h_{n,i-1}, \sum_{i=1}^{j} h_{n,i-1})$, where $h_{n,0} = 0$. Then $h = \sum_{i=1}^{k} h_{n,i}$.

One can use a simple diagonalization argument to construct a sequence $\{c_n\}$ which has the following property.

PROPOSITION 4.6. *There is a sequence $\{c_n\}$ such that if $((\alpha_1, \beta_1), \ldots \ldots, (\alpha_k, \beta_k)) \in \mathbb{Z}^{2k}$, with $\alpha_i \geq \beta_i$ and $(\alpha_i, \beta_i) \neq (0, 0)$, for $1 \leq i \leq k$, then there exists $n \in \mathbb{N}$ such that*

$$c_n = ((\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k)).$$

A $\mathbb{Z}^2$-*multistep action* is defined by giving a sequence $\{c_n\}$ as in Proposition 4.6, called the cutting sequence, and a sequence of grids $\{G_n\}$ where $G_{n+1}$ is a $c_n$-multistep cut of $G_n$. Put $G_0 = [0, 1)$ and $h_0 = 1$.

The next result follows from Proposition 4.1.

PROPOSITION 4.7. *Let $T$ be the multistep action sequence of cuts $c_n$ as defined above. Then $T$ is defined on an infinite measure space.*

The following lemma shows that the multistep action satisfies a much stronger version of the triangle property.

LEMMA 4.8. *Let $k > 0$ and $((\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k)) \in \mathbb{Z}^{2k}$, with $\alpha_i \geq \beta_i$ and $(\alpha_i, \beta_i) \neq (0, 0)$, for $1 \leq i \leq k$. Given positive integers $n$ and $t_i$, $i = 1, \ldots, k$, there exists an integer $H = H(n, t_1, \ldots, t_k) > 0$ such that given any intervals $I_1, \ldots, I_k$ and $J_1, \ldots, J_k$ in $G_n$ so that $J_i \in \triangle(I_i, t_i)$, we have*

$$\mu(T^{H(\alpha_i,\beta_i)} I_i \cap J_i) \geq \frac{1}{(16k)^t} \mu(J_i),$$

*where $t = \max\{t_i : 1 \leq i \leq k\}$.*

Proof. There exists a strictly increasing sequence $\{r_j\} \subset \mathbb{N}$ and an infinite sequence $\{s_j\} \subset \{c_n\}$ so that

$$s_j = ((j\alpha_1, j\beta_1), \ldots, (j\alpha_k, j\beta_k)),$$

and $G_{r_j+1}$ is an $s_j$-multistep cut of $G_{r_j}$. Also, note that $G_{r_j+1}$ contains $16k$ copies of $G_{r_j}$ and for all $1 \leq i \leq k$,

$$\begin{aligned}
T^{jh_{r_j}(\alpha_i,\beta_i)} G^{(0,0)}_{r_j,i} &= G^{(1,0)}_{r_j,i}, \\
T^{jh_{r_j}(\alpha_i,\beta_i)} G^{(1,0)}_{r_j,i} &= T^{(-1,0)} G_{r_j,i} G^{(2,0)}_{r_j,i}, \\
T^{jh_{r_j}(\alpha_i,\beta_i)} G^{(0,1)}_{r_j,i} &= T^{(-1,-1)} G^{(1,1)}_{r_j,i}
\end{aligned}$$

and $\mu(G^{(a,b)}_{r_j,i}) = \frac{1}{16k} \mu(G_{r_j})$ for $(a, b) \in \mathcal{SQ}(4)$ and $1 \leq i \leq k$.

Let $j$ be the smallest integer so that $r_j > n$. Set $H = \sum_{l=j}^{j+t-1} lh_{r_l}$. We now define $I_{l,i}$ recursively using the same idea as in Lemma 4.2 to obtain $I_{t,j}$ with $I_{t,j} \subset T^{H(\alpha_j,\beta_j)}I$, and $\mu(I_{t,j} \cap J) \geq \frac{1}{(16k)^t}\mu(J)$. ■

Lemma 4.8 can be generalized to the cases when $\alpha_i < 0$ or $\beta_i < 0$ by appropriately redefining the $t$-triangle in each case. Now the next theorem follows from Lemma 4.8 using the same argument as in Theorem 4.3.

THEOREM 4.9. *Let $T$ be the multistep $\mathbb{Z}^2$-action as defined above. Then $T$ is power weakly mixing.*

**5. Nonsingular type III $\mathbb{Z}^d$-actions.** In this section we construct ergodic nonsingular type III free actions. The type III$_\lambda$ examples, $0 < \lambda < 1$, can be seen as $\mathbb{Z}^2$ versions of the type III$_\lambda$ Chacon transformations of [JS], in the same way as the constructions in [PR] generalize to $\mathbb{Z}^2$ the classic (finite measure preserving) Chacon transformation. It is easy to see how to change these constructions to obtain type III$_1$ examples. However, for the type III$_0$ examples we use a modification of the staircase construction.

As explained in [PR], there are several choices for the arrangement of the spacers in a Chacon $\mathbb{Z}^2$-action. For the type III$_\lambda$ examples, $0 < \lambda \leq 1$, that we construct, the basis transformations are not isomorphic, and we obtain infinite ergodic index for $T^{(1,0)}$, while $T^{(0,1)}$ is not ergodic. The proof of Theorem 5.9 follows techniques from [AFS2], where the nonsingular Chacon transformations of [JS] are shown to be power weakly mixing (in the first version of the present paper the authors had only shown ergodicity of the basis transformations). One could modify our construction to a nonsingular multistep action as before to obtain power weak mixing for $T^{(1,0)}$ but we omit the details. We note that the nonsingular Chacon transformations of [JS] were shown to have trivial centralizer, while in our examples the centralizer contains an isomorphic copy of $\mathbb{Z}^2$ (we do not know if the containment is proper).

For the III$_0$ examples we go back to a modification of the original $\mathbb{Z}^2$-staircase of [AS] and so have to use an unbounded sequence $\{c_n\}$ of cuts, and hence only obtain weak mixing for the basis transformations; our method to show ergodicity of products does not seem to apply to an unbounded sequence of cuts, and in this case we only show that the basis transformations are weakly mixing.

**5.1.** *A nonsingular type* III$_\lambda$ *Chacon $\mathbb{Z}^2$-action.* We let $G_0 = \{[0,1)\}$, $h_0 = 1$, $0 < \lambda < 1$. Assume $G_n$ has been defined. $G_{n+1}$ is the grid of length $h_{n+1} = 3h_n + 1$ that contains 9 copies of $G_n$ so that for $(i,j) \in \mathcal{SQ}(3)$, $\mu(G_n^{(i,j)}) = \alpha_{ij}\mu(G_n)$ where $\alpha_{ij} = 1/(5+4\lambda)$ if $i+j$ is even and $\lambda/(5+4\lambda)$

if $i+j$ is odd. We arrange the copies of $G_n$ as follows:

$$\begin{array}{ll}
\mathrm{Loc}_{G_{n+1}}(G_n^{(0,0)}) = (0,0), & \mathrm{Loc}_{G_{n+1}}(G_n^{(1,0)}) = (h_n+1,0), \\
\mathrm{Loc}_{G_{n+1}}(G_n^{(2,0)}) = (2h_n+1,0), & \mathrm{Loc}_{G_{n+1}}(G_n^{(0,1)}) = (0,h_n+1), \\
\mathrm{Loc}_{G_{n+1}}(G_n^{(1,1)}) = (h_n+1,h_n), & \mathrm{Loc}_{G_{n+1}}(G_n^{(2,1)}) = (2h_n+1,h_n+1), \\
\mathrm{Loc}_{G_{n+1}}(G_n^{(0,2)}) = (0,2h_n+1), & \mathrm{Loc}_{G_{n+1}}(G_n^{(1,2)}) = (h_n+1,2h_n+1), \\
\mathrm{Loc}_{G_{n+1}}(G_n^{(2,2)}) = (2h_n+1,2h_n+1). &
\end{array}$$

The rest of the grid is filled up with spacers chosen so that $T^{(1,0)}$ and $T^{(0,1)}$ are measure preserving when they go from an interval in $G_n^{(i,j)}$ to a spacer in $G_{n+1}$, where $(i,j)\in\mathcal{SQ}(3)$. If the length of the spacer remains undefined at this stage, choose its length so that $T^{(1,1)}$ is measure preserving from an interval in $G_n^{(i,j)}$ to the spacer (this only happens for $(i,j)=(0,0)$). One checks that there are no conflicts. We leave it to the reader to verify that this defines an ergodic nonsingular $\mathbb{Z}^2$-action on a finite measure space. Finally, the measure is normalized so that $\mu(X)=1$; let $\gamma$ be such that

$$\mu([0,1))=\gamma.$$

Figure 3 shows a step in the construction of a Chacon type $\mathrm{III}_\lambda$ action. The relative sizes of the intervals are not shown in this figure.

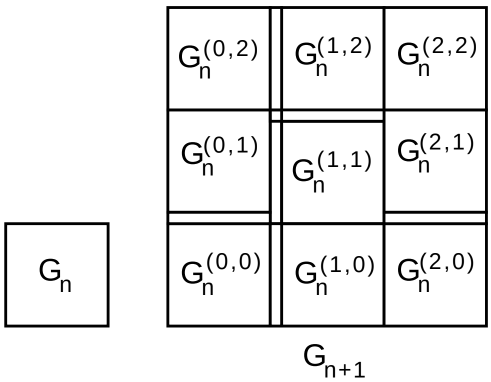


Fig. 3. A Chacon type $\mathrm{III}_\lambda$ $\mathbb{Z}^2$-action

PROPOSITION 5.1. *The nonsingular Chacon $\mathbb{Z}^2$-action is of type* $\mathrm{III}_\lambda$, $0<\lambda<1$.

Proof. Given $A$ with $\mu(A)>0$, choose $I$ in some grid $G_n$ such that

$$\frac{\mu(A\cap I)}{\mu(I)} > \left(1-\frac{1}{2}\left(\frac{\lambda}{5+4\lambda}\right)\right).$$

Let $I_n^{(i,j)} = I\cap G_n^{(i,j)}$ for $(i,j)\in\mathcal{SQ}(3)$. Then $\mu(I_n^{(i,j)}\cap A) > \frac{1}{2}\mu(I_n^{(i,j)})$ for all $(i,j)$. By construction, $T^{(h_n,0)}I_n^{(1,0)} = I_n^{(2,0)}$, so that $\mu(A\cap T^{(-h_n,0)}A)>0$.

Moreover, $\mu(I_n^{(2,0)}) = \lambda^{-1}\mu(I_n^{(1,0)})$, and since $T^{(h_n,0)}$ is an affine transformation from $I_n^{(1,0)}$ onto $I_n^{(2,0)}$, $\omega_{T^{(h_n,0)}}(x) = \lambda^{-1}$ for a.a. $x \in I_n^{(1,0)}$. Thus $\lambda \in r(T)$.

Now we prove that $r(T) \neq \mathbb{R}^+$. For a.a. $x \in X$, for any $g \in G \setminus \{e\}$, there exists a grid $G_n$ where $x$ and $T^g x$ reside in two different intervals. Call these two intervals $I_0$ and $I_1$, respectively. $T^g$ is an affine map from $I_0$ to $I_1$, so $\omega_{T^g}$ is a constant on $I_0$ equal to $\mu(I_1)/\mu(I_0)$.

Furthermore, $\mu(I_0)$ and $\mu(I_1)$ can be written as

$$\mu(I_0) = \left(\frac{1}{5+4\lambda}\right)^a \left(\frac{\lambda}{5+4\lambda}\right)^b \gamma = \lambda^b \left(\frac{1}{5+4\lambda}\right)^n \gamma,$$

$$\mu(I_1) = \left(\frac{1}{5+4\lambda}\right)^c \left(\frac{\lambda 1}{5+4\lambda}\right)^d \gamma = \lambda^d \left(\frac{1}{5+4\lambda}\right)^n \gamma,$$

for some positive integers $a$, $b$, $c$, and $d$ where $a+b=c+d=n$. Therefore $\mu(I_1)/\mu(I_0) = \lambda^{b-d}$. Since these ratios dictate the only possible values for $\omega_{T^g}$, $T$ is of type $\mathrm{III}_\lambda$. ■

For the transformation $T^{(1,0)}$, let

$$B_n^{(1,0)} = \{I \in G_n : \mathrm{Loc}_{G_n}(I) = (0,k),\ 0 \le k < h_n\}.$$

Similarly, let

$$B_n^{(0,1)} = \{I \in G_n : \mathrm{Loc}_{G_n}(I) = (k,0),\ 0 \le k < h_n\}.$$

PROPOSITION 5.2. *For all* $n \ge 1$ *and a.a.* $x \in B_n^{(1,0)}$, $\omega_{T^{(h_n-1,0)}}(x) = 1$, $\omega_{T^{(1,0)}}(x) = 1$, *and* $\omega_{T^{(0,1)}}(x) \in \{\lambda^{-1}, 1, \lambda\}$. *Also, for a.a.* $x \in B_n^{(0,1)}$, $\omega_{T^{(0,h_n-1)}}(x) = 1$, $\omega_{T^{(0,1)}}(x) \in \{\lambda^{-1}, 1, \lambda\}$ *and* $\omega_{T^{(1,0)}}(x) \in \{\lambda^{-1}, 1, \lambda\}$.

Proof. The statement follows by induction by verifying it for $G_1$ and then from the nature of the construction. ■

PROPOSITION 5.3. *For a.a.* $x \in X$, $\omega_{T^{(1,0)}}(x) \in \{\lambda^{-2}, \lambda^{-1}, 1, \lambda, \lambda^2\}$ *and* $\omega_{T^{(0,1)}}(x) \in \{\lambda^{-1}, 1, \lambda, \}$, *and* $\omega_{T^{(1,1)}}(x) \in \{\lambda^{-2}, \lambda^{-1}, 1, \lambda, \lambda^2\}$.

Proof. We show the $T^{(1,0)}$ case; the other cases are analogous. We show by induction on $n \ge 1$ that if $x \in G_n$ and $T^{(1,0)}x$ is defined in $G_n$ then $\omega_{T^{(1,0)}}(x) \in \{\lambda^{-2}, \lambda^{-1}, 1, \lambda, \lambda^2\}$. The case $n = 1$ is clear from the definition of $G_1$. Now assume the induction hypothesis for $n$. Let $x \in I$ for $I \in G_{n+1}$ with $T^{(1,0)}x$ defined in $G_{n+1}$. If for some $(i,j) \in \mathcal{SQ}(4)$, $x \in G_n^{(i,j)}$ and $T^{(1,0)}x \in G_n^{(i,j)}$ then the induction hypothesis completes the proof. Now assume that $x \in G_n^{(1,j)}$ and $T^{(1,0)}x \in G_n^{(2,j)}$, $j = 0,1,2$; the remaining cases are simpler or analogous. Let $y = T^{(-h_n+1,0)}x$ and $J = T^{(-h_n+1,0)}I$; so $y \in B_n^{(1,0)} \cap G_n^{(1,j)}$. By the placement of the grids, $T^{(h_n,\delta)}G_n^{(1,j)} = G_n^{(2,j)}$ (where $\delta = 0$ if $j = 0,2$ and $\delta = 1$ when $j = 1$). Also, $\mu(G_n^{(2,j)}) = \beta\mu(G_n^{(1,j)})$ where

$\beta \in \{\lambda^{-1}, \lambda\}$. It follows that $\omega_{T^{(h_n,\delta)}}(y) = \mu(T^{(h_n,\delta)}J)/\mu(J) \in \{\lambda^{-1}, \lambda\}$. Then

$$\omega_{T^{(h_n,\delta)}}(y) = \omega_{T^{(h_n-1,0)}}(y)\omega_{T^{(1,0)}}(x)\omega_{T^{(0,\delta)}}(T^{(1,0)}x).$$

By Proposition 5.2, $\omega_{T^{(h_n-1,0)}}(y) = 1$. Also, since $T^{(1,0)}x \in B_n^{(1,0)}$ and $\delta = 0$ or 1, by Proposition 5.2 again, $\omega_{T^{(0,\delta)}}(T^{(1,0)}x) \in \{\lambda^{-1}, 1, \lambda\}$. So $\omega_{T^{(1,0)}}(x) \in \{\lambda^{-2}, \lambda^{-1}, 1, \lambda, \lambda^2\}$ (worst case—in fact one can argue that $\lambda^{-2}$ does not occur). ■

The idea of (implicit) use of the cocycle relation in the following proof comes from [JS] and [AFS2].

LEMMA 5.4. *Let $T$ be the $\mathbb{Z}^2$-action on $(X, \mathfrak{B}, \mu)$ as defined above. For a.a. $x \in X$ and all $n \geq 0$, $\omega_{T^{(h_n,0)}}(x) \in \{\lambda^{-6}, \lambda^{-5}, \ldots, \lambda^6\}$ and $\omega_{T^{(0,h_n)}}(x) \in \{\lambda^{-4}, \lambda^{-3}, \ldots, \lambda^4\}$.*

Proof. First assume $x \in G_n$. Let $k \geq n$ be the smallest integer so that $T^{(h_n,0)}x$ is defined in $G_k$. Let $G_n^1$ be the $G_n$-copy in $G_k$, so that $x \in G_n^1$. There exists another $G_n$-copy, call it $G_n^2$, so that $T^{(h_n+i,j)}G_n^1 = G_n^2$ and $\mu(G_n^1) = \beta\mu(G_n^2)$ where $0 \leq i \leq 1$, $-1 \leq j \leq 1$, and $\beta \in \{\lambda^{-1}, 1, \lambda\}$; thus $\omega_{T^{(h_n+i,j)}}(x) = \beta^{-1}$. Using the cocycle relation for the Radon–Nikodym derivatives and Proposition 5.3, we get $\omega_{T^{(h_n,0)}}(x) \in \{\lambda^{-4}, \ldots, \lambda^4\}$. Finally, if $x \notin G_n$ we note that $T^{(i,j)}x \in G_n$ for some $(i,j) = (1,0), (1,1)$, or $(0,1)$. Another application of the cocycle relation gives the desired result. Finally, for the case of $\omega_{T^{(0,h_n)}}(x)$, we note that $T^{(0,h_n+j)}G_n^1 = G_n^2$, $j = 0, 1$. ■

The proof of the following corollary just uses the fact that $l$ is the sum of grid lengths.

COROLLARY 5.5. *For $l = \sum_{i=0}^{2t-1} h_{n+i}$, $n \geq 0$ and $t > 0$,*

$$\lambda^{12t} \leq \omega_{T^{(l,0)}}(x) \leq \lambda^{-12t} \quad \text{and} \quad \lambda^{8t} \leq \omega_{T^{(0,l)}}(x) \leq \lambda^{-8t}.$$

The next lemma shows that the basis transformation $T^{(1,0)}$ has the triangle property. The proof is as the proof of Lemma 4.2, only that in this case one must take into account that after each iteration the measure of the intervals is reduced in the worst case by $\lambda/(5+4\lambda)$.

LEMMA 5.6. *Let $T$ be the $\mathrm{III}_\lambda$ Chacon $\mathbb{Z}^2$-action as defined above. Let $n$ and $t$ be positive integers. Then there exists an integer $l$ given by*

$$l = \sum_{k=0}^{2t-1} h_{n+k}$$

*such that if $I, J \in G_n$ and $J \in \triangle(I,t)$ then*

$$\mu(T^{(l,0)}I \cap J) \geq \frac{\lambda^{2t}}{(5+4\lambda)^{2t}}\mu(J).$$

The lemma below follows directly from the construction of the action.

LEMMA 5.7. *Let $I$ and $J$ be two intervals in a grid $G_m$ and $n > m$. Let $I_v$ and $J_v$ be any two subintervals of $I$ and $J$, respectively, in the grid $G_n$, such that $I_v$ and $J_v$ are in the same copy of $G_m$ in $G_n$. If $\mu(I) = \lambda^k \mu(J)$ then $\mu(I_v) = \lambda^k \mu(J_v)$.*

The next proposition follows from Lemma 5.4 and the proof of Lemma 5.6.

PROPOSITION 5.8. *Let $T$ be the Chacon $\mathbb{Z}^2$-action given by the above construction. Then the transformations $T^{(1,0)}$ and $T^{(0,1)}$ are partially rigid under the sequence $r_n = h_n$ and $\eta \geq \lambda/(5 + 4\lambda)$.*

THEOREM 5.9. *Let $T$ be the $\mathrm{III}_\lambda$ Chacon $\mathbb{Z}^2$-action as defined above. The basis transformation $T^{(1,0)}$ has infinite ergodic index. Furthermore, $T^{(0,1)}$ is not ergodic but has infinite conservative index.*

Proof. The proof starts (with the same notation) as the proof of Theorem 4.3. Now choose $0 < \delta < \lambda^{2t}/(5 + 4\lambda)^{2t}$, $l = \sum_{k=0}^{2t-1} h_{m+k}$ as in Lemma 5.6. By Corollary 5.5,

$$\frac{d\mu_k \circ S^l}{d\mu_k} \leq \lambda^{-12kt} \quad \text{a.e.} \tag{2}$$

Let $\alpha_i$ be such that $\mu(I_i) = \alpha_i \mu(J_i)$ and $\alpha = \prod_{j=1}^k \alpha_i$. Then $\mu_k(I) = \alpha \mu_k(J)$. Let $\beta = \alpha \lambda^{-12kt}$. As in Theorem 4.3 there exist indices $U''$ and $V''$ and rectangles $I_u$ and $J_v$ so that for all $u \in U''$ and $v \in V''$, $I_u = I'_1 \times \ldots \times I'_k$ is $(1 - \delta^k/(3\beta))$-full of $A$ and $J_v = J'_1 \times \ldots \times J'_k$ is $(1 - \delta^k/3)$-full of $B$, $I'_1, \ldots, I'_k$ and $J'_1, \ldots, J'_k$ are in the same grid $G_n$ and for each $i$, $I'_i$ and $J'_i$ are in the same $G_m$-copy in $G_n$, it follows that $J'_i \in \triangle(I'_i, t)$. Also, if $I'' = \bigcup_{u \in U''} I_u$ and $J'' = \bigcup_{v \in V''} J_v$, then

$$\mu_k(I'' \triangle I) < \tfrac{1}{2}\mu_k(I) \quad \text{and} \quad \mu_k(J'' \triangle J) < \tfrac{1}{2}\mu_k(J).$$

Since these unions cover more than $1/2$ of $I$ and $J$ respectively, by Lemma 5.7 we have

$$\mu_k\Big(\bigcup_{u \in U''} I_u \triangle I\Big) < \tfrac{1}{2}\mu_k(I) \quad \text{and} \quad \mu_k\Big(\bigcup_{v \in V''} J_v \triangle J\Big) < \tfrac{1}{2}\mu_k(J).$$

Thus, there must exist at least one index $w \in U'' \cap V''$. Since $l$ is defined as in Lemma 5.6,

$$\mu_k(S^l I_w \cap J_w) > \delta^k \mu(J_w).$$

Also, using (3), we get

$$\mu_k(S^l(I_w \setminus A)) = \int_{I_w \setminus A} \frac{d\mu_k \circ S^l}{d\mu_k}\, d\mu_k \leq \lambda^{-12kt} \mu_k(I_w \setminus A)$$
$$\leq \lambda^{-12kt} \frac{\delta^k}{3\beta} \mu_k(I_w) = \frac{\delta^k}{3} \mu_k(J_w).$$

Therefore,

$$\begin{aligned}\mu_k(S^l A\cap B) &\geq \mu_k(S^l I_w\cap J_w)-\mu_k((S^l I_w\cap J_w)\setminus(S^l A\cap B))\\&\geq \mu_k(S^l I_w\cap J_w)-\mu_k(S^l I_w\setminus A)-(S^l J_w\setminus B)\\&\geq \delta^k\mu_k(J_w)-\frac{\delta^k}{3}\mu_k(J_w)-\frac{\delta^k}{3}\mu_k(J_w)>0. \ \blacksquare\end{aligned}$$

**5.2.** *Type* $\mathrm{III}_1$. This is a direct consequence of the previous example. In the construction of the grids, for even $n$ use $\lambda_1$ and for odd $n$ use $\lambda_2$ such that $\log\lambda_1$ and $\log\lambda_2$ are irrationally related.

**5.3.** *Type* $\mathrm{III}_0$. The process of defining the $\mathrm{III}_0$ staircase $\mathbb{Z}^2$-actions is similar to the construction of the infinite measure preserving staircase action; in this case the number of cuts $c_n$ is unbounded. Given a positive integer c, a grid $H$ is defined to be a *staircase c-cut* of a grid $G$, of length $g$, if $G\subset H$ and $H$ is a grid of least size that contains $(c+1)^2$ copies of $G$ located at

$$(ig+i(i-1)/2+ij,\, jg+j(j-1)/2+ij)$$

for $(i,j)\in\mathcal{SQ}(h)$. The length of $H$ is $h=(c+1)g+c(c-1)/2+c^2$.

The cutting sequence $c_n$ is defined to be

$$c_n=\begin{cases}2^{2^n} & \text{for } n \text{ even},\\ c & \text{for } n \text{ odd}.\end{cases}$$

Given a grid $G_n$, $G_{n+1}$ is a staircase $c_n$-cut of $G_n$. For odd $n$,

$$\mu(G_n^{(i,j)})=\frac{1}{(c+1)^2}\mu(G_n)\qquad\text{for } (i,j)\in\mathcal{SQ}(c+1).$$

For even $n$, $\mu(G_n^{(0,0)})=\frac{1}{2}\mu(G_n)$, and

$$\mu(G_n^{(i,j)})=\frac{1}{2}\frac{1}{(c_n+1)^2-1}\mu(G_n)\qquad\text{for } (i,j)\in\mathcal{SQ}(c_n+1)\setminus\{(0,0)\}.$$

This creates one "thick" subgrid, the subgrid of $G_n$ with the large pieces of intervals, and many "thin" subgrids. Choose the length of the spacers so that the transformations $T^{(1,0)},T^{(0,1)}$ are measure preserving when they go from an interval of $G_{n+1}$ into a spacer of $G_{n+1}$ (one checks that there are no conflicts). This defines a nonsingular ergodic $\mathbb{Z}^2$-action.

LEMMA 5.10. *$T$ is of type* $\mathrm{III}_0$.

Proof. We first show that for any $\varepsilon>0$ and $\mu(A)>0$ there exists $l\in\mathbb{Z}^2$ such that $\mu(T^l A\cap A\cap\{x:\omega_l(x)<\varepsilon\})>0$. Now there exists an interval $I$ in some grid $G_p$ with $p$ even such that $\mu(I\cap A)>\frac{3}{4}\mu(I)$ and $1/((2^{2^p}+1)^2-1)<\varepsilon$. Let $I_1=I\cap G_{p+1}^{(0,0)}$. It follows that $\mu(I_1)=\frac{1}{2}\mu(I)$ and

$\mu(I_1 \cap A) > \frac{1}{2}\mu(I_1)$. There must exist another copy of $I$, call it $I_2$, so that

$$\mu(I_2) = \frac{1}{2} \cdot \frac{1}{(2^{2^p}+1)^2 - 1}\mu(I) \quad \text{and} \quad \mu(I_2 \cap A) > \frac{1}{2}\mu(I_2).$$

Let $l = \mathrm{Loc}(I_2) - \mathrm{Loc}(I_1)$. Then $T^l I_1 = I_2$ and $\mu(T^l A \cap A) > 0$. Since $\omega_l(x)$ is constant over intervals,

$$\omega_l(x) = \frac{\mu(I_2)}{\mu(I_1)} = \frac{1}{(2^{2^p}+1)^2 - 1} < \varepsilon.$$

Thus $0 \in r(T)$.

Now assume that there exists $q \in r(T)$ with $q \in (0,1)$. Let $\varepsilon > 0$ be such that $q - 2\varepsilon > 0$ and $q + \varepsilon < 1$. For any $A$ of positive measure, there exists $l \in \mathbb{Z}^2$ so that

$$\mu(T^l A \cap A \cap \{x : \omega_l(x) \in N_\varepsilon(q)\}) > 0.$$

Consider an interval $I \in G_p$ where $p$ is even and $1/((2^{2^p}+1)^2 - 1) < \varepsilon$. Let $I_1$ and $I_2$ be subintervals of $I \in G_{p+i}$ for some $i > 0$. We will show that $\mu(I_2)/\mu(I_1) \notin N_\varepsilon(q)$.

Let $l \in \mathbb{Z}^2$ so that $T^l(I_1) = I_2$. We may assume that $\mu(I_1) \geq \mu(I_2)$. So $\omega_l(x) \in [0,1]$. Let $J = \{m : I_1$ was in a thin cut of $G_m\}$ and $K = \{m : I_2$ was in a thin cut of $G_m\}$ where $p \leq m < p+i$. Then the lengths of $I_1$ and $I_2$ are given by

$$\mu(I_1) = \left(\frac{1}{2}\right)^{\lceil i/2 \rceil} \cdot \prod_{j \in J} \frac{1}{((2^{2^j}+1)^2 - 1)} \cdot \left(\frac{1}{4^2}\right)^{\lfloor i/2 \rfloor}, \tag{3}$$

$$\mu(I_2) = \left(\frac{1}{2}\right)^{\lceil i/2 \rceil} \cdot \prod_{k \in K} \frac{1}{((2^{2^k}+1)^2 - 1)} \cdot \left(\frac{1}{4^2}\right)^{\lfloor i/2 \rfloor}. \tag{4}$$

Since $\omega_l(x) = \mu(I_2)/\mu(I_1)$, from (3) and (4),

$$\omega_l(x) = \frac{\prod_{j \in J}((2^{2^j}+1)^2 - 1)}{\prod_{k \in K}((2^{2^k}+1)^2 - 1)}.$$

If $J = K$, then $\omega_l(x) = 1 \notin N_\varepsilon(q)$. Thus, there exists an even $n$ such that either $I_1 \in G_n^{(0,0)}$ and $I_2 \notin G_n^{(0,0)}$ or $I_2 \in G_n^{(0,0)}$ and $I_1 \notin G_n^{(0,0)}$. Let $N$ be the largest such $n$. We may assume without loss of generality that $N \in K$. This ensures that $\mu(I_2) < \mu(I_1)$. Note that if $j > N$ and $j \in J$, then $j \in K$ by the construction of $N$. In the calculation of $\omega_l(x)$ these terms will cancel. Thus, let $J' = J \setminus \{j \in J : j > N\}$. Using the property that $(2^{2^N}+1)(2^{2^N}-1) = \prod_{a=0}^{N}(2^{2^a}+1)$, one can verify $\omega_l(x) < \varepsilon$. Hence assuming $q \in f(T)$ and $q \in (0,1)$ results in a contradiction, and so $T$ is type $\mathrm{III}_0$. ∎

THEOREM 5.11. *For the type* $\mathrm{III}_0$ $\mathbb{Z}^2$*-action defined above, the transformations* $T^{(1,0)}$ *and* $T^{(0,1)}$ *are weakly mixing.*

Proof. The proof of ergodicity of the basis transformations is omitted as it follows the idea in the proof of Theorem 5.9; however, the proof is simpler and does not need the estimate analogous to Corollary 5.5. It remains to show that the only $L^\infty$ eigenvalue of $T^{(1,0)}$ is 1. Let $f \in L^\infty$ be such that $f(T^{(1,0)}(x)) = \lambda f(x)$. For all $\varepsilon > 0$, there is a set $A$ of positive measure such that

$$\left|\frac{f(x)}{f(y)} - 1\right| < \frac{\varepsilon}{3}$$

for all $x, y \in A$. Choose an interval $I$ in some odd grid $G_n$ such that

$$\mu(A \cap I) > (1 - 1/3^t)\mu(A).$$

Cut and tile $G_n$. Each subgrid of $G_n$ in $G_{n+1}$ contains a piece of $I$ that is more than 2/3 full of $A$. Consider $G_n^{(0,0)}$, $G_n^{(1,0)}$, and $G_n^{(2,0)}$. Note that $G_n^{(1,0)}$ is not shifted relative to $G_n^{(0,0)}$ and $G_n^{(2,0)}$ is shifted only 1 unit relative to $G_n^{(1,0)}$ in the direction in which $T^{(1,0)}$ maps.

Thus, there must exist some $x \in A$ such that

$$T^{(h_n,0)}(x) \in A \quad \text{and} \quad T^{(2h_n+1,0)}(x) \in A.$$

By definition of $A$, $|\lambda^{h_n} - 1| < \varepsilon/3$, which implies

$$|\lambda^{2h_n} - 1| < |\lambda^{2h_n} - \lambda^{h_n}| + |\lambda^{h_n} - 1| < 2\varepsilon/3.$$

Again using $A$, we get $|\lambda^{2h_n+1} - 1| < \varepsilon/3$. Combining these two inequalities gives

$$|\lambda - 1| < |\lambda^{2h_n+1} - \lambda^{2h_n}| \le |\lambda^{2h_n+1} - 1| + |\lambda^{2h_n} - 1| < \varepsilon.$$

Thus, $\lambda = 1$. ∎

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