C O L L O Q U I U M   M A T H E M A T I C U M

VOL. 82 1999 NO. 2

IMMERSIONS OF MODULE VARIETIES

BY

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Abstract. We show that a homomorphism of algebras is a categorical epimorphism if and only if all induced morphisms of the associated module varieties are immersions. This enables us to classify all minimal singularities in the subvarieties of modules from homogeneous standard tubes.

1. Introduction and main results. Throughout the paper k will be a fixed algebraically closed field and all the algebras considered are associative k-algebras with identities. For any algebra A and $d \geq 1$ we denote by $\mathbb{M}_d(A)$ the algebra of square matrices of degree d with coefficients in A.

Let A be a finitely generated algebra, that is, A is isomorphic to the quotient of a finitely generated free (non-commutative) algebra $k\langle X_1, \ldots, X_n\rangle$ by a two-sided ideal I. For any natural number d we define an affine variety $\mathrm{mod}_A(d)$ of (left) A-module structures on k^d as follows:

$$\mathrm{mod}_A(d) = \{m = (m_1, \ldots, m_n) \in (\mathbb{M}_d(k))^n : \varrho(m_1, \ldots, m_n) = 0,\ \varrho \in I\}.$$

The general linear group $\mathrm{Gl}_d(k)$ acts on $\mathrm{mod}_A(d)$ by conjugation and the orbits of this action correspond to the isomorphism classes of d-dimensional left A-modules. The variety $\mathrm{mod}_A(d)$ depends on the choice of the representation $k\langle X_1, \ldots, X_n\rangle/I$ of A only up to a $\mathrm{Gl}_d(k)$-equivariant isomorphism.

Let $\varphi : A \to B$ be a homomorphism of finitely generated algebras. Then there are induced regular $\mathrm{Gl}_d(k)$-equivariant morphisms of affine varieties $\varphi^{(d)} : \mathrm{mod}_B(d) \to \mathrm{mod}_A(d)$ for all $d \geq 1$. An interesting problem is to find homomorphisms of algebras inducing regular morphisms of affine varieties with nice properties.

Examples of nice regular morphisms are immersions. Recall some definitions from [9]. By a *variety* we mean a quasi-affine variety, that is, a locally closed subset of an affine variety. We say that a regular morphism $\mu : X \to Y$ of varieties is an *immersion* (respectively, *closed immersion*) if μ gives an isomorphism of X with a locally closed subset of Y (respectively, a closed subset of Y).

1991 *Mathematics Subject Classification*: 14B05, 14E25, 14L30, 16B50, 16G10, 16G70.

On the other hand, well-known examples of homomorphisms of algebras are categorical epimorphisms. A homomorphism $\varphi : A \to B$ of rings with identity is called an *epimorphism in the category of rings*, or briefly a *categorical epimorphism*, if for given homomorphisms of rings $\beta_1, \beta_2 : B \to C$ such that $\beta_1\varphi = \beta_2\varphi$, we have $\beta_1 = \beta_2$ (see [14]). Our first main result explains connections between categorical epimorphisms of algebras and immersions of module varieties.

THEOREM 1. *Let $\varphi : A \to B$ be a homomorphism of algebras. Assume that the algebra A is finitely generated and the algebra B is finite-dimensional. Then the following conditions are equivalent*:

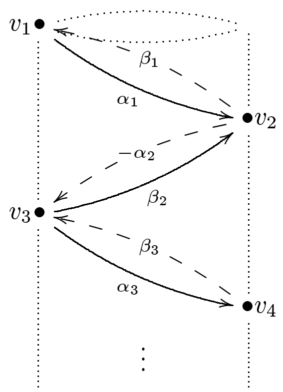
(i) *φ is a categorical epimorphism.*
(ii) *$\varphi^{(d)} : \mathrm{mod}_B(d) \to \mathrm{mod}_A(d)$ is an injective morphism for all $d \geq 1$.*
(iii) *$\varphi^{(d)}$ is an immersion for all $d \geq 1$.*

This theorem is useful for problems concerning the geometry of module varieties like the types of singularities in closures of orbits. Following Hesselink (see (1.7) in [10], (8.1) in [1] and (2.1) in [12]) we call two pointed varieties $(\mathcal{X}, x_0)$ and $(\mathcal{Y}, y_0)$ *smoothly equivalent* if there are smooth morphisms $f : \mathcal{Z} \to \mathcal{X}$, $g : \mathcal{Z} \to \mathcal{Y}$ and a point $z_0 \in \mathcal{Z}$ with $f(z_0) = x_0$, $g(z_0) = y_0$. This is an equivalence relation and equivalence classes will be denoted by $\mathrm{Sing}(\mathcal{X}, x_0)$. If $\mathrm{Sing}(\mathcal{X}, x_0) = \mathrm{Sing}(\mathcal{Y}, y_0)$ then the variety $\mathcal{X}$ is regular or normal at x_0 if and only if the same is true for the variety $\mathcal{Y}$ at y_0 (see Section 17 in [8] for more information about smooth morphisms).

In the case of module varieties we wish to investigate the types of singularities $\mathrm{Sing}(\overline{\mathrm{Gl}_d(k)m}, n)$ of closures of orbits, where $n \in \overline{\mathrm{Gl}_d(k)m}$. Especially, it is of interest to classify the types of *minimal singularities*, that is, $\mathrm{Sing}(\overline{\mathrm{Gl}_d(k)m}, n)$, where $n \in \overline{\mathrm{Gl}_d(k)m} \setminus \mathrm{Gl}_d(k)m$ and there is no point $p \in \mathrm{mod}_A(d)$ such that $\overline{\mathrm{Gl}_d(k)n} \subsetneq \overline{\mathrm{Gl}_d(k)p} \subsetneq \overline{\mathrm{Gl}_d(k)m}$. In case $A = k[X]/(X^n)$ all types of minimal singularities in $\mathrm{mod}_A(d)$ are known and we list them in Section 4.3. For some results concerning minimal singularities we refer to [15], [1], [12] and [5].

For any finite-dimensional algebra A one defines the *Auslander–Reiten quiver* Γ_A of A, that is, a translation quiver whose vertices are the isomorphism classes of indecomposable finite-dimensional left A-modules and the arrows correspond to irreducible maps (see [3] for more details). It will cause no confusion if we identify the isomorphism classes of modules with their representatives. A connected component Γ of the Auslander–Reiten quiver Γ_A of A is said to be a *homogeneous tube* if it is isomorphic to the translation quiver (V_0, V_1, τ) with the set of vertices $V_0 = \{v_i : i \geq 1\}$, the set of arrows $V_1 = \{\alpha_i : v_i \to v_{i+1}, \beta_i : v_{i+1} \to v_i : i \geq 1\}$ and the translation $\tau : V_0 \to V_0$ given by $\tau(v_i) = v_i$ for all $i \geq 1$ (cf. Section 3.1 in [13]).

An A-module in Γ corresponding to v_1 is said to be *quasi-simple.*



Homogeneous tubes are important components of the Auslander–Reiten quivers of tame algebras. A finite-dimensional algebra A is called *tame* if for any $d \geq 1$ there is a finite number of A-$k[X]$-bimodules $M_1, \ldots, M_{n(d)}$ which are free of rank d as right $k[X]$-modules and such that every indecomposable left A-module of dimension d is isomorphic to $M_i \otimes_{k[X]} k[X]/(X-\lambda)$ for some $1 \leq i \leq n(d)$ and $\lambda \in k$. Crawley-Boevey proved in [6] that if the algebra A is tame, then all but finitely many isomorphism classes of indecomposable left A-modules of any dimension d lie in homogeneous tubes of Γ_A. Furthermore, if A is a tame hereditary algebra, then all connected components of Γ_A are standard and all but finitely many of them are homogeneous tubes. A connected component Γ of the Auslander–Reiten quiver Γ_A of A is called *standard* if the category of A-modules from Γ is equivalent to the mesh category of Γ (see [13] for details).

Let $\mathrm{mod}_\Gamma(d)$ denote the sum of orbits in $\mathrm{mod}_A(d)$ corresponding to the modules from $\mathrm{add}(\Gamma)$, for any component Γ of Γ_A. Our second main result enables us to classify all minimal singularities in $\mathrm{mod}_\Gamma(d)$, where Γ is a standard homogeneous tube.

THEOREM 2. *Let Γ be a standard homogeneous tube in the Auslander–Reiten quiver of a finite-dimensional algebra A. Let $d = d'e$, where $d' \geq 1$, $e = \dim_k T_1$ and T_1 is a quasi simple module in Γ. Then*:

(i) $\mathrm{mod}_\Gamma(d)$ *is a locally closed subset of* $\mathrm{mod}_A(d)$.

(ii) *The types of singularities* (*respectively*, *minimal singularities*) *in the varieties* $\mathrm{mod}_\Gamma(d)$ *and* $\mathrm{mod}_{k[X]/(X^r)}(d')$ *coincide provided* $r \geq d'$.

Observe that $\mathrm{mod}_\Gamma(d)$ is an empty set if d is not divisible by $e = \dim_k T_1$.

The rest of the paper is organized as follows. In Section 2 we consider algebraic conditions for ring homomorphisms to be categorical epimorphisms. Section 3 is devoted to the proof of Theorem 1. In Section 4 we show some applications of Theorem 1, in particular, we prove Theorem 2.

For background on the representation theory of algebras we refer to [3], [13] and for algebraic geometry to [9].

The author wishes to express his thanks to D. Simson for helpful comments concerning categorical epimorphisms.

2. Categorical epimorphisms. For any ring R (with an identity) we denote by $\operatorname{Mod} R$ the category of left R-modules and by $\operatorname{Hom}_R(-,-)$ the functor of R-homomorphisms of left R-modules. Furthermore, any ring homomorphism $\varphi : A \to B$ induces a faithful functor $\varphi^* : \operatorname{Mod} B \to \operatorname{Mod} A$ and we may consider any left B-module as a left A-module. In that way, $\operatorname{Hom}_B(X,Y)$ becomes a subset of $\operatorname{Hom}_A(X,Y)$ for any $X,Y \in \operatorname{Mod} B$. Lemma 3 collects some equivalent conditions for epimorphisms in the category of rings (see Section 1 in [11]).

LEMMA 3. *Let $\varphi : A \to B$ be a homomorphism of unitary rings. Then the following conditions are equivalent*:

(i) *φ is a categorical epimorphism.*

(ii) $\operatorname{Hom}_A(X,Y) = \operatorname{Hom}_B(X,Y)$ *for all* $X,Y \in \operatorname{Mod} B$, *that is*, *the functor* $\varphi^* : \operatorname{Mod} B \to \operatorname{Mod} A$ *induced by φ is full.*

(iii) *The canonical surjective homomorphism of B-bimodules $B \otimes_A B \to B$, $b \otimes b' \mapsto bb'$, is an isomorphism.*

We now consider the category of algebras (over k). For an algebra A we have the canonical functor $D = \operatorname{Hom}_k(-,k) : \operatorname{Mod} A \to \operatorname{Mod} A^{\mathrm{op}}$, where A^{op} denotes the opposite algebra of A, or equivalently, $\operatorname{Mod} A^{\mathrm{op}}$ denotes the category of right A-modules. Since any algebra B is a left and a right B-module, the k-space $\operatorname{Hom}_B(B, DB)$ is defined. Furthermore, for any homomorphism $\varphi : A \to B$ we may treat $\operatorname{Hom}_A(B, DB)$ as a k-space containing the subspace $\operatorname{Hom}_B(B, DB)$. We shall need the following fact.

LEMMA 4. *Let $\varphi : A \to B$ be a homomorphism of algebras. Then φ is a categorical epimorphism if and only if* $\operatorname{Hom}_A(B, DB) = \operatorname{Hom}_B(B, DB)$.

Proof. The homomorphism $\eta : B \otimes_A B \to B$, $b \otimes b' \mapsto bb'$, is bijective if and only if so is the homomorphism $D(\eta) : DB \to D(B \otimes_A B)$. Since the functor $B \otimes_A (-)$ is left adjoint to the functor $\operatorname{Hom}_k(B_A, -)$, we have the following isomorphisms:

$$D(B \otimes_A B) = \operatorname{Hom}_k(B \otimes_A B, k) \simeq \operatorname{Hom}_A(B, \operatorname{Hom}_k(B,k)) = \operatorname{Hom}_A(B, DB),$$

and similarly,

$$DB \simeq D(B \otimes_B B) \simeq \operatorname{Hom}_B(B, DB).$$

Observe that, applying the above isomorphisms, $D(\eta)$ corresponds to the inclusion $\operatorname{Hom}_B(B, DB) \subseteq \operatorname{Hom}_A(B, DB)$. Hence, η is an isomorphism of B-bimodules if and only if $\operatorname{Hom}_A(B, DB) = \operatorname{Hom}_B(B, DB)$. Lemma 3 now completes the proof. ∎

3. Proof of Theorem 1. Throughout this section we fix a homomorphism $\varphi : A \to B$, where A is a finitely generated algebra and B is a finite-dimensional algebra. The implication (iii)$\Rightarrow$(ii) is obvious.

(ii)$\Rightarrow$(i). Let $d = \dim_k(B \oplus DB) < \infty$. Take a point $x \in \mathrm{mod}_B(d)$ such that its orbit $\mathrm{Gl}_d(k)x$ corresponds to the isomorphism class of the left B-module $B \oplus DB$. We set $y = \varphi^{(d)}(x)$, where $\varphi^{(d)} : \mathrm{mod}_B(d) \to \mathrm{mod}_A(d)$ is the induced regular morphism. Then the orbit $\mathrm{Gl}_d(k)y$ corresponds to the isomorphism class of the left A-module $B \oplus DB$. Since the morphism $\varphi^{(d)}$ is injective we get a bijection $\mathrm{Gl}_d(k)x \to \mathrm{Gl}_d(k)y$, and consequently, equality of the isotropy groups: $\mathrm{Gl}_d(k)_x = \mathrm{Gl}_d(k)_y$. Observe that the affine algebraic group $\mathrm{Gl}_d(k)_x$ is isomorphic to the group $\mathrm{Aut}_B(B \oplus DB)$ of B-module automorphisms of $B \oplus DB$. Furthermore, $\mathrm{Aut}_B(B \oplus DB)$ is a non-empty open subset of the space $\mathrm{End}_B(B \oplus DB)$ of B-endomorphisms of $B \oplus DB$. This implies that $\dim \mathrm{Aut}_B(B \oplus DB) = \dim_k \mathrm{End}_B(B \oplus DB)$. Hence, $\dim \mathrm{Gl}_d(k)_x = \dim_k \mathrm{End}_B(B \oplus DB)$, and similarly, $\dim \mathrm{Gl}_d(k)_y = \dim_k \mathrm{End}_A(B \oplus DB)$. Consequently, $\mathrm{End}_B(B \oplus DB) = \mathrm{End}_A(B \oplus DB)$. Since $\mathrm{Hom}_B(X, Y) \subseteq \mathrm{Hom}_A(X, Y)$ for any B-modules X, Y and

$$\begin{aligned}
&\mathrm{Hom}_B(B, B) \oplus \mathrm{Hom}_B(B, DB) \oplus \mathrm{Hom}_B(DB, B) \oplus \mathrm{Hom}_B(DB, DB) \\
&\quad = \mathrm{End}_B(B \oplus DB) = \mathrm{End}_A(B \oplus DB) \\
&\quad = \mathrm{Hom}_A(B, B) \oplus \mathrm{Hom}_A(B, DB) \oplus \mathrm{Hom}_A(DB, B) \oplus \mathrm{Hom}_A(DB, DB),
\end{aligned}$$

we get $\mathrm{Hom}_B(B, DB) = \mathrm{Hom}_A(B, DB)$. Hence φ is a categorical epimorphism by Lemma 4.

(i)$\Rightarrow$(iii). Let $d \geq 1$. We divide the proof into several steps.

STEP (a). *If the homomorphism $\varphi : A \to B$ is surjective, then the induced regular morphism $\varphi^{(d)} : \mathrm{mod}_B(d) \to \mathrm{mod}_A(d)$ is a closed immersion.*

Proof. The algebra A is finitely generated, hence $A \simeq k\langle X_1, \ldots, X_n\rangle / I$. Observe that $B \simeq k\langle X_1, \ldots, X_n\rangle / J$ for some two-sided ideal J containing I. By definition, the affine varieties $\mathrm{mod}_B(d)$ and $\mathrm{mod}_A(d)$ are the sets of points in $(\mathbb{M}_d(k))^n$ satisfying polynomial equations induced by J and I, respectively. Since $J \supseteq I$, $\mathrm{mod}_B(d)$ becomes a closed subset of $\mathrm{mod}_A(d)$, and $\varphi^{(d)}$ is an inclusion. ∎

STEP (b). *We may assume that A is a subalgebra of B and φ is an inclusion.*

Proof. By our assumptions and Lemma 3, the canonical homomorphism $B \otimes_A B \to B$ is injective. Let $\overline{A} = A/\ker \varphi$. Then the homomorphism $\varphi : A \to B$ factorizes through $\overline{A}$ and we get the induced regular morphisms of affine varieties:

$$\mathrm{mod}_B(d) \to \mathrm{mod}_{\overline{A}}(d) \to \mathrm{mod}_A(d).$$

By (a), the latter is a closed immersion. Since the B-bimodules $B \otimes_A B$ and $B \otimes_{\overline{A}} B$ are isomorphic to each other, we may assume that $A = \overline{A}$, $A \subseteq B$ and $\varphi : A \to B$ is an inclusion. ■

We choose a linear basis $1 = b_0, b_1, \ldots, b_s$ of B, where $s = \dim_k B - 1$, such that $1 = b_0, b_1, \ldots, b_t$ form a basis of A and $t = \dim_k A - 1$. We also choose the following representations of the algebras A and B:

$$A \simeq k\langle X_1, \ldots, X_s\rangle/I \quad \text{and} \quad B \simeq k\langle X_1, \ldots, X_t\rangle/J$$

for some two-sided ideals I and J, where the generator X_i corresponds to b_i for all $1 \le i \le s$. Then $\varphi^{(d)}(x_1, \ldots, x_s) = (x_1, \ldots, x_t)$ for any point $x = (x_1, \ldots, x_s) \in \mathrm{mod}_B(d)$.

Let $\mathbb{M}_{\alpha\times\beta}(C)$ denote the space of all matrices with α rows, β columns and coefficients in an algebra C.

STEP (c). *There are matrices $Y' \in \mathbb{M}_{\alpha\times 1}(A)$, $Y'' \in \mathbb{M}_{\alpha\times s}(A)$, $Z \in \mathbb{M}_{s\times\alpha}(B)$, for some $\alpha \ge 1$, such that*

$$Z \cdot Y'' = 1_s, \qquad Y'' \cdot [b_1, \ldots, b_s]^{\mathrm{T}} = Y', \tag{3.1}$$

$$Y' = [b_1, \ldots, b_t|*]^{\mathrm{T}}, \qquad Y'' = \begin{bmatrix} 1_t & 0 \\ * & * \end{bmatrix}, \qquad Z = \begin{bmatrix} 1_t & 0 \\ 0 & * \end{bmatrix} \tag{3.2}$$

where M^{T} means the matrix transpose to M.

Proof. We quote Lemma 6.4 in [7] formulated for commutative rings. This useful lemma remains true in the non-commutative case.

LEMMA 5. *Let M be a right A-module and N be a left A-module generated by a family of elements $\{n_l\}_{l\in L}$. Then an element $\sum_{l\in L} m_l \otimes n_l \in M \otimes_A N$ equals zero if and only if there exist elements $\{m'_j\}_{j\in J}$ of M and elements $\{a_{j,l} \in R : j \in J,\ l \in L\}$ such that*

$$\sum_{j\in J} m'_j a_{j,l} = m_l \quad \text{for all } l \in L \quad \text{and} \quad \sum_{l\in L} a_{j,l} n_l = 0 \quad \text{for all } j \in J.$$

Since the canonical homomorphism $B \otimes_A B \to B$ is injective, it follows that $b_i \otimes 1 = 1 \otimes b_i$ for all $1 \le i \le s$. Applying Lemma 5 to the right A-module B, the left A-module B generated by elements $\{1, b_1, \ldots, b_s\}$, and to the elements $(-b_i) \otimes 1 + 1 \otimes b_i = 0 \in B \otimes_A B$, $1 \le i \le s$, we get, for all $1 \le i \le s$, a matrix Y_i with $s+1$ columns and coefficients in A and a matrix Z_i with one row and coefficients in B such that

$$Z_i \cdot Y_i = [-b_i, 0, \ldots, 0, 1, 0, \ldots, 0] \quad \text{and} \quad Y_i \cdot [1, b_1, \ldots, b_s]^{\mathrm{T}} = 0.$$

We may assume that $Y_i = [-b_i, 0, \ldots, 0, 1, 0, \ldots, 0]$ and $Z_i = [1]$ for all

$1 \le i \le t$. We collect together the matrices Y_i and Z_i as follows:

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \dots \\ Y_s \end{bmatrix} \quad \text{and} \quad Z = \begin{bmatrix} Z_1 & 0 & \dots & 0 \\ 0 & Z_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & Z_s \end{bmatrix}.$$

Then $Y \in \mathbb{M}_{\alpha\times(s+1)}(A)$ and $Z \in \mathbb{M}_{s\times\alpha}(B)$ for some $\alpha \ge 1$. Moreover,

$$Z \cdot Y = \begin{bmatrix} -b_1 & 1 & 0 & \dots & 0 \\ -b_2 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -b_s & 0 & 0 & \dots & 1 \end{bmatrix} \quad \text{and} \quad Y \cdot \begin{bmatrix} 1 \\ b_1 \\ \dots \\ b_s \end{bmatrix} = 0.$$

Let $Y = [-Y', Y'']$, where $Y' \in \mathbb{M}_{\alpha\times 1}(A)$ and $Y'' \in \mathbb{M}_{\alpha\times s}(A)$. Then we obtain the formulas (3.1). Since $Y_i = [-b_i, 0, \dots, 0, 1, 0, \dots, 0]$ and $Z_i = [1]$ for all $1 \le i \le t$, the matrices Y', Y'' and Z have the form (3.2). ■

Any point m of $\operatorname{mod}_A(d)$ determines a homomorphism of algebras $m : A \to \mathbb{M}_d(k)$, for simplicity of notation denoted by the same letter. Furthermore, for any $u, v \ge 1$, we may extend m to a map $\widehat{m} = \widehat{m}(u, v) : \mathbb{M}_{u\times v}(A) \to \mathbb{M}_{ud\times vd}(k)$ such that

$$\widehat{m}\left(\begin{bmatrix} a_{1,1} & \dots & a_{1,v} \\ \dots & \dots & \dots \\ a_{u,1} & \dots & a_{u,v} \end{bmatrix}\right) = \begin{bmatrix} m(a_{1,1}) & \dots & m(a_{1,v}) \\ \dots & \dots & \dots \\ m(a_{u,1}) & \dots & m(a_{u,v}) \end{bmatrix}.$$

STEP (d). *The set*

$$\mathcal{C} = \{m \in \operatorname{mod}_A(d) : \operatorname{rank}(\widehat{m}(Y'')) = \operatorname{rank}[\widehat{m}(Y''), \widehat{m}(Y')] = sd\}$$

is locally closed in $\operatorname{mod}_A(d)$ *and* $\operatorname{im}\varphi^{(d)} \subseteq \mathcal{C}$.

Proof. Indeed, the condition $\operatorname{rank}(\widehat{m}(Y'')) = \operatorname{rank}[\widehat{m}(Y''), \widehat{m}(Y')] = sd$ is equivalent to the vanishing and non-vanishing of some minors of the matrix $[\widehat{m}(Y''), \widehat{m}(Y')]$. This defines a locally closed subset of $\operatorname{mod}_A(d)$.

Let $n \in \operatorname{mod}_B(d)$ and $m = \varphi^{(d)}(n) \in \operatorname{mod}_A(d)$. The equalities (3.1) imply that $\widehat{n}(Z) \cdot \widehat{n}(Y'') = \widehat{n}(1_s) = 1_{sd} \in \mathbb{M}_{sd}(k)$. Therefore $\operatorname{rank}(\widehat{n}(Y'')) = sd$. Since Y'' is a matrix with coefficients in $A \subseteq B$, we have $\widehat{m}(Y'') = \widehat{n}(Y'')$, and hence $\operatorname{rank}(\widehat{m}(Y'')) = sd$. The equalities (3.1) also imply that

$$\widehat{n}(Y'') \cdot \widehat{n}([b_1, \dots, b_s]^{\mathrm{T}}) = \widehat{n}(Y') \in \mathbb{M}_{\alpha d\times d}(k).$$

It follows that $\operatorname{rank}(\widehat{n}(Y'')) = \operatorname{rank}[\widehat{n}(Y''), \widehat{n}(Y')]$, by classical linear algebra. Since Y' and Y'' have coefficients in $A \subseteq B$, we see that $\widehat{m}(Y') = \widehat{n}(Y')$, $\widehat{m}(Y'') = \widehat{n}(Y'')$, hence $\operatorname{rank}(\widehat{m}(Y'')) = \operatorname{rank}[\widehat{m}(Y''), \widehat{m}(Y')]$, and finally $m \in \mathcal{C}$. ■

STEP (e). *There is an injective regular morphism* $\eta : \mathcal{C} \to (\mathbb{M}_d(k))^s$ *such that* $\eta(m) = (x_1, \ldots, x_s)$ *for all* $m \in \mathcal{C}$, *where*

$$\widehat{m}(Y'') \cdot [x_1, \ldots, x_s]^{\mathrm{T}} = \widehat{m}(Y'). \tag{3.3}$$

Proof. The existence of a regular map $\eta : \mathcal{C} \to (\mathbb{M}_d(k))^s$ is a consequence of elementary linear algebra and the fact that $\operatorname{rank}(\widehat{m}(Y'')) = \operatorname{rank}[\widehat{m}(Y''), \widehat{m}(Y')]$ equals the number of columns of $\widehat{m}(Y'') \in \mathbb{M}_{\alpha d \times sd}(k)$.

The set $\mathcal{C}$ is a finite union of open subsets defined by the condition that fixed sd rows in $\widehat{m}(Y'')$ are linearly independent. From Cramer's rule, η is a regular morphism on each of these open subsets. Therefore, η is a regular morphism.

Let $m = (m_1, \ldots, m_t) \in \mathcal{C}$ and $\eta(m) = (x_1, \ldots, x_s)$. The combination of the formulas (3.2) and (3.3) yields

$$\widehat{m}\left(\begin{bmatrix} 1_t & 0 \\ * & * \end{bmatrix}\right) \cdot [x_1, \ldots, x_s]^{\mathrm{T}} = \widehat{m}([b_1, \ldots, b_t|*]^{\mathrm{T}})$$

and hence $\widehat{m}([1_t, 0]) \cdot [x_1, \ldots, x_s]^{\mathrm{T}} = \widehat{m}([b_1, \ldots, b_t]^{\mathrm{T}})$. This implies that $[1_{td}, 0] \cdot [x_1, \ldots, x_s]^{\mathrm{T}} = [m(b_1), \ldots, m(b_t)]^{\mathrm{T}}$, hence that $[x_1, \ldots, x_t]^{\mathrm{T}} = [m_1, \ldots, m_t]^{\mathrm{T}}$. Thus the map η is injective. ∎

STEP (f). *For any* $n \in \operatorname{mod}_B(d)$ *we have* $\eta(\varphi_d(n)) = n$.

Proof. Let $n = (n_1, \ldots, n_s) \in \operatorname{mod}_B(d)$, $m = \varphi^{(d)}(n) = (n_1, \ldots, n_t) \in \operatorname{mod}_A(d)$ and $\eta(m) = (x_1, \ldots, x_s) \in \operatorname{mod}_B(d)$. Since

$$\widehat{m}(Y'') \cdot [n_1, \ldots, n_s]^{\mathrm{T}} = \widehat{m}(Y') = \widehat{m}(Y'') \cdot [x_1, \ldots, x_s]^{\mathrm{T}}$$

and the rank of $\widehat{m}(Y'')$ equals the number of its columns, we find that $n_i = x_i$ for all $1 \le i \le s$. This implies that $\eta(m) = n$. ∎

STEP (g). *The set* $\mathcal{D} = \eta^{-1}(\operatorname{mod}_B(d))$ *is locally closed in* $\operatorname{mod}_A(d)$ *and the morphisms* $\varphi^{(d)}$ *and* η *induce mutually inverse isomorphisms of* $\operatorname{mod}_B(d)$ *and* $\mathcal{D}$.

Proof. The set $\mathcal{D}$ is closed in $\mathcal{C}$. Hence $\mathcal{D}$ is a locally closed subset of $\operatorname{mod}_A(d)$, by (d). Moreover, $\operatorname{im} \varphi^{(d)}$ is contained in $\mathcal{D}$, by (f). It remains to show that $\varphi^{(d)}(\eta(m)) = m$ for all $m \in \mathcal{D}$. Indeed, this follows from the equality $\eta\varphi^{(d)}\eta(m) = \eta(m)$ and since the map η is injective, by (e) and (f). ∎

The final step shows that $\varphi^{(d)}$ is an immersion, which completes the proof of Theorem 1. ∎

4. Geometry of modules from standard homogeneous tubes

4.1. *Finite tubes.* Let A be an algebra and $r \ge 2$. A *finite tube over* A *of height* r is a sequence $\mathcal{T} = (T_i, \alpha_j, \beta_j : 1 \le i \le r,\ 1 \le j < r)$ such that T_i is a finite-dimensional A-module for any $1 \le i \le r$, $\alpha_j : T_j \to T_{j+1}$,

$\beta_j : T_{j+1} \to T_j$ are A-homomorphisms for any $1 \le j < r$, and the short sequences

$$0 \to T_1 \xrightarrow{\alpha_1} T_2 \xrightarrow{\beta_1} T_1 \to 0, \tag{4.1}$$

$$0 \to T_i \xrightarrow{(\alpha_i,\beta_{i-1})^{\mathrm{T}}} T_{i+1} \oplus T_{i-1} \xrightarrow{(\beta_i,-\alpha_{i-1})} T_i \to 0, \quad 1 < i < r, \tag{4.2}$$

are exact.

The Auslander–Reiten quiver of the algebra $k[X]/(X^r)$ is a finite tube of height r. The same conclusion holds for the algebras $\mathbb{M}_e(k[X]/(X^r))$, $e \ge 1$, which are Morita equivalent to $k[X]/(X^r)$.

Lemma 6. *Let $A = k\langle X_1, \ldots, X_n\rangle/I$ be a finitely generated algebra. Let $\mathcal{T} = (T_i, \alpha_j, \beta_j)$ be a finite tube over A of height r and $e = \dim_k T_1$. Then there are $y_1, \ldots, y_r \in (\mathbb{M}_e(k))^n$ such that the A-module T_i corresponds to the point*

$$t_i = \begin{bmatrix} y_1 & y_2 & y_3 & \ldots & y_i \\ 0 & y_1 & y_2 & \ldots & y_{i-1} \\ 0 & 0 & y_1 & \ldots & y_{i-2} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & y_1 \end{bmatrix}$$

of $\mathrm{mod}_A(ei)$ *for all* $1 \le i \le r$.

Proof. By (4.1), α_1 is a monomorphism. If α_{i-1} is a monomorphism then so is α_i for all $1 < i < r$, by (4.2). Therefore, α_i is a monomorphism, and similarly, β_i is an epimorphism for all $1 \le i < r$. Moreover, the exactness of the sequences (4.1) and (4.2) leads to

$$\beta_1\alpha_1 = 0, \qquad \beta_i\alpha_i = \alpha_{i-1}\beta_{i-1}, \quad 1 < i < r. \tag{4.4}$$

Let $\gamma = \alpha_{r-1}\beta_{r-1} \in \mathrm{End}_A(T_r)$. Applying the formulas (4.4) several times we get

$$\gamma^i = (\alpha_{r-1}\beta_{r-1})^i = \alpha_{r-1}\ldots\alpha_{r-i}\beta_{r-i}\ldots\beta_{r-1}$$

for all $1 \le i < r$ and $\gamma^r = 0$. Observe that the short sequence

$$0 \to T_i \xrightarrow{\alpha_{r-1}\ldots\alpha_i} T_r \xrightarrow{\beta_{r-i}\ldots\beta_{r-1}} T_{r-i} \to 0$$

is exact for all $1 \le i < r$. Then T_i is isomorphic to

$$\ker\gamma^i = \ker(\beta_{r-i}\ldots\beta_{r-1}) = \mathrm{im}(\alpha_{r-1}\ldots\alpha_i) = \mathrm{im}\,\gamma^{r-i}.$$

Let $\varepsilon_1, \ldots, \varepsilon_e$ be a basis of $\ker\gamma = \mathrm{im}\,\gamma^{r-1}$. We inductively choose ε_j satisfying $\gamma(\varepsilon_j) = \varepsilon_{j-e}$ for all $e < j \le er$. Observe that the residue classes of the elements ε_j in $\ker\gamma^i/\ker\gamma^{i-1} = \mathrm{im}\,\gamma^{r-i}/\mathrm{im}\,\gamma^{r-i+1}$, where $e(i-1) < j \le ei$ and $1 \le i \le r$, form a basis. Hence $\{\varepsilon_i : 1 \le i \le er\}$ is a basis of T_r.

Furthermore, γ in this basis has the form

$$\widetilde{\gamma} = \begin{bmatrix} 0 & 1_e & 0 & \dots & 0 \\ 0 & 0 & 1_e & \dots & 0 \\ \hdotsfor{5} \\ 0 & 0 & 0 & \dots & 1_e \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

Let t_r be the point of $\operatorname{mod}_A(er)$ corresponding to T_r in the above basis. We may consider t_r as a homomorphism from A to $\mathbb{M}_{er}(k)$. The map γ is a homomorphism of A-modules, thus the matrices $\widetilde{\gamma}, t_r(a) \in \mathbb{M}_{er}(k)$ commute for all $a \in A$. This implies that

$$t_r = \begin{bmatrix} y_1 & y_2 & \dots & y_r \\ 0 & y_1 & \dots & y_{r-1} \\ \hdotsfor{4} \\ 0 & 0 & \dots & y_1 \end{bmatrix}$$

for some $y_i \in (\mathbb{M}_e(k))^n$, $1 \le i \le r$. The epimorphism $\beta_i \dots \beta_{r-1} : T_r \to T_i$ maps the set $\{\varepsilon_j : e(r-i) < j \le er\}$ into a basis $\mathcal{B}_i$ of T_i for all $1 \le i < r$. Let t_i be the point of $\operatorname{mod}_A(ei)$ determined by T_i and the basis $\mathcal{B}_i$ for all $1 \le i < r$. Then t_i is obtained from t_r by removing the first $e(r-i)$ rows and columns for all $1 \le i < r$, which completes the proof. ■

The following proposition shows that finite tubes of height r are strongly related to the algebras $\mathbb{M}_e(k[X]/(X^r))$, $e \ge 1$.

PROPOSITION 7. *Let A be a finitely generated algebra and $\mathcal{T} = (T_i, \alpha_j, \beta_j)$ be a finite tube over A of height r. Let $B = \mathbb{M}_e(k[X]/(X^r))$ with $e = \dim_k T_1$. Then*:

(i) *There are*, *up to isomorphism*, *r indecomposable B-modules $U_1, \dots, U_r$ with $\dim_k U_i = ei$ for all $1 \le i \le r$.*

(ii) *There is a homomorphism of algebras $\varphi : A \to B$ such that the A-module $\varphi^*(U_i)$ is isomorphic to T_i, where $\varphi^* : \operatorname{Mod} B \to \operatorname{Mod} A$ is the functor induced by φ and $1 \le i \le r$.*

Proof. The first part is well known since the algebra B is Morita equivalent to $k[X]/(X^r)$. Take an epimorphism $\pi : k\langle X_1, \dots, X_{e^2+1}\rangle \to B$ such that $x_1 = \pi(X_1), \dots, x_{e^2} = \pi(X_{e^2})$ form a basis of $\mathbb{M}_e(k) \subseteq B$ and $\pi(X_{e^2+1}) = \overline{X} \in B$. Then $B \simeq k\langle X_1, \dots, X_{e^2+1}\rangle / \ker \pi$ and the point $u_i = (v_1, \dots, v_{e^2+1}) \in \operatorname{mod}_B(ei)$ satisfying

$$v_j = \begin{bmatrix} x_j & 0 & \dots & 0 \\ 0 & x_j & \dots & 0 \\ \hdotsfor{4} \\ 0 & 0 & \dots & x_j \end{bmatrix}, \quad 1 \le j \le e^2, \quad v_{e^2+1} = \begin{bmatrix} 0 & 1_e & 0 & \dots & 0 \\ 0 & 0 & 1_e & \dots & 0 \\ \hdotsfor{5} \\ 0 & 0 & 0 & \dots & 1_e \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix},$$

corresponds to the B-module U_i for all $1 \le i \le r$.

Applying Lemma 6 we deduce that the A-module T_r determines in some basis a point t_r of the form (4.3), for some $y_1, \dots, y_r \in (\mathbb{M}_e(k))^n$ and $n \geq 1$. We may consider $y_1, \dots, y_r$ as linear maps from A to $\mathbb{M}_e(k)$. Then the map $\varphi : A \to B$ given by

$$\varphi(a) = y_1(a) + y_2(a) \cdot \overline{X} + \ldots + y_r(a) \cdot \overline{X}^{r-1}, \qquad a \in A,$$

is easily seen to be a homomorphism of algebras. Furthermore, $\varphi^{(ei)}(u_i) = t_i$ for all $1 \leq i \leq r$. This implies that $\varphi^*(U_i)$ is isomorphic to T_i for all $1 \leq i \leq r$, and the proposition follows. ∎

COROLLARY 8. *Let A be a finitely generated algebra and $\mathcal{T} = (T_i, \alpha_j, \beta_j)$ be a finite tube over A of height r. Let $B = \mathbb{M}_e(k[X]/(X^r))$. If $\dim_k \operatorname{End}_A(T_r) = r$ then the morphisms $\varphi^{(d)} : \operatorname{mod}_B(d) \to \operatorname{mod}_A(d)$ are immersions for all $d \geq 1$.*

Proof. There is a B-module isomorphism $B \simeq (U_r)^e$. Moreover, U_r is an indecomposable projective and injective B-module. Thus the left B-modules B and DB are isomorphic to each other. Furthermore, $\varphi^*(U_r) \simeq T_r$, by Proposition 7. This implies that

$$\begin{aligned}\dim_k \operatorname{Hom}_A(B, DB) &= \dim_k \operatorname{End}_A((T_r)^e) \\ &= e^2 \dim_k \operatorname{End}_A(T_r) = e^2 r = e^2 \dim_k \operatorname{End}_B(U_r) \\ &= \dim_k \operatorname{End}_B((U_r)^e) = \dim_k \operatorname{Hom}_B(B, DB).\end{aligned}$$

Our claim follows from Lemma 4 and Theorem 1. ∎

4.2. *Proof of Theorem 2.* Let $\Gamma = (T_i, \alpha_i, \beta_i)_{i \geq 1}$ be a standard homogeneous tube of the Auslander–Reiten quiver of A. Fix natural numbers r, d' satisfying $r \geq d'$. Then the sequence $\mathcal{T} = (T_i, \alpha_j, \beta_j : 1 \leq i \leq r,\ 1 \leq j < r)$ is a finite tube over A of height r. Let $B = \mathbb{M}_e(k[X]/(X^r))$. Applying Proposition 7 we get a homomorphism $\varphi : A \to B$ of algebras such that $\operatorname{im} \varphi^{(d)}$ is the union of orbits corresponding to modules from $\operatorname{add}(T_1, \dots, T_r)$. Since $\dim_k T_i = ei > d$ for any $i > r$, we get $\operatorname{mod}_\Gamma(d) = \operatorname{im} \varphi^{(d)}$.

Since Γ is a standard tube, $\dim_k \operatorname{End}_A(T_r) = r$, and consequently, $\varphi^{(d)}$ is an immersion, by Corollary 8. This implies that $\operatorname{mod}_\Gamma(d)$ is isomorphic to $\operatorname{mod}_B(d)$ and is a locally closed subset of $\operatorname{mod}_A(d)$.

The algebra B is Morita equivalent to the algebra $C = k[X]/(X^r)$. In particular, there is an equivalence $\mathcal{F} : \operatorname{Mod} C \to \operatorname{Mod} B$ of additive categories. Bongartz proved in [4] that there is a closed immersion $\mu : \operatorname{mod}_C(d') \to \operatorname{mod}_B(d)$ which induces a bijection between the set of $\mathrm{Gl}_{d'}(k)$-orbits in $\operatorname{mod}_C(d')$ and the set of $\mathrm{Gl}_d(k)$-orbits in $\operatorname{mod}_B(d)$. This bijection is compatible with $\mathcal{F}$ and preserves and reflects closures, inclusions and the types of singularities occurring in orbit closures (see also Section 4.2 in [5]). Therefore, $\operatorname{mod}_\Gamma(d)$ and $\operatorname{mod}_C(d')$ have the same types of singularities and minimal singularities. ∎

4.3. *List of minimal singularities.* We present here all types of minimal singularities in the module varieties over the algebra $k[X]/(X^r)$. They were investigated in [12] (for a field k of characteristic zero) and in [5] (in the general case). These types form two series:

- $a_n = \mathrm{Sing}(\{\text{nilpotent matrices in } \mathbb{M}_{(n+1)\times(n+1)}(k) \text{ of rank } \le 1\}, 0)$,
- $A_n = \mathrm{Sing}(\{\text{nilpotent matrices in } \mathbb{M}_{(n+1)\times(n+1)}(k)\}, L_n)$,

where $1 \le n < r$ and

$$L_n = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

is a nilpotent matrix of rank $n-1$. In the above list all types are pairwise different except $A_1 = a_1$.

Note that $A_n = \mathrm{Sing}(\{X^{n+1} + YZ = 0\}, 0)$ is the type of one of the Kleinian (or simple) singularities. This can be obtained using transverse slices (see Section 5.1 in [15] and [2]).

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Received 8 June 1999;
revised 19 July 1999