MARCZEWSKI–BURSTIN-LIKE CHARACTERIZATIONS
OF σ-ALGEBRAS, IDEALS, AND MEASURABLE FUNCTIONS

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Abstract. \( \mathcal{L} \) denotes the Lebesgue measurable subsets of \( \mathbb{R} \) and \( \mathcal{L}_0 \) denotes the sets of Lebesgue measure 0. In 1914 Burstin showed that a set \( M \subseteq \mathbb{R} \) belongs to \( \mathcal{L} \) if and only if every perfect \( P \in \mathcal{L} \setminus \mathcal{L}_0 \) has a perfect subset \( Q \in \mathcal{L} \setminus \mathcal{L}_0 \) which is a subset of or misses \( M \) (a similar statement omitting “is a subset of or” characterizes \( \mathcal{L}_0 \)). In 1935, Marczewski used similar language to define the \( \sigma \)-algebra \( (s) \) which we now call the “Marczewski measurable sets” and the \( \sigma \)-ideal \( (s^0) \) which we call the “Marczewski null sets”. \( M \in (s) \) if every perfect set \( P \) has a perfect subset \( Q \) which is a subset of or misses \( M \). \( M \in (s^0) \) if every perfect set \( P \) has a perfect subset \( Q \) which misses \( M \).

In this paper, it is shown that there is a collection \( G \) of \( G_\delta \) sets which can be used to give similar “Marczewski–Burstin-like” characterizations of the collections \( B_w \) (sets with the Baire property in the wide sense) and \( FC \) (first category sets). It is shown that no collection of \( F_\sigma \) sets can be used for this purpose. It is then shown that no collection of Borel sets can be used in a similar way to provide Marczewski–Burstin-like characterizations of \( B_r \) (sets with the Baire property in the restricted sense) and \( AFC \) (always first category sets). The same is true for \( U \) (universally measurable sets) and \( U_0 \) (universal null sets).

Marczewski–Burstin-like characterizations of the classes of measurable functions are also discussed.

1. Measurable sets. \( \mathcal{L} \) denotes the Lebesgue measurable subsets of \( \mathbb{R} \) and \( \mathcal{L}_0 \) denotes the sets of Lebesgue measure 0. In 1914 Burstin \(^{(1)}\) [3] showed that a set \( M \subseteq \mathbb{R} \) belongs to \( \mathcal{L} \) if and only if every perfect \( P \in \mathcal{L} \setminus \mathcal{L}_0 \) has a perfect subset \( Q \in \mathcal{L} \setminus \mathcal{L}_0 \) which is a subset of or misses \( M \).

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\(^{(1)}\) Burstin’s paper contains a number of erroneous statements indicating that it was not understood that there exists an \( M \subseteq \mathbb{R} \) such that both \( M \) and \( M^c \) intersect every interval in an \( \mathcal{L} \setminus \mathcal{L}_0 \) set. However, the result referred to here is correct.
FC denotes the first category sets in some Polish space $X$ and $B_w$ denotes the sets with the “Baire property in the wide sense”. The $\sigma$-algebra $B_w$ was defined (for $X$ a perfect subset of $\mathbb{R}$) by Nikodym [9] who showed that the class of functions $f : X \to \mathbb{R}$ having the “property of Baire in the wide sense” (i.e. $f|(X \setminus F)$ is continuous for some FC subset $F$ of $X$) was precisely the class of functions which were measurable with respect to this $\sigma$-algebra. Kuratowski [4] extended these results to complete metric spaces $X$, with a simplified definition of $B_w$. Nikodym’s definition of $M \in B_w$ was equivalent to saying that $M$ is residual in every open set $U \subseteq X$ in which it is categorically dense (i.e. of second category in every open subset of $U$). Kuratowski’s equivalent simplified definition was that $M = (U \setminus F) \cup N$ for some open $U \subseteq X$ and first category sets $F \subseteq U$ and $N \subseteq U^c$. Kuratowski also defined the $\sigma$-algebra, $B_r$, of sets $M$ which have the Baire property in the restricted sense (i.e. $M \cap P$ has property $B_w$ relative to $P$ for every perfect $P$) and used this class of sets to characterize the class of functions going by a similar name.

Ruziewicz [11] showed that every $f : \mathbb{R} \to \mathbb{R}$ is a composition of two $L$-measurable functions and both of the functions used in his proof are also $B_w$-measurable. Sierpiński [12] showed that the similar result for compositions of $B_r$-measurable functions did not hold by constructing a certain class of functions which contained the $B_r$-measurable functions, was closed under compositions, and did not contain all functions. Marczewski [5] described a $\sigma$-algebra of sets he denoted by $(s)$ and showed that the class of functions described by Sierpiński was precisely the class of $(s)$-measurable functions.

We now call $(s)$ the class of “Marczewski measurable sets”. $M \in (s)$ if every perfect set $P$ has a perfect subset $Q$ which is a subset of or misses $M$. $(s^0)$ denotes the “Marczewski null sets”. $M \in (s^0)$ if every perfect set $P$ has a perfect subset $Q$ which misses $M$.

A collection $G$ of Borel subsets of $X$ is said to be the basis for a Marczewski–Burstin-like (or MB-like) characterization of a given $\sigma$-algebra, $S$, of subsets of $X$ provided $\emptyset \notin G$ and

(i) $M \in S \iff \forall P \in G, \exists Q \in G, \ Q \subseteq P : Q \subseteq M$ or $Q \cap M = \emptyset$.

$G$ provides an MB-like characterization of a given $\sigma$-ideal, $I$, on $X$ provided $\emptyset \notin G$ and

(ii) $M \in I \iff \forall P \in G, \exists Q \in G, \ Q \subseteq P : Q \cap M = \emptyset$.

The following theorem is a generalization of Burstin’s result. For a proof of (1) see Lemma 3.6 of [10] (proof of (2) is similar). Also see the theorems and corollaries given in Section 2 of [7].

**Theorem 1.** If $X$ is a Polish space having no isolated points, $\Sigma$ is the collection of sets which are measurable with respect to the completion $\overline{\mu}$ of
a finite nonatomic Borel measure $\mu$ on $X$, and $\Sigma_0$ consists of the sets in $\Sigma$ which have measure zero, then

(1) $M \in \Sigma$ if and only if every perfect $P \in \Sigma \setminus \Sigma_0$ has a perfect subset $Q \in \Sigma \setminus \Sigma_0$ which is a subset of or misses $M$, and

(2) $M \in \Sigma_0$ if and only if every perfect $P \in \Sigma \setminus \Sigma_0$ has a perfect subset $Q \in \Sigma \setminus \Sigma_0$ which misses $M$.

The theorem provides MB-like characterizations of $\Sigma$ and $\Sigma_0$ based upon a special collection, $G = \{\text{perfect } P \mid P \in L \setminus L_0\}$, of closed sets. It is natural to ask if it might be possible to use a “better” collection for $G$, perhaps a special collection of “clopen” sets if $X$ is zero-dimensional, for example. The following theorem shows that this is generally not the case.

**Theorem 2.** If $X$ is a Polish space having no isolated points, $\Sigma$ is the collection of sets which are measurable with respect to the completion $\overline{\mu}$ of a finite nonatomic Borel measure $\mu$ on $X$, and $\Sigma_0$ consists of the sets in $\Sigma$ which have measure zero, then there is no MB-like characterization of either $\Sigma$ or $\Sigma_0$ based upon any collection of open sets.

**Proof.** Assume the hypothesis of the theorem and let $G$ be any collection of open subsets of $X$. If $D$ is a countable dense subset of $X$, then $D \in \Sigma_0$ and every element of $G$ intersects $D$, so there can be no MB-like characterization of $\Sigma_0$ based on $G$. On the other hand, suppose $G$ is a basis for an MB-like characterization of $\Sigma$. Since $\mu$ is assumed to be nonatomic, there is a perfect set $M \in \Sigma \setminus \Sigma_0$ which is nowhere dense in $X$. Suppose $U \in G$. Then $U$ must have a subset $V \in G$ which is a subset of or misses $M$. Since no open set is a subset of $M$, $V$ must miss $M$. This implies that $M \in \Sigma_0$, which is false, so there can be no MB-like characterization of $\Sigma$ based on $G$. $\blacksquare$

Note that the definitions of $(s)$ and $(s^0)$ are MB-like statements based upon a special collection $G$ of closed sets. No collection of open sets would suffice.

**Theorem 3.** If $X$ is a Polish space having no isolated points, then there is no MB-like characterization of $(s)$ or $(s^0)$ based upon any collection of open sets.

**Proof.** Proof is the same as that of the previous theorem except that $M$ can be taken to be any perfect nowhere dense subset of $X$. $\blacksquare$

An MB-like characterization of the FC and $B_\infty$ subsets of a Polish space $X$ will now be given. It is based upon a special collection of $G_\delta$ sets

$$G_{B_\infty} = \{P \subseteq X \mid \exists \text{ an open } U \supseteq P : U \setminus P \in FC \cap F_\sigma\}.$$
Theorem 4. If $X$ is a Polish space then

1. $M \in B_w$ if and only if every set $P \in G_{B_w}$ has a subset $Q \in G_{B_w}$ which is a subset of or misses $M$, and

2. $M \in FC$ if and only if every set $P \in G_{B_w}$ has a subset $Q \in G_{B_w}$ which misses $M$.

Proof. First suppose $M \in B_w$. Then $M = (U_1 \setminus F_1) \cup G_1$, where $U_1$ is open and $F_1 \subseteq U_1$ and $G_1 \subseteq U_1^c$ are both FC. Suppose $P \in G_{B_w}$. Then $P = U_2 \setminus F_2$, where $U_2$ is open and $F_2 \subseteq U_2$ is $F_\sigma$ and FC. Let $H_1$ be an $F_\sigma$ and FC subset of $U_1$ containing $F_1$ and let $K_1$ be an $F_\sigma$ and FC set containing $G_1$. If $U_1 \cap U_2 \neq \emptyset$, then $(U_1 \cap U_2) \cap [(H_1 \cup K_1 \cup F_2) \cap (U_1 \cap U_2)]$ is the desired subset $Q \in G_{B_w}$ of $P$ which is a subset of $M$. If $U_1 \cap U_2$ is empty, which is the case if $M$ is FC (i.e. when $U_1 = \emptyset$), then $U_2 \setminus (K_1 \cup F_2)$ is the desired subset $Q \in G_{B_w}$ of $P$ which misses $M$. Thus, the “$\Rightarrow$” implications of (1) and (2) have been proved.

On the other hand, suppose $M$ satisfies the property of (1) involving the $G_{B_w}$ sets. Let $U_1, U_2, \ldots$ be a countable basis of nonempty open sets for the space. For each $i$, $U_i$ is itself an element of $G_{B_w}$, so let $Q_i = V_i \setminus F_i \in G_{B_w}$ be a subset of $U_i$ which is a subset of or misses $M$ (assume without loss of generality that $V_i$ is an open subset of $U_i$ and $F_i$ is an $F_\sigma$ and FC subset of $V_i$). Let $U = \bigcup \{V_i \mid Q_i \subseteq M\}$. Then $N_1 = U \setminus M$ is a subset of $F_1 \cup F_2 \cup \ldots$ so $N_1 \in FC$. Also $N_2 = M \setminus U$ is FC. Otherwise there would exist a $U_i$ in which $N_2$ is categorically dense. Then $Q_i = V_i \setminus F_i$ must be a subset of $M$ and $\emptyset \neq V_i \subseteq U$, which is a contradiction. Therefore, $M = (U \setminus N_1) \cup N_2 \in B_w$ and the “$\Leftarrow$” implication of (1) is proved.

If $M$ satisfies the property of (2) involving the $G_{B_w}$ sets, it follows that every subset of $M$ satisfies the similar property of (1) and that $M$ is “hereditarily” $B_w$ and therefore FC. Thus, the “$\Leftarrow$” implication of (2) is proved.

One now wonders if the collection of $G_\delta$ sets used in the previous theorem could be replaced with a “better” collection of Borel sets, perhaps some collection of ambiguous $F_\sigma G_\delta$ sets. The following shows that the answer is “no”.

Theorem 5. If $X$ is a Polish space having no isolated points, then there is no MB-like characterization of either $B_w$ or FC based upon any collection of $F_\sigma$ sets.

Proof. Let $X$ be a Polish space and let $G$ be any collection of $F_\sigma$ subsets of $X$.

Suppose $G$ provides a basis for an MB-like characterization of the FC sets. If there existed an FC set $P \in G$, then $P$ would have to have a subset $Q \in G$ which misses $P$, which is impossible. Therefore, all $P \in G$ are second category. Let $D$ be a countable dense subset of $X$ and let $P \in G$. Then
\[ D \in \text{FC} \] and so \( P \) must have a subset \( Q \in G \) which misses \( D \). Since \( Q \) is \( F_\sigma \) and every closed subset of \( Q \) misses \( D \) (and is therefore nowhere dense), it follows that \( Q \) is also FC, which is a contradiction.

Suppose \( G \) provides a basis for an MB-like characterization of \( B_w \). Let \( B \) be a Bernstein subset of \( X \). Then \( B \notin B_w \), so there exists a \( P \in G \) such that every subset \( Q \in G \) of \( P \) intersects both \( B \) and \( B^c \). Let \( F \) be an \( F_\sigma \) and FC set which is perfectly dense in \( X \) (i.e. every open set contains a perfect subset of \( F \)). Then the set \( F_1 = F \cap B \in \text{FC} \subseteq B_w \), so \( P \) has a subset \( Q \in G \) which is a subset of or misses \( F_1 \). If \( Q \subseteq F_1 \), then \( Q \) must be countable because \( F_1 \) has no perfect subsets. If \( Q \cap F_1 = \emptyset \), then \( Q \) would necessarily be the union of countably many closed sets, each of which is nowhere dense because it misses \( F_1 \) which is dense in \( X \). In either case, \( Q_1 = Q \cap B \in \text{FC} \subseteq B_w \). Both \( Q_1 \) and \( Q_2 = Q \cap B^c \) are nonempty. Since \( Q_1 \in B_w \), \( Q \) must have a subset \( Q_3 \in G \) which is either a subset of or misses \( Q_1 \). Both of these possibilities would contradict the fact that \( Q_3 \) would necessarily have to intersect both \( B \) and \( B^c \).

It is fairly easy to show that the collection \( G_{B_w} \) is a “category base” [8], \( C \), with respect to which the FC sets and the \( B_w \) sets are the \( C \)-meagre and \( C \)-Baire sets, respectively. However, there are much better collections of sets which form such category bases. The open sets or the regular closed sets will work, as will the clopen sets in zero-dimensional spaces. The fact that one cannot replace the collection \( G_{B_w} \) with even a collection of \( F_\sigma \) sets illustrates the fact that the connection between a basis, \( G \), of Borel sets used in an MB-like characterization and the \( \sigma \)-algebra and the \( \sigma \)-ideal being characterized is much closer than in the category base theory.

The question of whether or not there is a collection \( G \) of Borel sets which can be used as a basis for a simultaneous MB-like characterization of the \( (\sigma \text{-ideal, } \sigma \text{-algebra}) \) pair, \( (\text{AFC}, B_r) \), in a Polish space \( X \) with no isolated points will now be answered. It would seem at first that the collection

\[ G = \{ P \mid P = Q \setminus R, \; Q \text{ perfect, } R \subseteq P \text{ an } F_\sigma \text{ set which is FC relative to } Q \} \]

would serve. It is the case that if a set \( M \) satisfies the right hand side of (i) (respectively (ii)) in the definitions of MB-like characterizations for this collection, \( G \), then it follows that \( M \) is \( B_r \) (respectively AFC). However, the forward implications of (i) and (ii) fail to hold. In fact, it will be shown that there is no collection \( G \) of Borel sets which can be used as a basis for a simultaneous MB-like characterization of the \( \sigma \)-ideal, \( \sigma \)-algebra pair, \( (\text{AFC}, B_r) \). This will provide a partial solution to Problem 1.1 of [1] where it is asked (using different language and notation) whether there is a field of subsets of \( \mathbb{R} \) for which no collection \( G \) of (Borel or non-Borel) subsets of \( \mathbb{R} \) can form the basis for an MB-like characterization. There is also no collection \( G \) of Borel sets which can be used as a basis for a simultaneous MB-like
characterization of the (σ-ideal, σ-algebra) pair, \((U_0, U)\), of universal null sets and universally measurable sets. These last two collections were defined by Marczewski [6] as follows: \(U = \{ M \mid M \) is measurable with respect to the completion, \(\overline{\mu}\), of every Borel measure, \(\mu\), on \(X\)\} and \(U_0 = \{ M \mid M \) has measure 0 with respect to the completion, \(\overline{\mu}\), of every nonatomic Borel measure, \(\mu\), on \(X\)\}. It will be shown that if there were such a simultaneous MB-like characterization of either of these pairs, \((I, S)\), then the pair would have to satisfy the “Marczewski Hull Condition”,

\[
\forall Z \subseteq X, \exists M \in S : Z \subseteq M \quad \text{and} \quad \forall N \in S : Z \subseteq N, \; M \setminus N \in I,
\]

which was shown by John Walsh in [13] not to be the case. The proof will mimic the proof of Theorem 3 and Corollary 1 of [13] (see [2] for another paper where these proofs have been useful in showing a different result).

**Lemma 1.** Let \(I \subseteq S\) be a (σ-ideal, σ-algebra) pair on a Polish space \(X\) which has no isolated points, such that \(I\) contains all of the countable subsets of \(X\) but none of the perfect subsets of \(X\) while \(S\) contains the Borel subsets of \(X\). Assume \(G\) is a collection of Borel sets that forms the basis for a simultaneous MB-like characterization of \(I\) and \(S\). Then every uncountable Borel set \(P\) contains \(\mathfrak{c}\)-many disjoint subsets \(\{Q_\alpha \mid \alpha < \mathfrak{c}\}\) from \(G\).

**Proof.** Let \(P\) be an uncountable Borel set. Then \(P\) contains \(\mathfrak{c}\)-many disjoint perfect sets \(Q'_\alpha\). Each \(Q'_\alpha\) is in \(S\) but not in \(I\). Since \(Q'_\alpha \notin I\), it follows that there exists a \(P'_\alpha \in G\) such that every subset of \(P'_\alpha\) which is in \(G\) intersects \(Q'_\alpha\). Since \(Q'_\alpha\) is in \(S\), it follows that there is a subset \(Q_\alpha\) of \(P'_\alpha\) which is a subset of or misses \(Q'_\alpha\), and it must be a subset of \(Q'_\alpha\). Finally, \(\{Q_\alpha \mid \alpha < \mathfrak{c}\}\) is the desired collection of disjoint subsets of \(P\). \(\blacksquare\)

**Theorem 6.** Let \(I \subseteq S\) be a σ-ideal and σ-algebra on \(X\) satisfying all the hypotheses of Lemma 1 above. Then, if \(Z \subseteq X\), there exists \(Y \in S\) such that \(Z \subseteq Y\) and if \(P \in G\) is a subset of \(X \setminus Z\), then \(|P \cap Y| < \mathfrak{c}\).

**Proof.** We proceed as in the proof of Theorem 3 of [13], letting \(A = \{ A \in G \mid A \subseteq Z \text{ or else every } C \in G \text{ for which } C \subseteq A \text{ intersects both } Z \text{ and } Z^c \}\) and \(B = \{ B \in G \mid B \cap Z = \emptyset\}\). Following Walsh, it is noted that if \(A\) is empty, then \(Z\) will belong to \(I\). This is because if \(A = \emptyset\), then for every \(A \in G\), \(A \subseteq Z\) and there exists some \(C \in G\) for which \(C \subseteq A\) such that either \(C \cap Z = \emptyset\) or \(C \subseteq Z\) (note that the latter case is impossible). It would follow that \(Z \in I\) and the theorem follows. Therefore, it can be assumed that \(A \neq \emptyset\). For similar reasons, it may be assumed that \(B \neq \emptyset\). Lemma 1 was needed to prove that \(|A| = |B| = \mathfrak{c}\). This follows from the fact that if \(P\) belongs to either \(A\) or \(B\), then \(P \in G\) and \(P\) would have to be uncountable, otherwise \(G\) could not be the basis for an MB-like characterization of \(I\), which contains all of the countable subsets of \(X\). Therefore, \(P\) will contain \(\mathfrak{c}\)-many disjoint subsets which are also in \(G\), and each of these would be in \(A\).
or $B$ for the same reason $P$ was. Now, the rest of Walsh’s proof of Theorem 3 of [13] can be changed slightly by replacing the perfect sets by the sets from $G$ and the general theorem presented here is proved. □

**Corollary 1.** Let $I \subset S$ be a $\sigma$-ideal and $\sigma$-algebra on $X$ satisfying all the hypotheses of Lemma 1 above. Then $(I, S)$ satisfies the Marczewski Hull Condition.

**Proof.** Similar to proof of Corollary 1 of [13].

**Corollary 2.** There is no collection $G$ of Borel sets which forms a basis for a simultaneous MB-like characterization of AFC and $B_r$ in a Polish space $X$ with no isolated points. No such collection exists for $U_0$ and $U$ either.

**Proof.** $(\text{AFC, } B_r)$ and $(U_0, U)$ both satisfy the hypotheses of Lemma 1 above and it was shown [13] that neither satisfies the Marczewski Hull Condition. □

**2. Measurable functions.** Marczewski invented the $\sigma$-algebra $(s)$ to show the following.

**Theorem 7.** Given a Polish space $X$, a separable metric space $Y$, and a function $f : X \to Y$, the following are equivalent:

1. $f$ is $(s)$-measurable,
2. every perfect $P \subseteq X$ has a perfect subset $Q$ such that $f|Q$ is continuous.

The second statement describes the class of functions used by Sierpiński in [12] (with $X = Y = \mathbb{R}$). Similar theorems for the Lebesgue measurable and $B_w$ measurable functions will now be given. The next theorem includes the Lebesgue measurable case (also see Theorem 5, Sec. III, Ch. 5 of [8]).

**Theorem 8.** If $X$ is a Polish space having no isolated points, $\Sigma$ is the collection of sets which are measurable with respect to the completion $\overline{\mu}$ of a finite nonatomic Borel measure $\mu$ on $X$, $\Sigma_0$ consists of the sets in $\Sigma$ which have measure zero, and $f : X \to \mathbb{R}$, then the following are equivalent:

1. $f$ is $\Sigma$-measurable,
2. every perfect $P \in \Sigma \setminus \Sigma_0$ has a perfect subset $Q \in \Sigma \setminus \Sigma_0$ such that $f|Q$ is continuous.

**Proof.** The $(1) \Rightarrow (2)$ implication follows immediately from Lusin’s Theorem, so suppose $f : X \to \mathbb{R}$ is not $\Sigma$-measurable. Then there exist $t \in \mathbb{R}$ such that $[f < t]$ (notation for $\{x \mid f(x) < t\}$) $\notin \Sigma$. It follows that

$$\mu^c([f < t]) + \mu^c([t \leq f]) > \mu(X)$$
(μ₀ denotes the outer μ-measure), and this in turn implies that there exists an s < t such that

\[ μ₀([f ≤ s]) + μ₀([t ≤ f]) > μ(X). \]

Suppose the last assertion fails. Then

\[ μ₀([f ≤ t − 1/n]) + μ₀([t ≤ f]) ≤ μ(X) \]

for n = 1, 2, ... It follows that for each n, one could choose an open set \( U_n \) such that \([f ≤ t − 1/n] \subseteq U_n \) and such that

\[ μ(U_n) ≤ μ(X) − μ₀([t ≤ f]) + 1/n. \]

Note that for each n,

\[ [f < t − 1/n] \subseteq U_n ∩ U_{n+1} \cap \ldots, \]

so

\[ [f < t] \subseteq \bigcap_{n=1}^{∞} \bigcap_{k=n}^{∞} U_k = \liminf_{n→∞} U_n, \]

\[ μ₀([f < t]) ≤ μ(\liminf_{n→∞} U_n) ≤ \liminf_{n→∞} μ(U_n) \]

\[ ≤ \liminf_{n→∞} (μ(X) − μ₀([t ≤ f]) + 1/n) \]

and

\[ μ₀([f < t]) ≤ μ(X) − μ₀([t ≤ f]), \]

which is a contradiction, so the assertion is true.

Let s < t be such that \( μ₀([f ≤ s]) + μ₀([t ≤ f]) − μ(X) = ε > 0. \) Let \( G_1 \) and \( G_2 \) be \( G_δ \) sets containing \([f ≤ s] \) and \([t ≤ f] \), respectively, such that \( μ(G_1) + μ(G_2) = μ(X) + ε \) and let \( G = G_1 ∩ G_2. \) It follows that

\[ μ(G) = μ₀(G ∩ [f ≤ s]) = μ₀(G ∩ [t ≤ f]) = ε. \]

Now \( G \) has a perfect subset \( P ∈ Σ \setminus Σ_0 \) and if \( Q \) is any perfect subset of \( P \) in \( Σ \setminus Σ_0, \) then \( Q \) intersects both \([f ≤ t] \) and \([s ≤ f] \) in sets of positive outer measure. Therefore, \( f|Q \) has a point \( x \) of discontinuity for every perfect \( Q ∈ Σ \setminus Σ_0 \) (one can choose \( x \) to be any μ-density point of \( Q). \]

Theorem 9. Given a Polish space \( X, \) a separable metric space \( Y, \) and a function \( f : X → Y, \) the following are equivalent:

1. \( f \) is \( B_{σ} \)-measurable,
2. every \( P ∈ G_{B_{σ}} \) has a subset \( Q ∈ G_{B_{σ}} \) such that \( f|Q \) is continuous.

Proof. Suppose \( f : X → \mathbb{R} \) is \( B_{σ} \)-measurable and \( P ∈ G_{B_{σ}}. \) Then there is a residual set \( R ⊆ X \) such that \( f|R \) is continuous. Assume without loss of generality that \( R = X \setminus F, \) where \( F \) is an \( F_σ \) and FC set, so that \( R ∈ G_{B_{σ}}. \)

We have \( P = U \setminus N \) for some open \( U \) and some \( F_σ \) and FC set \( N ⊆ U. \) Then
\[ Q = R \cap P = U \setminus (F \cup N) \in G_{Bw} \text{ and } f|Q \text{ is continuous. This establishes the (1)⇒(2) implication.} \]

Suppose \( f : X \to \mathbb{R} \) is not \( B_w \)-measurable. There exists a \( t \in \mathbb{R} \) such that \( [f < t] \notin B_w \). Then there is an open \( U \subseteq X \) in which \( [f < t] \) is categorically dense, but not residual. As \( U \cap [t \leq f] \) is not FC, there exists an open \( V \subseteq U \) in which \( [f < t] \) and \( [t \leq f] \) are both categorically dense. It follows that there exists an \( n \) such that \( [f \leq t - 1/n] \) and \( [t \leq f] \) are both categorically dense in some open \( W \subseteq V \). Otherwise, for every \( n \), \( [f \leq t - 1/n \cap V \) would be FC, so that \( [f < t] \cap V \) would be FC, which is false. Let \( n \) and \( W \) be as described. The open set \( W \) is itself in \( G_{Bw} \). Suppose there were a \( Q \in G_{Bw} \), \( Q \subseteq W \), such that \( f|Q \) were continuous. Then \( Q = W_1 \setminus F \) for some open set \( W_1 \) and FC set \( F \subseteq W_1 \). We have \( W \cap W_1 \neq \emptyset \) because \( Q \subseteq W \). Both \( [f \leq t - 1/n] \) and \( [t \leq f] \) are categorically dense in \( W \cap W_1 \) and therefore both are dense in \( Q \). It follows that \( f|Q \) is discontinuous at every point of \( Q \). □

**Remark.** One could say that the theorems of this section show that the collections \( P_1 \) of all perfect subsets of \( X \), \( P_2 \) of all perfect sets \( P \in \Sigma \setminus \Sigma_0 \), and the collection \( G_{Bw} \) are collections of Borel sets which can be used to provide “MB-like” characterizations of the collections of \( (s)\)-measurable, \( \Sigma \)-measurable, and \( B_w \)-measurable functions, respectively. By considering characteristic functions and using the theorems of the previous section, it can be shown that one could not replace \( P_1 \) or \( P_2 \) with collections of open sets or replace \( G_{Bw} \) with any collection of \( F_\sigma \) sets and accomplish the same results. For similar reasons, it follows that there are no collections of Borel sets which can be used to provide MB-like characterizations of the \( B_r \)-measurable or \( U \)-measurable functions.

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Received 22 March 1999; revised 15 July 1999