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MARCZEWSKI–BURSTIN-LIKE CHARACTERIZATIONS OF σ-ALGEBRAS, IDEALS, AND MEASURABLE FUNCTIONS

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Abstract. $\mathcal{L}$ denotes the Lebesgue measurable subsets of $\mathbb{R}$ and $\mathcal{L}_0$ denotes the sets of Lebesgue measure 0. In 1914 Burstin showed that a set $M \subseteq \mathbb{R}$ belongs to $\mathcal{L}$ if and only if every perfect $P \in \mathcal{L} \setminus \mathcal{L}_0$ has a perfect subset $Q \in \mathcal{L} \setminus \mathcal{L}_0$ which is a subset of or misses M (a similar statement omitting "is a subset of or" characterizes $\mathcal{L}_0$). In 1935, Marczewski used similar language to define the σ-algebra (s) which we now call the "Marczewski measurable sets" and the σ-ideal (s^0) which we call the "Marczewski null sets". $M \in (s)$ if every perfect set P has a perfect subset Q which is a subset of or misses M. $M \in (s^0)$ if every perfect set P has a perfect subset Q which misses M.

In this paper, it is shown that there is a collection G of G_δ sets which can be used to give similar "Marczewski–Burstin-like" characterizations of the collections B_{w} (sets with the Baire property in the wide sense) and FC (first category sets). It is shown that no collection of F_σ sets can be used for this purpose. It is then shown that no collection of Borel sets can be used in a similar way to provide Marczewski–Burstin-like characterizations of B_{r} (sets with the Baire property in the restricted sense) and AFC (always first category sets). The same is true for U (universally measurable sets) and U_0 (universal null sets).

Marczewski–Burstin-like characterizations of the classes of measurable functions are also discussed.

1. Measurable sets.

$\mathcal{L}$ denotes the Lebesgue measurable subsets of $\mathbb{R}$ and $\mathcal{L}_0$ denotes the sets of Lebesgue measure 0. In 1914 Burstin [1] [3] showed that a set $M \subseteq \mathbb{R}$ belongs to $\mathcal{L}$ if and only if every perfect $P \in \mathcal{L} \setminus \mathcal{L}_0$ has a perfect subset $Q \in \mathcal{L} \setminus \mathcal{L}_0$ which is a subset of or misses M.

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(1) Burstin's paper contains a number of erroneous statements indicating that it was not understood that there exists an $M \subseteq \mathbb{R}$ such that both M and M^{c} intersect every interval in an $\mathcal{L} \setminus \mathcal{L}_0$ set. However, the result referred to here is correct.

FC denotes the first category sets in some Polish space X and B_{w} denotes the sets with the "Baire property in the wide sense". The σ-algebra B_{w} was defined (for X a perfect subset of $\mathbb{R}$) by Nikodym [9] who showed that the class of functions $f : X \to \mathbb{R}$ having the "property of Baire in the wide sense" (i.e. $f|(X \setminus F)$ is continuous for some FC subset F of X) was precisely the class of functions which were measurable with respect to this σ-algebra. Kuratowski [4] extended these results to complete metric spaces X, with a simplified definition of B_{w}. Nikodym's definition of $M \in B_{\mathrm{w}}$ was equivalent to saying that M is residual in every open set $U \subseteq X$ in which it is categorically dense (i.e. of second category in every open subset of U). Kuratowski's equivalent simplified definition was that $M = (U \setminus F) \cup N$ for some open $U \subseteq X$ and first category sets $F \subseteq U$ and $N \subseteq U^{\mathrm{c}}$. Kuratowski also defined the σ-algebra, B_{r}, of sets M which have the Baire property in the restricted sense (i.e. $M \cap P$ has property B_{w} relative to P for every perfect P) and used this class of sets to characterize the class of functions going by a similar name.

Ruziewicz [11] showed that *every* $f : \mathbb{R} \to \mathbb{R}$ is a composition of two $\mathcal{L}$-measurable functions and both of the functions used in his proof are also B_{w}-measurable. Sierpiński [12] showed that the similar result for compositions of B_{r}-measurable functions did not hold by constructing a certain class of functions which contained the B_{r}-measurable functions, was closed under compositions, and did not contain all functions. Marczewski [5] described a σ-algebra of sets he denoted by (s) and showed that the class of functions described by Sierpiński was precisely the class of (s)-measurable functions.

We now call (s) the class of "Marczewski measurable sets". $M \in (s)$ if every perfect set P has a perfect subset Q which is a subset of or misses M. (s^0) denotes the "Marczewski null sets". $M \in (s^0)$ if every perfect set P has a perfect subset Q which misses M.

A collection G of Borel subsets of X is said to be the basis for a *Marczewski–Burstin-like* (or *MB-like*) characterization of a given σ-algebra, S, of subsets of X provided $\emptyset \notin G$ and

(i) $M \in S \;\Leftrightarrow\; \forall P \in G,\ \exists Q \in G,\ Q \subseteq P : Q \subseteq M$ or $Q \cap M = \emptyset$.

G provides an MB-like characterization of a given σ-ideal, I, on X provided $\emptyset \notin G$ and

(ii) $M \in I \;\Leftrightarrow\; \forall P \in G,\ \exists Q \in G,\ Q \subseteq P : Q \cap M = \emptyset$.

The following theorem is a generalization of Burstin's result. For a proof of (1) see Lemma 3.6 of [10] (proof of (2) is similar). Also see the theorems and corollaries given in Section 2 of [7].

Theorem 1. *If X is a Polish space having no isolated points*, *Σ is the collection of sets which are measurable with respect to the completion $\overline{\mu}$ of* *a finite nonatomic Borel measure* μ *on* X, *and* Σ_0 *consists of the sets in* Σ *which have measure zero*, *then*

(1) $M \in \Sigma$ *if and only if every perfect* $P \in \Sigma \setminus \Sigma_0$ *has a perfect subset* $Q \in \Sigma \setminus \Sigma_0$ *which is a subset of or misses* M, *and*

(2) $M \in \Sigma_0$ *if and only if every perfect* $P \in \Sigma \setminus \Sigma_0$ *has a perfect subset* $Q \in \Sigma \setminus \Sigma_0$ *which misses* M.

The theorem provides MB-like characterizations of Σ and Σ_0 based upon a special collection, $G = \{\text{perfect } P \mid P \in \mathcal{L} \setminus \mathcal{L}_0\}$, of closed sets. It is natural to ask if it might be possible to use a "better" collection for G, perhaps a special collection of "clopen" sets if X is zero-dimensional, for example. The following theorem shows that this is generally not the case.

THEOREM 2. *If* X *is a Polish space having no isolated points*, Σ *is the collection of sets which are measurable with respect to the completion* $\overline{\mu}$ *of a finite nonatomic Borel measure* μ *on* X, *and* Σ_0 *consists of the sets in* Σ *which have measure zero*, *then there is no MB-like characterization of either* Σ *or* Σ_0 *based upon any collection of open sets.*

Proof. Assume the hypothesis of the theorem and let G be any collection of open subsets of X. If D is a countable dense subset of X, then $D \in \Sigma_0$ and every element of G intersects D, so there can be no MB-like characterization of Σ_0 based on G. On the other hand, suppose G is a basis for an MB-like characterization of Σ. Since μ is assumed to be nonatomic, there is a perfect set $M \in \Sigma \setminus \Sigma_0$ which is nowhere dense in X. Suppose $U \in G$. Then U must have a subset $V \in G$ which is a subset of or misses M. Since no open set is a subset of M, V must miss M. This implies that $M \in \Sigma_0$, which is false, so there can be no MB-like characterization of Σ based on G. ∎

Note that the *definitions* of (s) and (s^0) are MB-like statements based upon a special collection G of closed sets. No collection of open sets would suffice.

THEOREM 3. *If* X *is a Polish space having no isolated points*, *then there is no MB-like characterization of* (s) *or* (s^0) *based upon any collection of open sets.*

Proof. Proof is the same as that of the previous theorem except that M can be taken to be any perfect nowhere dense subset of X. ∎

An MB-like characterization of the FC and B_{w} subsets of a Polish space X will now be given. It is based upon a special collection of G_δ sets

$$G_{B_{\mathrm{w}}} = \{P \subseteq X \mid \exists \text{ an open } U \supseteq P : U \setminus P \in FC \cap F_\sigma\}.$$

THEOREM 4. *If X is a Polish space then*

(1) $M \in B_{\mathrm{w}}$ *if and only if every set* $P \in G_{B_{\mathrm{w}}}$ *has a subset* $Q \in G_{B_{\mathrm{w}}}$ *which is a subset of or misses* M, *and*

(2) $M \in \mathrm{FC}$ *if and only if every set* $P \in G_{B_{\mathrm{w}}}$ *has a subset* $Q \in G_{B_{\mathrm{w}}}$ *which misses* M,

Proof. First suppose $M \in B_{\mathrm{w}}$. Then $M = (U_1 \setminus F_1) \cup G_1$, where U_1 is open and $F_1 \subset U_1$ and $G_1 \subseteq U_1^{\mathrm{c}}$ are both FC. Suppose $P \in G_{B_{\mathrm{w}}}$. Then $P = U_2 \setminus F_2$, where U_2 is open and $F_2 \subseteq U_2$ is F_σ and FC. Let H_1 be an F_σ and FC subset of U_1 containing F_1 and let K_1 be an F_σ and FC set containing G_1. If $U_1 \cap U_2 \neq \emptyset$, then $(U_1 \cap U_2) \setminus [(H_1 \cup K_1 \cup F_2) \cap (U_1 \cap U_2)]$ is the desired subset $Q \in G_{B_{\mathrm{w}}}$ of P which is a subset of M. If $U_1 \cap U_2$ is empty, which is the case if M is FC (i.e. when $U_1 = \emptyset$), then $U_2 \setminus (K_1 \cup F_2)$ is the desired subset $Q \in G_{B_{\mathrm{w}}}$ of P which misses M. Thus, the "$\Rightarrow$" implications of (1) and (2) have been proved.

On the other hand, suppose M satisfies the property of (1) involving the $G_{B_{\mathrm{w}}}$ sets. Let $U_1, U_2, \ldots$ be a countable basis of nonempty open sets for the space. For each i, U_i is itself an element of $G_{B_{\mathrm{w}}}$, so let $Q_i = V_i \setminus F_i \in G_{B_{\mathrm{w}}}$ be a subset of U_i which is a subset of or misses M (assume without loss of generality that V_i is an open subset of U_i and F_i is an F_σ and FC subset of V_i). Let $U = \bigcup\{V_i \mid Q_i \subseteq M\}$. Then $N_1 = U \setminus M$ is a subset of $F_1 \cup F_2 \cup \ldots$ so $N_1 \in \mathrm{FC}$. Also $N_2 = M \setminus U$ is FC. Otherwise there would exist a U_i in which N_2 is categorically dense. Then $Q_i = V_i \setminus F_i$ must be a subset of M and $\emptyset \neq V_i \subseteq U$, which is a contradiction. Therefore, $M = (U \setminus N_1) \cup N_2 \in B_{\mathrm{w}}$ and the "$\Leftarrow$" implication of (1) is proved.

If M satisfies the property of (2) involving the $G_{B_{\mathrm{w}}}$ sets, it follows that every subset of M satisfies the similar property of (1) and that M is "hereditarily" B_{w} and therefore FC. Thus, the "$\Leftarrow$" implication of (2) is proved. ■

One now wonders if the collection of G_δ sets used in the previous theorem could be replaced with a "better" collection of Borel sets, perhaps some collection of ambiguous $F_\sigma G_\delta$ sets. The following shows that the answer is "no".

THEOREM 5. *If X is a Polish space having no isolated points*, *then there is no MB-like characterization of either* B_{w} *or* FC *based upon any collection of* F_σ *sets.*

Proof. Let X be a Polish space and let G be any collection of F_σ subsets of X.

Suppose G provides a basis for an MB-like characterization of the FC sets. If there existed an FC set $P \in G$, then P would have to have a subset $Q \in G$ which misses P, which is impossible. Therefore, all $P \in G$ are second category. Let D be a countable dense subset of X and let $P \in G$. Then

$D \in \mathrm{FC}$ and so P must have a subset $Q \in G$ which misses D. Since Q is F_σ and every closed subset of Q misses D (and is therefore nowhere dense), it follows that Q is also FC, which is a contradiction.

Suppose G provides a basis for an MB-like characterization of B_{w}. Let B be a Bernstein subset of X. Then $B \notin B_{\mathrm{w}}$, so there exists a $P \in G$ such that every subset $Q \in G$ of P intersects both B and B^{c}. Let F be an F_σ and FC set which is perfectly dense in X (i.e. every open set contains a perfect subset of F). Then the set $F_1 = F \cap B \in \mathrm{FC} \subseteq B_{\mathrm{w}}$, so P has a subset $Q \in G$ which is a subset of or misses F_1. If $Q \subseteq F_1$, then Q must be countable because F_1 has no perfect subsets. If $Q \cap F_1 = \emptyset$, then Q would necessarily be the union of countably many closed sets, each of which is nowhere dense because it misses F_1 which is dense in X. In either case, $Q_1 = Q \cap B \in FC \subset B_{\mathrm{w}}$. Both Q_1 and $Q_2 = Q \cap B^{\mathrm{c}}$ are nonempty. Since $Q_1 \in B_{\mathrm{w}}$, Q must have a subset $Q_3 \in G$ which is either a subset of or misses Q_1. Both of these possibilities would contradict the fact that Q_3 would necessarily have to intersect both B and B^{c}. ∎

It is fairly easy to show that the collection $G_{B_{\mathrm{w}}}$ is a "category base" [8], $\mathcal{C}$, with respect to which the FC sets and the B_{w} sets are the $\mathcal{C}$-meagre and $\mathcal{C}$-Baire sets, respectively. However, there are much better collections of sets which form such category bases. The open sets or the regular closed sets will work, as will the clopen sets in zero-dimensional spaces. The fact that one cannot replace the collection $G_{B_{\mathrm{w}}}$ with even a collection of F_σ sets illustrates the fact that the connection between a basis, G, of Borel sets used in an MB-like characterization and the σ-algebra and the σ-ideal being characterized is much closer than in the category base theory.

The question of whether or not there is a collection G of Borel sets which can be used as a basis for a simultaneous MB-like characterization of the (σ-ideal, σ-algebra) pair, $(\mathrm{AFC}, B_{\mathrm{r}})$, in a Polish space X with no isolated points will now be answered. It would seem at first that the collection

$$G = \{P \mid P = Q \setminus R,\ Q \text{ perfect},\ R \subseteq P \text{ an } F_\sigma \text{ set which is FC relative to } Q\}$$

would serve. It is the case that if a set M satisfies the right hand side of (i) (respectively (ii)) in the definitions of MB-like characterizations for this collection, G, then it follows that M is B_{r} (respectively AFC). However, the forward implications of (i) and (ii) fail to hold. In fact, it will be shown that there is no collection G of Borel sets which can be used as a basis for a simultaneous MB-like characterization of the σ-ideal, σ-algebra pair, $(\mathrm{AFC}, B_{\mathrm{r}})$. This will provide a partial solution to Problem 1.1 of [1] where it is asked (using different language and notation) whether there is a field of subsets of $\mathbb{R}$ for which no collection G of (Borel or non-Borel) subsets of $\mathbb{R}$ can form the basis for an MB-like characterization. There is also no collection G of Borel sets which can be used as a basis for a simultaneous MB-like characterization of the (σ-ideal, σ-algebra) pair, (U_0, U), of universal null sets and universally measurable sets. These last two collections were defined by Marczewski [6] as follows: $U = \{M \mid M$ is measurable with respect to the completion, $\overline{\mu}$, of every Borel measure, μ, on $X\}$ and $U_0 = \{M \mid M$ has measure 0 with respect to the completion, $\overline{\mu}$, of every nonatomic Borel measure, μ, on $X\}$. It will be shown that if there were such a simultaneous MB-like characterization of either of these pairs, (I, S), then the pair would have to satisfy the "Marczewski Hull Condition",

$$\forall Z \subseteq X,\ \exists M \in S : Z \subseteq M \quad \text{and} \quad \forall N \in S : Z \subseteq N,\ M \setminus N \in I,$$

which was shown by John Walsh in [13] not to be the case. The proof will mimic the proof of Theorem 3 and Corollary 1 of [13] (see [2] for another paper where these proofs have been useful in showing a different result).

LEMMA 1. *Let $I \subseteq S$ be a (σ-ideal, σ-algebra) pair on a Polish space X which has no isolated points, such that I contains all of the countable subsets of X but none of the perfect subsets of X while S contains the Borel subsets of X. Assume G is a collection of Borel sets that forms the basis for a simultaneous MB-like characterization of I and S. Then every uncountable Borel set P contains $\mathfrak{c}$-many disjoint subsets $\{Q_\alpha \mid \alpha < \mathfrak{c}\}$ from G.*

Proof. Let P be an uncountable Borel set. Then P contains $\mathfrak{c}$-many disjoint perfect sets Q'_α. Each Q'_α is in S but not in I. Since $Q'_\alpha \notin I$, it follows that there exists a $P'_\alpha \in G$ such that every subset of P'_α which is in G intersects Q'_α. Since Q'_α is in S, it follows that there is a subset Q_α of P'_α which is a subset of or misses Q'_α, and it must be a subset of Q'_α. Finally, $\{Q_\alpha \mid \alpha < \mathfrak{c}\}$ is the desired collection of disjoint subsets of P. ∎

THEOREM 6. *Let $I \subset S$ be a σ-ideal and σ-algebra on X satisfying all the hypotheses of Lemma 1 above. Then, if $Z \subseteq X$, there exists $Y \in S$ such that $Z \subseteq Y$ and if $P \in G$ is a subset of $X \setminus Z$, then $|P \cap Y| < \mathfrak{c}$.*

Proof. We proceed as in the proof of Theorem 3 of [13], letting $\mathcal{A} = \{A \in G \mid A \subseteq Z$ or else every $C \in G$ for which $C \subseteq A$ intersects both Z and $Z^{c}\}$ and $\mathcal{B} = \{B \in G \mid B \cap Z = \emptyset\}$. Following Walsh, it is noted that if $\mathcal{A}$ is empty, then Z will belong to I. This is because if $\mathcal{A} = \emptyset$, then for every $A \in G$, $A \not\subseteq Z$ *and* there exists some $C \in G$ for which $C \subseteq A$ such that either $C \cap Z = \emptyset$ or $C \subseteq Z$ (note that the latter case is impossible). It would follow that $Z \in I$ and the theorem follows. Therefore, it can be assumed that $\mathcal{A} \neq \emptyset$. For similar reasons, it may be assumed that $\mathcal{B} \neq \emptyset$. Lemma 1 was needed to prove that $|\mathcal{A}| = |\mathcal{B}| = \mathfrak{c}$. This follows from the fact that if P belongs to either $\mathcal{A}$ or $\mathcal{B}$, then $P \in G$ and P would have to be uncountable, otherwise G could not be the basis for an MB-like characterization of I, which contains all of the countable subsets of X. Therefore, P will contain $\mathfrak{c}$-many disjoint subsets which are also in G, and each of these would be in $\mathcal{A}$

or $\mathcal{B}$ for the same reason P was. Now, the rest of Walsh's proof of Theorem 3 of [13] can be changed slightly by replacing the perfect sets by the sets from G and the general theorem presented here is proved. ■

COROLLARY 1. *Let $I \subset S$ be a σ-ideal and σ-algebra on X satisfying all the hypotheses of Lemma* 1 *above. Then (I, S) satisfies the Marczewski Hull Condition.*

P r o o f. Similar to proof of Corollary 1 of [13].

COROLLARY 2. *There is no collection G of Borel sets which forms a basis for a simultaneous MB-like characterization of* AFC *and B_{r} in a Polish space X with no isolated points. No such collection exists for U_0 and U either.*

P r o o f. $(\mathrm{AFC}, B_{\mathrm{r}})$ and (U_0, U) both satisfy the hypotheses of Lemma 1 above and it was shown [13] that neither satisfies the Marczewski Hull Condition. ■

2. Measurable functions. Marczewski invented the σ-algebra (s) to show the following.

THEOREM 7. *Given a Polish space X, a separable metric space Y, and a function $f : X \to Y$, the following are equivalent*:

(1) *f is (s)-measurable*,

(2) *every perfect $P \subseteq X$ has a perfect subset Q such that $f|Q$ is continuous.*

The second statement describes the class of functions used by Sierpiński in [12] (with $X = Y = \mathbb{R}$). Similar theorems for the Lebesgue measurable and B_{w} measurable functions will now be given. The next theorem includes the Lebesgue measurable case (also see Theorem 5, Sec. III, Ch. 5 of [8]).

THEOREM 8. *If X is a Polish space having no isolated points*, *Σ is the collection of sets which are measurable with respect to the completion $\overline{\mu}$ of a finite nonatomic Borel measure μ on X*, *Σ_0 consists of the sets in Σ which have measure zero*, *and $f : X \to \mathbb{R}$*, *then the following are equivalent*:

(1) *f is Σ-measurable*,

(2) *every perfect $P \in \Sigma \setminus \Sigma_0$ has a perfect subset $Q \in \Sigma \setminus \Sigma_0$ such that $f|Q$ is continuous.*

P r o o f. The (1)⇒(2) implication follows immediately from Lusin's Theorem, so suppose $f : X \to \mathbb{R}$ is not Σ-measurable. Then there exist $t \in \mathbb{R}$ such that $[f < t]$ (notation for $\{x \mid f(x) < t\}) \notin \Sigma$. It follows that

$$\mu^{\mathrm{o}}([f < t]) + \mu^{\mathrm{o}}([t \leq f]) > \mu(X)$$

(μ^{o} denotes the outer μ-measure), and this in turn implies that there exists an $s < t$ such that

$$\mu^{\mathrm{o}}([f \le s]) + \mu^{\mathrm{o}}([t \le f]) > \mu(X).$$

Suppose the last assertion fails. Then

$$\mu^{\mathrm{o}}([f \le t - 1/n]) + \mu^{\mathrm{o}}([t \le f]) \le \mu(X)$$

for $n = 1, 2, \ldots$ It follows that for each n, one could choose an open set U_n such that $[f \le t - 1/n] \subseteq U_n$ and such that

$$\mu(U_n) \le \mu(X) - \mu^{\mathrm{o}}([t \le f]) + 1/n.$$

Note that for each n,

$$[f < t - 1/n] \subseteq U_n \cap U_{n+1} \cap \ldots,$$

so

$$[f < t] \subseteq \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} U_k = \liminf_{n\to\infty} U_n,$$

$$\begin{aligned}\mu^{\mathrm{o}}([f < t]) &\le \mu(\liminf_{n\to\infty} U_n) \le \liminf_{n\to\infty} \mu(U_n)\\ &\le \liminf_{n\to\infty}(\mu(X) - \mu^{\mathrm{o}}([t \le f]) + 1/n)\end{aligned}$$

and

$$\mu^{\mathrm{o}}([f < t]) \le \mu(X) - \mu^{\mathrm{o}}([t \le f]),$$

which is a contradiction, so the assertion is true.

Let $s < t$ be such that $\mu^{\mathrm{o}}([f \le s]) + \mu^{\mathrm{o}}([t \le f]) - \mu(X) = \varepsilon > 0$. Let G_1 and G_2 be G_δ sets containing $[f \le s]$ and $[t \le f]$, respectively, such that $\mu(G_1) + \mu(G_2) = \mu(X) + \varepsilon$ and let $G = G_1 \cap G_2$. It follows that

$$\mu(G) = \mu^{\mathrm{o}}(G \cap [f \le s]) = \mu^{\mathrm{o}}(G \cap [t \le f]) = \varepsilon.$$

Now G has a perfect subset $P \in \Sigma \setminus \Sigma_0$ and if Q is any perfect subset of P in $\Sigma \setminus \Sigma_0$, then Q intersects both $[f \le t]$ and $[s \le f]$ in sets of positive outer measure. Therefore, $f|Q$ has a point x of discontinuity for every perfect $Q \in \Sigma \setminus \Sigma_0$ (one can choose x to be any μ-density point of Q). ■

THEOREM 9. *Given a Polish space X, a separable metric space Y, and a function $f : X \to Y$, the following are equivalent*:

(1) *f is B_{w}-measurable*,

(2) *every $P \in G_{B_{\mathrm{w}}}$ has a subset $Q \in G_{B_{\mathrm{w}}}$ such that $f|Q$ is continuous.*

Proof. Suppose $f : X \to \mathbb{R}$ is B_{w}-measurable and $P \in G_{B_{\mathrm{w}}}$. Then there is a residual set $R \subseteq X$ such that $f|R$ is continuous. Assume without loss of generality that $R = X \setminus F$, where F is an F_σ and FC set, so that $R \in G_{B_{\mathrm{w}}}$. We have $P = U \setminus N$ for some open U and some F_σ and FC set $N \subseteq U$. Then

$Q = R \cap P = U \setminus (F \cup N) \in G_{B_w}$ and $f|Q$ is continuous. This establishes the (1)⇒(2) implication.

Suppose $f : X \to \mathbb{R}$ is not B_w-measurable. There exists a $t \in \mathbb{R}$ such that $[f < t] \notin B_w$. Then there is an open $U \subseteq X$ in which $[f < t]$ is categorically dense, but not residual. As $U \cap [t \le f]$ is not FC, there exists an open $V \subseteq U$ in which $[f < t]$ and $[t \le f]$ are both categorically dense. It follows that there exists an n such that $[f \le t - 1/n]$ and $[t \le f]$ are both categorically dense in some open $W \subseteq V$. Otherwise, for every n, $[f \le t - 1/n] \cap V$ would be FC, so that $[f < t] \cap V$ would be FC, which is false. Let n and W be as described. The open set W is itself in G_{B_w}. Suppose there were a $Q \in G_{B_w}$, $Q \subseteq W$, such that $f|Q$ were continuous. Then $Q = W_1 \setminus F$ for some open set W_1 and FC set $F \subseteq W_1$. We have $W \cap W_1 \neq \emptyset$ because $Q \subseteq W$. Both $[f \le t - 1/n]$ and $[t \le f]$ are categorically dense in $W \cap W_1$ and therefore both are dense in Q. It follows that $f|Q$ is discontinuous at every point of Q. ■

REMARK. One could say that the theorems of this section show that the collections $\mathcal{P}_1$ of all perfect subsets of X, $\mathcal{P}_2$ of all perfect sets $P \in \Sigma \setminus \Sigma_0$, and the collection G_{B_w} are collections of Borel sets which can be used to provide "MB-like" characterizations of the collections of (s)-measurable, Σ-measurable, and B_w-measurable functions, respectively. By considering characteristic functions and using the theorems of the previous section, it can be shown that one could not replace $\mathcal{P}_1$ or $\mathcal{P}_2$ with collections of open sets or replace G_{B_w} with any collection of F_σ sets and accomplish the same results. For similar reasons, it follows that there are no collections of Borel sets which can be used to provide MB-like characterizations of the B_r-measurable or U-measurable functions.

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