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FEJÉR MEANS OF TWO-DIMENSIONAL FOURIER TRANSFORMS ON $H_p(\mathbb{R}\times\mathbb{R})$

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Abstract. The two-dimensional classical Hardy spaces $H_p(\mathbb{R}\times\mathbb{R})$ are introduced and it is shown that the maximal operator of the Fejér means of a tempered distribution is bounded from $H_p(\mathbb{R}\times\mathbb{R})$ to $L_p(\mathbb{R}^2)$ $(1/2 < p \le \infty)$ and is of weak type $(H_1^\sharp(\mathbb{R}\times\mathbb{R}), L_1(\mathbb{R}^2))$ where the Hardy space $H_1^\sharp(\mathbb{R}\times\mathbb{R})$ is defined by the hybrid maximal function. As a consequence we deduce that the Fejér means of a function $f \in H_1^\sharp(\mathbb{R}\times\mathbb{R}) \supset L\log L(\mathbb{R}^2)$ converge to f a.e. Moreover, we prove that the Fejér means are uniformly bounded on $H_p(\mathbb{R}\times\mathbb{R})$ whenever $1/2 < p < \infty$. Thus, in case $f \in H_p(\mathbb{R}\times\mathbb{R})$, the Fejér means converge to f in $H_p(\mathbb{R}\times\mathbb{R})$ norm $(1/2 < p < \infty)$. The same results are proved for the conjugate Fejér means.

1. Introduction. The Hardy–Lorentz spaces $H_{p,q}(\mathbb{R}\times\mathbb{R})$ of tempered distributions are endowed with the $L_{p,q}(\mathbb{R}^2)$ Lorentz norms of the non-tangential maximal function. Clearly, $H_p(\mathbb{R}\times\mathbb{R}) = H_{p,p}(\mathbb{R}\times\mathbb{R})$ are the usual Hardy spaces $(0 < p \le \infty)$.

In Zygmund [22] (Vol. II, p. 246) it is shown that the Fejér means $\sigma_T f$ of a one-dimensional function $f \in L_1(\mathbb{R})$ converge to f a.e. as $T \to \infty$. Moreover, the maximal operator of the Fejér means, $\sigma_* := \sup_{T>0}|\sigma_T|$, is of weak type $(1,1)$, i.e.

$$\sup_{\gamma>0} \gamma\lambda(\sigma_* f > \gamma) \le C\|f\|_1 \qquad (f \in L_1(\mathbb{R}))$$

(see Zygmund [22], Vol. I, p. 154 and Móricz [14]). Móricz [14] also verified that σ_* is bounded from $H_1(\mathbb{R})$ to $L_1(\mathbb{R})$. The author [19] proved that σ_* is also bounded from $H_{p,q}(\mathbb{R})$ to $L_{p,q}(\mathbb{R})$ whenever $1/2 < p < \infty$, $0 < q \le \infty$.

In [16] we investigated the Fejér means of two-parameter Fourier series and proved that $\sigma_* := \sup_{n,m\in\mathbb{N}}|\sigma_{n,m}|$ is bounded from $H_{p,q}(\mathbb{T}\times\mathbb{T})$ to

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$L_{p,q}(\mathbb{T}^2)$ $(3/4 < p \leq \infty, 0 < q \leq \infty)$ and is of weak type $(H_1^\sharp(\mathbb{T}\times\mathbb{T}), L_1(\mathbb{T}^2))$, i.e.

$$\sup_{\gamma>0} \gamma\lambda(\sigma_* f| > \gamma) \leq C\|f\|_{H_1^\sharp(\mathbb{T}\times\mathbb{T})} \qquad (f \in H_1^\sharp(\mathbb{T}\times\mathbb{T})).$$

Moreover, the Fejér means $\sigma_{n,m}f$ converge to f a.e. as $n, m \to \infty$ whenever $f \in H_1^\sharp(\mathbb{T}\times\mathbb{T}) \supset L\log L(\mathbb{T}^2)$ (see Weisz [15], [16] and Zygmund [22] for $L\log L(\mathbb{T}^2)$).

In this paper we sharpen and generalize these results for the Fejér means of two-dimensional Fourier transforms.

We show that the maximal operator σ_* is bounded from $H_{p,q}(\mathbb{R}\times\mathbb{R})$ to $L_{p,q}(\mathbb{R}^2)$ whenever $1/2<p<\infty$, $0<q\leq\infty$, and is of weak type $(H_1^\sharp(\mathbb{R}\times\mathbb{R}), L_1(\mathbb{R}^2))$. We introduce the conjugate distributions $\widetilde{f}^{(i,j)}$, the conjugate Fejér means $\widetilde{\sigma}_{T,U}^{(i,j)}$ and the conjugate maximal operators $\widetilde{\sigma}_*^{(i,j)}$ $(i,j = 0,1)$. We prove that the operator $\widetilde{\sigma}_*^{(i,j)}$ is also of type $(H_{p,q}(\mathbb{R}\times\mathbb{R}), L_{p,q}(\mathbb{R}^2))$ $(1/2 < p < \infty,\ 0 < q \leq \infty)$ and of weak type $(H_1^\sharp(\mathbb{R}\times\mathbb{R}), L_1(\mathbb{R}^2))$.

A usual density argument then implies that the Fejér means $\sigma_{T,U}f$ converge to f a.e. and the conjugate Fejér means $\widetilde{\sigma}_{T,U}^{(i,j)}f$ converge to $\widetilde{f}^{(i,j)}$ $(i,j = 0,1)$ a.e. as $T, U \to \infty$ provided that $f \in H_1^\sharp(\mathbb{R}\times\mathbb{R})$. Note that $\widetilde{f}^{(i,j)}$ is not necessarily in $H_1^\sharp(\mathbb{R}\times\mathbb{R})$ whenever f is.

We also prove that the operators $\sigma_{T,U}$ and $\widetilde{\sigma}_{T,U}^{(i,j)}$ $(T, U \in \mathbb{R})$ are uniformly bounded from $H_{p,q}(\mathbb{R}\times\mathbb{R})$ to $H_{p,q}(\mathbb{R}\times\mathbb{R})$ if $1/2 < p < \infty$, $0 < q \leq \infty$. From this it follows that $\sigma_{T,U}f \to f$ and $\widetilde{\sigma}_{T,U}^{(i,j)}f \to \widetilde{f}$ $(i,j = 0,1)$ in $H_{p,q}(\mathbb{R}\times\mathbb{R})$ norm as $T, U \to \infty$ whenever $f \in H_{p,q}(\mathbb{R}\times\mathbb{R})$ and $1/2 < p < \infty$, $0 < q \leq \infty$.

2. Hardy spaces and conjugate functions. Let $\mathbb{R}$ denote the real numbers, $\mathbb{R}_+$ the positive real numbers and let λ be the 2-dimensional Lebesgue measure. We also use the notation $|I|$ for the Lebesgue measure of the set I. We briefly write L_p for the real $L_p(\mathbb{R}^2,\lambda)$ space; the norm (or quasinorm) in this space is defined by $\|f\|_p := (\int_{\mathbb{R}^2} |f|^p\, d\lambda)^{1/p}$ $(0 < p \leq \infty)$.

The distribution function of a Lebesgue-measurable function f is defined by

$$\lambda(\{|f| > \varrho\}) := \lambda(\{x : |f(x)| > \varrho\}) \qquad (\varrho \geq 0).$$

The *weak L_p space* L_p^* $(0 < p < \infty)$ consists of all measurable functions f for which

$$\|f\|_{L_p^*} := \sup_{\varrho>0} \varrho\lambda(\{|f| > \varrho\})^{1/p} < \infty$$

and we set $L_\infty^* = L_\infty$.

The spaces L_p^* are special cases of the more general, Lorentz spaces $L_{p,q}$. In their definition another concept is used. For a measurable function f the *non-increasing rearrangement* is defined by

$$\widetilde{f}(t) := \inf\{\varrho : \lambda(\{|f| > \varrho\}) \le t\}.$$

The *Lorentz space* $L_{p,q}$ is defined as follows: for $0 < p < \infty$, $0 < q < \infty$,

$$\|f\|_{p,q} := \left(\int_0^\infty \widetilde{f}(t)^q t^{q/p} \, \frac{dt}{t} \right)^{1/q},$$

while for $0 < p \le \infty$,

$$\|f\|_{p,\infty} := \sup_{t>0} t^{1/p} \widetilde{f}(t).$$

Let

$$L_{p,q} := L_{p,q}(\mathbb{R}^2, \lambda) := \{f : \|f\|_{p,q} < \infty\}.$$

One can show the following equalities:

$$L_{p,p} = L_p, \qquad L_{p,\infty} = L_p^* \qquad (0 < p \le \infty)$$

(see e.g. Bennett–Sharpley [1] or Bergh–Löfström [2]).

Let f be a tempered distribution on $C^\infty(\mathbb{R}^2)$ (briefly $f \in \mathcal{S}'(\mathbb{R}^2) = \mathcal{S}'$). The *Fourier transform* of f is denoted by $\widehat{f}$. In the special case when f is an integrable function,

$$\widehat{f}(t, u) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} f(x, y) e^{-\imath t x} e^{-\imath u y} \, dx \, dy \qquad (t, u \in \mathbb{R})$$

where $\imath = \sqrt{-1}$.

For $f \in \mathcal{S}'$ and $t, u > 0$ let

$$F(x, y; t, u) := (f * P_t \times P_u)(x, y)$$

where $*$ denotes convolution and

$$P_t(x) := \frac{ct}{t^2 + x^2} \qquad (x \in \mathbb{R})$$

is the Poisson kernel.

For $\alpha > 0$ let

$$\Gamma_\alpha := \{(x, t) : |x| < \alpha t\},$$

a cone with vertex at the origin. We denote by $\Gamma_\alpha(x)$ ($x \in \mathbb{R}$) the translate of Γ_α with vertex at x. The non-tangential maximal function is defined by

$$F^*_{\alpha,\beta}(x, y) := \sup_{(x',t) \in \Gamma_\alpha(x),\, (y',u) \in \Gamma_\beta(y)} |F(x', y'; t, u)| \qquad (\alpha, \beta > 0).$$

For $0 < p, q \le \infty$ the *Hardy–Lorentz space* $H_{p,q}(\mathbb{R} \times \mathbb{R}) = H_{p,q}$ consists of all tempered distributions f for which $F^*_{\alpha,\beta} \in L_{p,q}$; we set

$$\|f\|_{H_{p,q}} := \|F^*_{1,1}\|_{p,q}.$$

For $0 < p < \infty$, $0 < q \le \infty$ Chang and Fefferman [3] and Lin [12] proved the equivalence $\|F^*_{\alpha,\beta}\|_{p,q} \sim \|F^*_{1,1}\|_{p,q}$ $(\alpha, \beta > 0)$. It is known that if $f \in H_p$ $(0 < p < \infty)$ then $f(x,y) = \lim_{t,u\to 0} F(x,y;t,u)$ in the sense of distributions (see Gundy–Stein [11], Chang–Fefferman [3]).

Let us introduce the hybrid Hardy spaces. For $f \in L_1$ and $t > 0$ let

$$G(x,y;t) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(v,y) P_t(x-v)\, dv$$

and

$$G^+_\alpha(x,y) := \sup_{(x',t)\in\Gamma_\alpha(x)} |G(x',y;t)| \qquad (0 < \alpha < 1).$$

We say that $f \in L_1$ is in the *hybrid Hardy–Lorentz space* $H^\sharp_{p,q}(\mathbb{R}\times\mathbb{R}) = H^\sharp_{p,q}$ if

$$\|f\|_{H^\sharp_{p,q}} := \|G^+_{1/2}\|_{p,q} < \infty.$$

The equivalences $\|G^+_\alpha\|_{p,q} \sim \|G^+_1\|_{p,q}$ $(\alpha > 0,\ 0 < p < \infty,\ 0 < q \le \infty)$ and

$$H_{p,q} \sim H^\sharp_{p,q} \sim L_{p,q} \qquad (1 < p < \infty,\ 0 < q \le \infty)$$

were proved in Fefferman–Stein [7], Gundy–Stein [11] and Lin [12]. Note that for $p = q$ the usual definitions of the Hardy spaces $H_{p,p} = H_p$ and $H^\sharp_{p,p} = H^\sharp_p$ are obtained.

The following interpolation result concerning Hardy–Lorentz spaces will be used several times in this paper (see Lin [12] and also Weisz [17]).

THEOREM A. *If a sublinear* (*resp. linear*) *operator* V *is bounded from* H_{p_0} *to* L_{p_0} (*resp. to* H_{p_0}) *and from* L_{p_1} *to* L_{p_1} $(p_0 \le 1 < p_1 < \infty)$ *then it is also bounded from* $H_{p,q}$ *to* $L_{p,q}$ (*resp. to* $H_{p,q}$) *if* $p_0 < p < p_1$ *and* $0 < q \le \infty$.

In this paper the constants C are absolute, while C_p (resp. $C_{p,q}$) depend only on p (resp. p and q) and may be different in different contexts.

One can prove similarly to the discrete case (see Weisz [16]) that $L\log L := L\log L(\mathbb{R}^2) \subset H^\sharp_1 \subset H_{1,\infty}$, more exactly,

$$\|f\|_{H_{1,\infty}} = \sup_{\varrho>0} \varrho\lambda(F^*_{1,1} > \varrho) \le C\|f\|_{H^\sharp_1} \qquad (f \in H^\sharp_1) \tag{1}$$

and

$$\|f\|_{H^\sharp_1} \le C + C\||f|\log^+|f|\|_1 \qquad (f \in L\log L)$$

where $\log^+ u = 1_{\{u>1\}} \log u$.

For a tempered distribution $f \in H_p$ $(0 < p < \infty)$ the *Hilbert transforms* or *conjugate distributions* $\widetilde{f}^{(1,0)}$, $\widetilde{f}^{(0,1)}$ and $\widetilde{f}^{(1,1)}$ are defined by

$$(\widetilde{f}^{(1,0)})^\wedge(t,u) := (-\imath \operatorname{sign} t)\widehat{f}(t,u) \qquad (t,u \in \mathbb{R})$$

(conjugate with respect to the first variable),

$$(\widetilde{f}^{(0,1)})^\wedge(t,u) := (-\imath \operatorname{sign} u)\widehat{f}(t,u) \qquad (t,u \in \mathbb{R})$$

(conjugate with respect to the second variable) and

$$(\widetilde{f}^{(1,1)})^\wedge(t,u) := (-\operatorname{sign}(tu))\widehat{f}(t,u) \qquad (t,u \in \mathbb{R})$$

(conjugate with respect to both variables). We use the notation $\widetilde{f}^{(0,0)} := f$.

Gundy and Stein [10], [11] verified that if $f \in H_p$ $(0 < p < \infty)$ then all conjugate distributions are also in H_p and

$$\|f\|_{H_p} = \|\widetilde{f}^{(i,j)}\|_{H_p} \qquad (i,j = 0,1). \tag{2}$$

Furthermore (see also Chang and Fefferman [3], Frazier [9], Duren [5]),

$$\|f\|_{H_p} \sim \|f\|_p + \|\widetilde{f}^{(1,0)}\|_p + \|\widetilde{f}^{(0,1)}\|_p + \|\widetilde{f}^{(1,1)}\|_p. \tag{3}$$

As is well known, if f is an integrable function then

$$\widetilde{f}^{(1,0)}(x,y) = \text{p.v.}\ \frac{1}{\pi}\int_{\mathbb{R}} \frac{f(x-t,y)}{t}\,dt := \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{\varepsilon<|t|} \frac{f(x-t,y)}{t}\,dt,$$

$$\widetilde{f}^{(0,1)}(x,y) = \text{p.v.}\ \frac{1}{\pi}\int_{\mathbb{R}} \frac{f(x,y-u)}{u}\,du,$$

$$\widetilde{f}^{(1,1)}(x,y) = \text{p.v.}\ \frac{1}{\pi^2}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{f(x-t,y-u)}{tu}\,dt\,du.$$

Moreover, the conjugate functions $\widetilde{f}^{(1,0)}$, $\widetilde{f}^{(0,1)}$ and $\widetilde{f}^{(1,1)}$ exist almost everywhere, but they are not integrable in general. Similarly, if $f \in H_1^\sharp$ then $\widetilde{f}^{(0,1)}$ and $\widetilde{f}^{(1,1)}$ are not necessarily in $H_1^\sharp$.

3. Fejér means. Suppose first that $f \in L_p$ for some $1 \le p \le 2$. It is known that under certain conditions

$$f(x,y) = \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \widehat{f}(t,u)e^{\imath xt}e^{\imath yu}\,dt\,du \qquad (x,y \in \mathbb{R}).$$

This motivates the definition of the *Dirichlet integral* $s_{t,u}f$:

$$s_{t,u}f(x,y) := \frac{1}{2\pi}\int_{-t}^{t}\int_{-u}^{u} \widehat{f}(v,w)e^{\imath xv}e^{\imath yw}\,dv\,dw \qquad (t,u>0).$$

The *conjugate Dirichlet integrals* are introduced by

$$\widetilde{s}_{t,u}^{(1,0)}f(x,y) := \frac{1}{2\pi}\int_{-t}^{t}\int_{-u}^{u} (-\imath \operatorname{sign} v)\widehat{f}(v,w)e^{\imath xv}e^{\imath yw}\,dv\,dw \qquad (t,u>0),$$

$$\widetilde{s}_{t,u}^{(0,1)}f(x,y) := \frac{1}{2\pi}\int_{-t}^{t}\int_{-u}^{u} (-\imath \operatorname{sign} w)\widehat{f}(v,w)e^{\imath xv}e^{\imath yw}\,dv\,dw \qquad (t,u>0)$$

and

$$\widetilde{s}_{t,u}^{(1,1)} f(x,y) := \frac{1}{2\pi} \int_{-t}^{t} \int_{-u}^{u} (-\operatorname{sign}(vw)) \widehat{f}(v,w) e^{\imath xv} e^{\imath yw} \, dv \, dw \qquad (t,u>0).$$

The *Fejér* and *conjugate Fejér means* are defined by

$$\widetilde{\sigma}_{T,U}^{(i,j)} f(x,y) := \frac{1}{TU} \int_0^T \int_0^U \widetilde{s}_{t,u}^{(i,j)} f(x,y) \, dt \, du \qquad (T,U>0;\ i,j=0,1).$$

We write $s_{t,u} f =: \widetilde{s}_{t,u}^{(0,0)} f$ and $\sigma_{T,U} f := \widetilde{\sigma}_{T,U}^{(0,0)} f$. It is easy to see that

$$s_{t,u} f(x,y) := \int_{\mathbb{R}} \int_{\mathbb{R}} f(x-v,y-w) \frac{\sin tv}{\pi v} \cdot \frac{\sin uw}{\pi w} \, dv \, dw$$

and

$$\sigma_{T,U} f(x,y) := \int_{\mathbb{R}} \int_{\mathbb{R}} f(x-t,y-u) K_T(t) K_U(u) \, dt \, du$$

where

$$K_T(t) := \frac{2}{\pi} \cdot \frac{\sin^2(Tt/2)}{Tt^2}$$

is the Fejér kernel. Note that

$$\int_{\mathbb{R}} K_T(t) \, dt = 1 \qquad (T>0) \tag{4}$$

(see Zygmund [22], Vol. II, pp. 250–251).

We extend the definition of the Fejér means and conjugate Fejér means to tempered distributions as follows:

$$\widetilde{\sigma}_{T,U}^{(i,j)} f := \widetilde{f}^{(i,j)} * (K_T \times K_U) \qquad (T,U>0;\ i,j=0,1).$$

One can show that $\widetilde{\sigma}_{T,U}^{(i,j)} f$ is well defined for all tempered distributions $f \in H_p$ $(0<p\le\infty)$ and for all functions $f \in L_p$ $(1 \le p \le \infty)$ (cf. Fefferman–Stein [7]).

The *maximal* and *maximal conjugate Fejér operators* are defined by

$$\widetilde{\sigma}_*^{(i,j)} f := \sup_{T,U>0} |\widetilde{\sigma}_{T,U}^{(i,j)} f| \qquad (i,j=0,1).$$

We again write $\sigma_* f := \widetilde{\sigma}_*^{(0,0)} f$.

4. The boundedness of the maximal Fejér operator. A function $a \in L_2$ is called a *rectangle p-atom* if there exists a rectangle $R \subset \mathbb{R}^2$ such that

(i) $\operatorname{supp} a \subset R$,
(ii) $\|a\|_2 \le |R|^{1/2-1/p}$,

(iii) for all $x, y \in \mathbb{R}$ and all $N \le [2/p - 3/2]$,

$$\int_{\mathbb{R}} a(x,y)x^N \, dx = \int_{\mathbb{R}} a(x,y)y^N \, dy = 0.$$

If I is an interval then let rI be the interval with the same center as I and with length $r|I|$ $(r \in \mathbb{N})$. For a rectangle $R = I \times J$ let $rR = rI \times rJ$.

An operator V which maps the set of tempered distributions into the collection of measurable functions will be called *p-quasi-local* if there exist a constant $C_p > 0$ and $\eta > 0$ such that for every rectangle p-atom a supported on the rectangle R and for every $r \ge 2$ one has

$$\int_{\mathbb{R}^2 \setminus 2^r R} |Ta|^p \, d\lambda \le C_p 2^{-\eta r}.$$

Although H_p cannot be decomposed into rectangle p-atoms, in the next theorem it is enough to take such atoms (see Weisz [16], Fefferman [8]).

THEOREM B. *Suppose that the operator V is sublinear and p-quasi-local for some $0 < p \le 1$. If V is bounded from L_2 to L_2 then*

$$\|Vf\|_p \le C_p \|f\|_{H_p} \qquad (f \in H_p).$$

Since the Fejér kernel is positive, we can prove the following inequality in the same way as in the discrete case (see Weisz [18]):

$$\|\sigma_* f\|_p \le C_p \|f\|_p \qquad (1 < p \le \infty). \tag{5}$$

Now we can formulate our main result.

THEOREM 1. *We have*

$$\|\sigma_* f\|_{p,q} \le C_{p,q} \|f\|_{H_{p,q}} \qquad (f \in H_{p,q}) \tag{6}$$

for every $1/2 < p < \infty$ and $0 < q \le \infty$. In particular, if $f \in H_1^\sharp$ then

$$\lambda(\sigma_* f > \varrho) \le \frac{C}{\varrho} \|f\|_{H_1^\sharp} \qquad (\varrho > 0). \tag{7}$$

Proof. First we will show that the operator σ_* is p-quasi-local for each $1/2 < p \le 1$. To this end let a be an arbitrary rectangle p-atom with support $R = I \times J$ and

$$2^{K-1} < |I| \le 2^K, \qquad 2^{L-1} < |J| \le 2^L \qquad (K, L \in \mathbb{Z}).$$

We can suppose that the center of R is zero. In this case

$$[-2^{K-2}, 2^{K-2}] \subset I \subset [-2^{K-1}, 2^{K-1}]$$

and

$$[-2^{L-2}, 2^{L-2}] \subset J \subset [-2^{L-1}, 2^{L-1}].$$

To prove the p-quasi-locality of the operator σ_* we have to integrate $|\sigma_* a|^p$ over

$$\mathbb{R}^2 \setminus 2^r R = (\mathbb{R} \setminus 2^r I) \times J \cup (\mathbb{R} \setminus 2^r I) \times (\mathbb{R} \setminus J) \cup I \times (\mathbb{R} \setminus 2^r J) \cup (\mathbb{R} \setminus I) \times (\mathbb{R} \setminus 2^r J)$$

where $r \geq 2$ is an arbitrary integer.

First we integrate over $(\mathbb{R} \setminus 2^r I) \times J$. Obviously,

$$\int_{\mathbb{R}\setminus 2^r I} \int_J |\sigma_* a(x,y)|^p \, dx \, dy \leq \sum_{|i|=2^{r-2}}^{\infty} \int_{i2^K}^{(i+1)2^K} \int_J |\sigma_* a(x,y)|^p \, dx \, dy.$$

For $x, y \in \mathbb{R}$ let

$$A_{1,0}(x,y) := \int_{-\infty}^{x} a(t,y)\, dt, \qquad A_{0,1}(x,y) := \int_{-\infty}^{y} a(x,u)\, du$$

and

$$A_{1,1}(x,y) := \int_{-\infty}^{x} \int_{-\infty}^{y} a(t,y)\, dt\, du.$$

By (iii) of the definition of the rectangle atom we can show that $\operatorname{supp} A_{k,l} \subset R$ and $A_{k,l}$ is zero at the vertices of R $(k, l = 0, 1)$. Moreover, using (ii) we can compute that

$$\|A_{k,l}\|_2 \leq |I|^k |J|^l (|I| \cdot |J|)^{1/2-1/p} \qquad (k, l = 0, 1). \tag{8}$$

Integrating by parts we can see that

$$|\sigma_{T,U} a(x,y)| = \left| \int_I \int_J A_{1,0}(t,u) K_T'(x-t) K_U(y-u)\, dt\, du \right| \leq \int_I \left| \int_J A_{1,0}(t,u) K_U(y-u)\, du \right| |K_T'(x-t)|\, dt.$$

Using the inequality

$$|K_T'(t)| \leq C/t^2 \qquad (T \in \mathbb{R}_+)$$

we get

$$|\sigma_{T,U} a(x,y)| \leq \int_I \left| \int_J A_{1,0}(t,u) K_U(y-u)\, du \right| \frac{C}{|x-t|^2}\, dt \leq \frac{C 2^{-2K}}{i^2} \int_I \left| \int_J A_{1,0}(t,u) K_U(y-u)\, du \right| dt$$

for $x \in [i2^K, (i+1)2^K)$. Hölder's inequality, the one-dimensional version of (5) and (8) imply

$$\begin{aligned}
&\int_J |\sigma_* a(x,y)|^p\,dy \\
&\quad\le \frac{C_p 2^{-2Kp}}{i^{2p}}|J|^{1-p}\Big(\int_I\int_J \sup_{U\in\mathbb{R}_+}\Big|\int_J A_{1,0}(t,u)K_U(y-u)\,du\Big|\,dy\,dt\Big)^p \\
&\quad\le \frac{C_p 2^{-2Kp}|J|^{1-p/2}}{i^{2p}}\Big(\int_I\Big(\int_{\mathbb{R}}\sup_{U\in\mathbb{R}_+}\Big|\int_J A_{1,0}(t,u)K_U(y-u)\,du\Big|^2\,dy\Big)^{1/2}dt\Big)^p \\
&\quad\le \frac{C_p 2^{-2Kp}|J|^{1-p/2}}{i^{2p}}\Big(\int_I\Big(\int_J |A_{1,0}(t,y)|^2\,dy\Big)^{1/2}dt\Big)^p \\
&\quad\le \frac{C_p 2^{-2Kp}|I|^{p/2}|J|^{1-p/2}}{i^{2p}}\Big(\int_I\int_J |A_{1,0}(t,y)|^2\,dy\,dt\Big)^{p/2} \\
&\quad\le \frac{C_p 2^{-2Kp}|I|^{2p-1}}{i^{2p}}.
\end{aligned}$$

Hence

$$\int_{\mathbb{R}\setminus 2^r I}\int_J |\sigma_* a(x,y)|^p\,dx\,dy \le C_p\sum_{i=2^{r-2}}^{\infty} 2^K\frac{2^{-K}}{i^{2p}} \le C_p 2^{-r(2p-1)}.$$

Next we integrate over $(\mathbb{R}\setminus 2^r I)\times(\mathbb{R}\setminus J)$:

$$\int_{\mathbb{R}\setminus 2^r I}\int_{\mathbb{R}\setminus J} |\sigma_* a(x,y)|^p\,dx\,dy \le \sum_{|i|=2^{r-2}}^{\infty}\sum_{|j|=1}^{\infty}\int_{i2^K}^{(i+1)2^K}\int_{j2^L}^{(j+1)2^L} |\sigma_* a(x,y)|^p\,dx\,dy.$$

Integrating by parts we obtain, for $x\in[i2^K,(i+1)2^K)$ and $y\in[i2^L,(i+1)2^L)$,

$$\begin{aligned}
|\sigma_{T,U}a(x,y)| &= \Big|\int_I\int_J A_{1,1}(t,u)K_T'(x-t)K_U'(y-u)\,dt\,du\Big| \\
&\le \frac{C2^{-2K}2^{-2L}}{i^2 j^2}\int_I\int_J |A_{1,1}(t,u)|\,dt\,du \\
&\le \frac{C2^{-2K}2^{-2L}|I|^{2-1/p}|J|^{2-1/p}}{i^2 j^2}.
\end{aligned}$$

Thus

$$\int_{\mathbb{R}\setminus 2^r I}\int_{\mathbb{R}\setminus J} |\sigma_* a(x,y)|^p\,dx\,dy \le C_p\sum_{|i|=2^{r-2}}^{\infty}\sum_{|j|=1}^{\infty} 2^{K+L}\frac{2^{-K}2^{-L}}{i^{2p}j^{2p}} \le C_p 2^{-r(2p-1)}.$$

The integrations over $I\times(\mathbb{R}\setminus 2^r J)$ and over $(\mathbb{R}\setminus I)\times(\mathbb{R}\setminus 2^r J)$ are similar. Hence σ_* is p-quasi-local. Theorem B implies (6) for $p=q$. Applying Theorem A and (5) we obtain (6).

Let us single out this result for $p=1$ and $q=\infty$. If $f\in H_1^\sharp$ then (1) implies

$$\|\sigma_* f\|_{1,\infty}=\sup_{\varrho>0}\gamma\lambda(\sigma_* f>\varrho)\le C\|f\|_{H_{1,\infty}}\le C\|f\|_{H_1^\sharp},$$

which shows (7). The proof of the theorem is complete. ∎

Note that Theorem 1 was proved for Fourier series and for $3/4<p<\infty$ by the author [16] with another method.

We can state the same for the maximal conjugate Fejér operator.

THEOREM 2. *For* $i,j=0,1$ *we have*

$$\|\widetilde{\sigma}_*^{(i,j)} f\|_{p,q}\le C_{p,q}\|f\|_{H_{p,q}}\qquad (f\in H_{p,q})$$

for every $1/2<p<\infty$ *and* $0<q\le\infty$. *In particular, if* $f\in H_1^\sharp$ *then*

$$\lambda(\widetilde{\sigma}_*^{(i,j)} f>\varrho)\le \frac{C}{\varrho}\|f\|_{H_1^\sharp}\qquad (\varrho>0).$$

P r o o f. By Theorem 1 for $p=q$ and (2) we obtain

$$\|\widetilde{\sigma}_*^{(i,j)} f\|_p=\|\sigma_*\widetilde{f}^{(i,j)}\|_p\le C_p\|\widetilde{f}^{(i,j)}\|_{H_p}=C_p\|f\|_{H_p}\qquad (f\in H_p)$$

for every $1/2<p<\infty$. Now Theorem 2 follows from Theorem A and (1). ∎

Since the set of those functions $f\in L_1$ whose Fourier transform has a compact support is dense in $H_1^\sharp$ (see Wiener [20]), the weak type inequalities of Theorems 1 and 2 and the usual density argument (see Marcinkiewicz–Zygmund [13]) imply

COROLLARY 1. *If* $f\in H_1^\sharp$ $(\supset L\log L)$ *and* $i,j=0,1$ *then*

$$\widetilde{\sigma}_{T,U}^{(i,j)} f\to \widetilde{f}^{(i,j)}\qquad a.e.\ as\ T,U\to\infty.$$

Note that $\widetilde{f}^{(i,j)}$ is not necessarily in $H_1^\sharp$ whenever f is.

Now we consider the norm convergence of $\sigma_{T,U}f$. It follows from (5) that $\sigma_{T,U}f\to f$ in L_p norm as $T,U\to\infty$ if $f\in L_p$ $(1<p<\infty)$. We are going to generalize this result.

THEOREM 3. *Assume that* $T,U\in\mathbb{R}_+$ *and* $i,j=0,1$. *Then*

$$\|\widetilde{\sigma}_{T,U}^{(i,j)} f\|_{H_{p,q}}\le C_{p,q}\|f\|_{H_{p,q}}\qquad (f\in H_{p,q})$$

for every $1/2<p<\infty$ *and* $0<q\le\infty$.

P r o o f. Since $(\sigma_{T,U}f)^{\sim(i,j)}=\widetilde{\sigma}_{T,U}^{(i,j)} f$, by Theorem 2 we have

$$\|(\sigma_{T,U}f)^{\sim(i,j)}\|_p\le C_p\|f\|_{H_p}\qquad (f\in H_p)$$

for all $T,U\in\mathbb{R}_+$ and $i,j=0,1$. (3) implies that

$$\|\sigma_{T,U}f\|_{H_p}\le C_p\|f\|_{H_p}\qquad (f\in H_p;\ T,U\in\mathbb{R}_+).$$

Hence, for $i, j = 0, 1$,

$$\|\widetilde{\sigma}_{T,U}^{(i,j)} f\|_{H_p} \le C_p \|f\|_{H_p} \qquad (f \in H_p;\ T, U \in \mathbb{R}_+).$$

which together with Theorem A implies Theorem 3.

COROLLARY 2. *Suppose that* $1/2 < p < \infty$, $0 < q \le \infty$ *and* $i, j = 0, 1$. *If* $f \in H_{p,q}$ *then*

$$\widetilde{\sigma}_{T,U}^{(i,j)} f \to \widetilde{f}^{(i,j)} \quad \textit{in } H_{p,q} \textit{ norm as } T, U \to \infty.$$

We suspect that Theorems 1, 2 and 3 are not true for $p \le 1/2$ though we could not find any counterexample.

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