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QUOTIENTS OF TORIC VARIETIES BY ACTIONS OF SUBTORI

BY

JOANNA ŚWIĘCICKA (WARSZAWA)

Abstract. Let X be an algebraic toric variety with respect to an action of an algebraic torus S. Let Σ be the corresponding fan. The aim of this paper is to investigate open subsets of X with a good quotient by the (induced) action of a subtorus $T \subset S$. It turns out that it is enough to consider open S-invariant subsets of X with a good quotient by T. These subsets can be described by subfans of Σ. We give a description of such subfans and also a description of fans corresponding to quotient varieties. Moreover, we give conditions for a subfan to define an open subset with a complete quotient space.

Introduction. Let X be a toric variety with respect to an action of a torus S and let T be a subtorus of S. In this paper we study quotients of open subsets of X by the induced action of T. If there exists a good quotient $q : U \to U//T$ and $V \subset U$ is a T-invariant subset such that the closures of T-orbits in U and in V coincide, then there exists a good quotient $q_1 : V \to V//T$ and the induced morphism of quotient spaces $U//T \to U//T$ is an open embedding. Such a $V \subset U$ is called a saturated subset of U. Any T-invariant open subset with a good quotient with respect to T is contained as a saturated subset in a T-maximal set, i.e. a set which is not properly contained as a saturated subset in any subvariety of X which admits a good quotient.

First we prove that any T-maximal set $U \subset X$ is a toric subvariety of X (see Corollary 2.4). Then for a given subtorus T of S we give a description of the fan of any toric variety X which admits a good quotient with respect to the induced action of T (Theorem 4.1). The good quotient of a toric variety is again a toric variety with respect to an action of some quotient of S/T. Theorem 4.1 gives the construction of the fan of $X//T$.

In the last section we give a description of the fan of any open, T-maximal subset U in X (see Theorem 5.2). This problem was solved in [2] in the particular case of $X = \mathbb{P}^n$, and in [5] in the case of a vector space.

Questions connected with quotients of toric varieties were also considered in [7] and [6]. In [7] only projective toric varieties were considered and the

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problem of existence of the Chow quotient was investigated. In [6], for a given toric variety X, the author found an open toric subvariety U of the affine toric variety $\mathbb{C}^n$ such that X is a quotient of U by an action of a subtorus $T \subset (\mathbb{C}^*)^n$ extended by a finite group. I was informed that also H. A. Hamm of Münster University considered similar problems and independently obtained results concerning good quotients of toric varieties.

I should also mention that Corollary 2.4 follows from Corollary to Theorem II in [1], but the proof of this general theorem is much more involved.

1. Notation and terminology. All varieties and algebraic spaces considered are assumed to be defined over the field $\mathbb{C}$ of complex numbers. Let G be an algebraic reductive group acting on an algebraic variety X and let Y be an algebraic variety with a trivial action of G.

DEFINITION 1.1. A G-morphism $q : X \to Y$ is said to be a *good quotient* if the following conditions are satisfied:

(i) q is affine,
(ii) $\mathcal{O}_Y = \pi_*(\mathcal{O}_X)^G$.

Let $U \subset V$ be G-invariant subvarieties of X. Then U is G-*saturated* in V if for any $x \in U$, the closures of the G-orbit Gx in U and in V coincide. We recall from [2]

DEFINITION 1.2. An open G-invariant subset U in X is called G-*maximal* if there exists a good quotient $U \to U//G$ and if U is maximal in X with respect to saturated inclusion in the family of all open, G-invariant subsets of X which admit a good quotient with respect to the action of G.

Let S be an algebraic torus and let $N(S)$ be the $\mathbb{Z}$-module of one-parameter subgroups of S. Denote by $E(S)$ the vector space $N(S) \otimes_{\mathbb{Z}} \mathbb{R}$. For any subtorus $T \subset S$ we shall consider $N(T)$ and $E(T)$ as embedded in $N(S)$ and $E(S)$ respectively.

By a *cone* we always mean a convex cone in $E(S)$ which is generated by a finite number of vectors from $N(S)$.

In the set of all strictly convex cones in $E(S)$ we have a (partial) order $\prec$: for any strictly convex cones σ, σ', $\sigma' \prec \sigma$ if and only if σ' is a face of σ. For any cone $\sigma \subset E(S)$, we denote its relative interior by σ°.

A collection $\Sigma = \{\sigma_1, \ldots, \sigma_m\}$ of strictly convex cones is a *fan* if

(i) for any $\sigma_i, \sigma_j \in \Sigma$, $\sigma_i \cap \sigma_j \prec \sigma_i$, and
(ii) if $\sigma_1 \in \Sigma$ and $\sigma \prec \sigma_1$ then $\sigma \in \Sigma$.

If Σ is a fan then we denote by $\Sigma_{\max}$ the subset of Σ consisting of all cones maximal with respect to $\prec$.

For any collection $\{\sigma_1, \ldots, \sigma_m\}$ of strictly convex cones satisfying

$$\sigma_i \cap \sigma_j \prec \sigma_i \quad \text{for } i, j = 1, \ldots, m, \tag{1}$$

we can define a fan $\Sigma(\sigma_1, \ldots, \sigma_m) = \bigcup_{i=1}^m \{\sigma : \sigma \prec \sigma_i\}$.

Let X be a toric variety with respect to an action of S (see [9]). It will be called an *S-toric variety*. Then in particular we have a distinguished point x_0 of the dense orbit and we consider S embedded in X by the morphism $i : S \to X$ where $i(s) = s \cdot x_0$. Let $\Sigma(X)$ be the fan corresponding to X. A strictly convex cone $\sigma \subset E(S)$ is contained in $\Sigma(X)$ if and only if there exists an open, S-invariant, affine subset $U(\sigma) \subset X$ such that σ is generated (as a cone with vertex 0) by all one-parameter subgroups $\alpha \in N(S)$ satisfying the following condition: $\lim_{t\to 0} \alpha(t)x_0$ exists in $U(\sigma)$. Moreover for any open, S-invariant subsets $U(\sigma_1)$, $U(\sigma_2)$, we have $U(\sigma_1) \subset U(\sigma_2)$ if and only if $\sigma_1 \prec \sigma_2$.

For any fan Σ in $E(S)$ there exists a unique (up to isomorphism) normal toric variety $U(\Sigma)$ corresponding to this fan. For any point $x \in U(\Sigma)$ there is a unique cone $\sigma(x)$ of minimal dimension such that $x \in U(\sigma)$. Then Sx is the unique closed orbit of S contained in $U(\sigma(x))$. The relative interior of $\sigma(x)$ will be denoted by $\sigma(x)^\circ$. It follows from the definition of $\sigma(x)$ that if $x = \lim_{t\to 0} \alpha(t)x_0$ for a one-parameter subgroup $\alpha \in E(S)$ then $\alpha \in \sigma(x)^\circ$ and the isotropy group S_x is generated by all one-dimensional subtori of S corresponding to the one-parameter subgroups $\alpha \in \operatorname{lin}(\sigma(x))$. Moreover

$$\sigma(x) \prec \sigma(y) \Leftrightarrow y \in \overline{Sx}. \tag{2}$$

Let $g : S \to S'$ be a homomorphism of algebraic tori and $\pi : E(S) \to E(S')$ be the linear map induced by the morphism of $\mathbb{Z}$-modules $N(S) \to N(S')$. Assume that X is a toric variety with respect to an action of S, Y a toric variety with respect to an action of S', and $q : X \to Y$ a morphism such that $q|S = g$ (we consider S and S' as subsets of X and Y respectively). Let Σ, Υ be the fans in $E(S)$ and $E(S')$ defining X and Y respectively. Then for any $\sigma \in \Sigma$ there exists a cone $\tau \in \Upsilon$ such that $\pi(\sigma) \subset \tau$.

LEMMA 1.3. *Let X, Y, π, Σ, Υ and $q : X \to Y$ be as above. Then q is affine if and only if for every $\tau \in \Upsilon$, there exists $\sigma \in \Sigma$ such that*

$$\forall \sigma' \in \Sigma : \quad \sigma' \subset \pi^{-1}(\tau) \Leftrightarrow \sigma' \prec \sigma. \tag{3}$$

Proof. Let V be an open, affine S'-invariant subvariety in Y. Then $V \simeq U(\tau)$ for a convex cone $\tau \in \Upsilon$. Assume first that q is affine. Then $q^{-1}(V)$ is affine and S-invariant and therefore it corresponds to a convex cone $\sigma \in \Sigma$. Obviously $\pi(\sigma) \subset \tau$. Let $\sigma' \in \Sigma$ and assume that $\sigma' \subset \pi^{-1}(\tau)$. Then $U(\sigma') \subset q^{-1}(V)$. Since $q^{-1}(V) = U(\sigma)$ this is equivalent to $\sigma' \prec \sigma$.

Assume now that there exists $\sigma \in \Sigma$ such that (3) is satisfied. The open set $q^{-1}(U(\tau))$ is an open subvariety of X, invariant under the action of S.

Therefore it is a toric variety (with the action of S) and it corresponds to a subfan $\Sigma' \subset \Sigma$. By assumption Σ' is the fan of faces of the cone σ and hence $q^{-1}(U(\tau))$ is the affine toric variety $U(\sigma)$. This proves that q is affine and ends the proof.

Let $\alpha : \mathbb{C}^* \to S$ be a one-parameter subgroup of S. In Section 3 we shall need

LEMMA 1.4. *Let $U(\Sigma)$ be an S-toric variety and $x, y \in U(\Sigma)$. Assume that $y = \lim_{t\to 0} \alpha(t)x$. Then $\sigma(x) \prec \sigma(y)$ and $(\sigma(x)^\circ + \{\alpha\}) \cap \sigma(y)^\circ \neq \emptyset$.*

Proof. In this case $y \in \overline{Sx}$, hence $\sigma(x) \prec \sigma(y)$ by (2). Let β be any one-parameter subgroup of S such that $\lim_{t\to 0} \beta(t)x_0 = sx$. Then $\beta \in \sigma(x)^\circ$. Consider the subtorus T_0 generated by $\alpha(\mathbb{C}^*)$ and $\beta(\mathbb{C}^*)$ in S. Let Y be the closure of the orbit T_0x_0 in $U(\Sigma)$. Then $x, y \in Y$, and $y \in \overline{T_0x}$. There exist $n, m \in \mathbb{N}$ such that $\lim_{t\to 0}(n\alpha + m\beta)(t)x_0 = sy$ for some $s \in T_0$. It follows that $(n\alpha + \sigma(x)^\circ) \cap \sigma(y)^\circ \neq \emptyset$. This implies that $(\alpha + \sigma(x)^\circ) \cap \sigma(y)^\circ \neq \emptyset$, completing the proof.

2. Two theorems on existence of good quotients. Assume that T is a torus contained in the torus S. Let X be a toric variety with respect to an action of S given by a fan Σ. Then we have an induced action of T on X. We shall prove (Corollary 2.4) that all T-maximal subsets U in X are S-invariant and therefore are also toric varieties with respect to the action of S. We first consider the general situation of actions of algebraic groups H and G on X.

THEOREM 2.1. *Let X be an algebraic variety and H, G be subgroups of* $\mathrm{Aut}(X)$. *Assume that H is connected, G is reductive and for any $h \in H$ and $g \in G$, $hgh^{-1} \in G$ (i.e. H normalizes G in* $\mathrm{Aut}(X)$). *Let U be an open, G-invariant subset of X such that there exists a good quotient $U \to U//G$. Then there exists a good quotient $H \cdot U \to H \cdot U//G$.*

Proof. Consider any points $x_1, x_2 \in H \cdot U$ and let $H_i = \{h \in H : hx_i \in U\}$ for $i = 1, 2$. Since U is open, the sets H_i, $i = 1, 2$, are open subsets of the connected group H. Hence there exists $h \in H$ such that $hx_i \in U$ for $i = 1, 2$ so $x_i \in h^{-1}U$ for $i = 1, 2$. The set $h^{-1}U$ is open. For any $g \in G$, there exists $g_1 \in G$ such that $gh^{-1} = h^{-1}g_1$ and so $h^{-1}U$ is G-invariant. As there exists a good quotient $U//G$, so does $h^{-1}U//G$. It follows from Theorem C of [4] that there exists a good quotient $H \cdot U \to H \cdot U//G$, completing the proof.

THEOREM 2.2. *Let X be an algebraic normal variety and H, G be algebraic subgroups of* $\mathrm{Aut}(X)$. *Assume that H is connected, G is reductive and for any $h \in H$ and $g \in G$, $hgh^{-1} = g$ (i.e. H centralizes G in* $\mathrm{Aut}(X)$).

Let U be an open, G-invariant subset of X such that there exists a good quotient $U \to U//G$. Then U is G-saturated in $H \cdot U$.

Proof. By Proposition 2.6 of [2] it is enough to prove that U is T_0-saturated in $H \cdot U$ for any one-dimensional subtorus T_0 of G. Let α be a one-parameter subgroup of G with $\alpha(\mathbb{C}^*) = T_0$. Assume that $x \in U$ and $\lim_{t\to 0} \alpha(t)x = y \in H \cdot U$. We show that $y \in U$. The point y is fixed under the action of T_0. Let X_0 be the irreducible component of X^{T_0} containing y. It follows from the definition that $x \in U \cap (X_0)^+$. The group H acts on X^{T_0} (as the action of H commutes with the action of T_0). Since H is connected, irreducible components of X^{T_0} are H-invariant. It follows that H acts on X_0 and on X_0^+.

Since $y \in H \cdot U$, there exists $h \in H$ such that $hy \in U$ and therefore $hy \in U \cap X_0$. In particular $U \cap X_0 \neq \emptyset$ and $hx \in (U \cap X_0)^+ \subset U \cap (X_0)^+$. The point y is in the closure of $H \cdot hx$ and therefore in the closure of $(U \cap X_0)^+$ in U. Since there exists a good quotient $U \to U//G$, the Reduction Theorem [3] implies that so does $q_0 : U \to U//T_0$. Then $F = q_0^{-1}(q_0(X_0 \cap U))$ is a closed subset in U. In particular $(X_0 \cap U)^+$ is closed in U. It follows that $x \in (U \cap X_0)^+$, hence $y \in U \cap X_0$ and therefore U is saturated in $H \cdot U$, completing the proof.

COROLLARY 2.3. *Let X, H and G be as in Theorem 2.2. Let U be any G-maximal set in X. Then U is H-invariant.*

Proof. This follows immediately from Theorems 2.1 and 2.2.

COROLLARY 2.4. *Let X be a toric variety with respect to an action of the torus S and let T be a subtorus of S. Assume that $U \subset X$ is a T-maximal subset of X. Then U is a toric variety with respect to the action of S.*

Proof. Since U is open and S-invariant by the previous corollary, it contains the open orbit of S in X.

COROLLARY 2.5. *Under the conditions of Corollary 2.4 the quotient $U//T$ is a toric variety with respect to the action of some quotient of the torus S.*

Proof. According to Corollary 2.4, S has an open orbit in U. Therefore a quotient of S (in fact a quotient of S/T) has a dense orbit in $U//T$ and $U//T$ is a normal variety.

3. Affine case. First assume that X is an affine toric variety (with respect to an action of S). Then X is defined by a strictly convex cone σ: $\Sigma_{\max} = \{\sigma\}$ (in $E(S) = N(S) \otimes_{\mathbb{Z}} \mathbb{R}$). Since X is affine, there exists a good quotient $q : X \to X//T$. The quotient $X//T$ is also affine. We shall describe the cone of the toric variety $X//T$.

As before, let $E(T) = N(T) \otimes \mathbb{R}$ be the linear subspace of the linear space $E(S)$ spanned by all one-parameter subgroups of T. Let E' be the vector space spanned by $E(T)$ and the elements from the face of σ of smallest dimension containing $\sigma \cap E(T)$. Moreover let T' be the subtorus in S generated by all one-parameter subgroups in E' (i.e. T' is generated by all elements $t \in \{\alpha(\mathbb{C}^*) : \alpha \in E' \cap N(S)\}$). Let $p : S \to S/T'$ be the quotient morphism. Then $E(S)/E' \simeq E(S/T')$ and if $\pi' : E(S) \to E(S)/E' \simeq E(S/T')$ is the quotient map then for any $\sigma \in \Sigma$, $\pi'(\sigma)$ is a (strictly) convex (rational) cone in $E(S/T')$.

PROPOSITION 3.1. *Let X be an affine toric variety with an action of S defined by a* (*strictly*) *convex cone σ in $E(S)$. Assume that T' and π' are as above. Then T' acts trivially on $X//T$, $X//T$ is a toric variety with respect to the action of S/T' and this toric variety is defined in $E(S/T')$ by the cone $\pi'(\sigma)$.*

Proof. Since the orbit $S \cdot q(x_0)$ is dense in $X//T$ and $X//T$ is normal, it is enough to show that there exists a point $y \in q^{-1}(q(x_0)) = \overline{Tx_0}$ such that the isotropy group of $q(y)$ equals T.

Let $y \in \overline{Tx_0}$ be a point with a closed T-orbit. Then $E(T) \cap \sigma(y)^\circ \neq \emptyset$. The isotropy group S_y is generated by one-parameter subgroups $\alpha \in \mathrm{lin}(\sigma(y))$ and it follows that $S_y \cdot T = T'$ acts trivially on $X//T$. On the other hand, if $s \in S$ acts trivially on $X//T$ then $sTy \subset Ty$ hence $s \in T \cdot S_y = T'$. Therefore $X//T$ is a toric variety with respect to the action of S/T'. Moreover $X//T$ is affine, because X is affine.

Let $\tau \subset E(S/T')$ be the strictly convex cone defining the toric variety $X//T'$. We prove that $\tau = \pi'(\sigma)$.

Notice that $\pi'(\sigma) \subseteq \tau$ since we have a morphism of the S-toric variety X into the S/T'-toric variety relative to the morphism of tori. Assume that $v \in (\tau \setminus \pi'(\sigma)) \cap N(S/T')$. This element v corresponds to a one-dimensional subtorus $T_v \subset S/T'$ such that the orbit $T_v \cdot q(x_0)$ is not closed in $X//T'$. Consider now the action of the torus $T_1 = p^{-1}(T_v)$ on X. We claim that the T'-invariant set $Z = T_1 x_0$ is closed. This follows from the fact that $(\pi')^{-1}(\mathrm{lin}(v)) \cap \sigma = E(T_1) \cap \sigma = \{0\}$. The quotient morphism q is closed and therefore the set $q(Z) = T_v \cdot q(x_0)$ is closed, contrary to the choice of v. This completes the proof.

In Section 5 we shall need the following easy lemma:

LEMMA 3.2. *Let $U(\sigma)$ be an affine S-toric variety, $T \subset S$, σ_1 a face of σ and let $\pi : E(S) \to E(S)/E(T)$ be the quotient map. The set $U(\sigma_1)$ is T-saturated in $U(\sigma)$ if and only if for any $\sigma_2 \prec \sigma$,*

$$\pi(\sigma_2^\circ) \cap \pi(\sigma_1) \neq \emptyset \Rightarrow \sigma_2 \prec \sigma_1. \tag{4}$$

Proof. Assume first that $U(\sigma_1)$ is T-saturated in X and $\sigma_2 \prec \sigma$. Suppose that $\pi(\sigma_2^\circ) \cap \pi(\sigma_1) \neq \emptyset$ and σ_2 is not a face of σ_1. Let $\sigma_3 \prec \sigma$ be the face of smallest dimension such that $\sigma_i \prec \sigma_3$ for $i = 1, 2$. It follows that

$\pi(\sigma_3^\circ) \cap \pi(\sigma_1) \neq \emptyset$. Then there exist $\alpha \in E(T)$ and $\beta \in \sigma_1$ such that $\alpha + \beta \in \sigma_3^\circ$. It follows that the limits $\lim_{t\to 0}(\alpha+\beta)(t)x_0$ and $\lim_{t\to 0}\beta(t)x_0$ exist in X. Let $y = \lim_{t\to 0}(\alpha+\beta)(t)x_0$ and $z = \lim_{t\to 0}\beta(t)x_0$. Let T_0 be the subtorus of S generated by α, β. It follows that $y \in U(\sigma_3)$, $z \in U(\sigma_1)$ and $T_0 y$ is the only closed T_0-orbit in $\overline{T_0 x_0}$, hence $y \in \overline{T_0 z}$. Since $\beta(\mathbb{C}^*) \in S_z$ we infer that $\lim_{t\to 0}\alpha(t)z \in T_0 y \subset U(\sigma_3) - U(\sigma_1)$. But this contradicts the assumption that $U(\sigma_1)$ is saturated in $U(\sigma)$ hence $\sigma_2 \prec \sigma_1$.

Assume now that for any face σ_2 of σ condition (4) is satisfied. We have to show that $U(\sigma_1)$ is T-saturated in $U(\sigma)$. It is enough to show that for any $z \in U(\sigma_1)$ and any one-parameter subgroup $\alpha \in E(T)$ if the limit $\lim_{t\to 0}\alpha(t)z$ exists in $U(\sigma)$ then $y = \lim_{t\to 0}\alpha(t)z \in U(\sigma_1)$. This follows from 1.4.

Remark 3.3. Condition (4) of Lemma 3.2 is equivalent to

$$\pi^{-1}(\sigma_1) \cap |\sigma| = \sigma_1. \tag{5}$$

4. Quotients of toric varieties. In this section we generalize the result of Section 3 to the case of any toric variety. In particular in Theorem 4.1 we give a necessary and sufficient condition for existence of a good quotient of a toric variety and a description of the fan of the quotient toric variety $U(\Sigma)//T$.

Let X be a toric variety with respect to an action of the torus S. Assume that X is defined by a fan Σ and $\Sigma_{\max} = \{\sigma_1, \ldots, \sigma_m\}$ and consider the induced action of T on X. We define the vector space $E_{T,\Sigma} \subset E(S)$ to be generated by $E(T)$ and by all $\sigma \in \Sigma$ such that $\sigma^\circ \cap E(T) \neq \emptyset$, and T' to be the subtorus of S generated by all (images of) one-parameter subgroups in $E_{T,\Sigma} \cap N(S)$. Then $E_{T,\Sigma} = E(T')$. Let $\pi : E(S) \to E(S)/E(T)$ and $\pi' : E(S) \to E(S)/E(T')$ be the quotient maps.

We shall prove the following

Theorem 4.1. *Let X, S, Σ, T, T', π and π' be as above. There exists a good quotient $q : X \to X//T$ if and only if for any $\sigma_i \in \Sigma_{\max}$,*

$$\pi^{-1}(\pi(\sigma_i)) \cap |\Sigma| = \sigma_i. \tag{6}$$

Moreover if (6) is satisfied then $X//T$ is a toric variety with respect to the action of S/T' corresponding to the fan Υ in $E(S)/E(T')$ with $\Upsilon_{\max} = \{\pi'(\sigma_i) : \sigma_i \in \Sigma_{\max}\}$.

Proof. Assume first that there exists a good quotient $q : X \to X//T$. Then $X//T$ is a toric variety with respect to the action of a quotient of torus S. We shall show that (6) is satisfied.

Let $V \subset X//T$ be any open, affine subvariety invariant with respect to the induced action of S. The set $q^{-1}(V)$ is an open S-invariant affine subvariety in X and therefore corresponds to a strictly convex cone $\sigma \in \Sigma$. Obviously

$q : U(\sigma) \to V$ is a good quotient of this affine toric variety and therefore we can use Proposition 3.1. It follows that $U(\sigma)//T$ is a toric variety with respect to an action of the quotient of S by the subtorus T'' generated by T and all one-parameter subgroups contained in the maximal face $\sigma'' \prec \sigma$ such that $E(T) \cap (\sigma'')^\circ \neq \emptyset$. As $X//G$ is a good quotient and V is an open subset of $X//T$, it follows that T'' acts trivially and S/T'' acts effectively on $X//T$. Therefore $T'' = T'$ and $E(T'') = E(T')$. Let Υ be the fan defining $X//T$ in $E(T')$.

The quotient morphism of toric varieties $X \to X//T$ induces a map of the corresponding fans. Let $\sigma_i \in \Sigma_{\max}$. There exists $\tau_j \in \Upsilon_{\max}$ such that $\pi'(\sigma_i) \subset \tau_j$. Then by Lemma 1.3, $(\pi')^{-1}(\tau_j) \cap |\Sigma|$ is a strictly convex cone in $\Sigma_{\max}$ containing σ_i. Since $\sigma_i \in \Sigma_{\max}$, we have $\sigma_i = (\pi')^{-1}(\tau_j) \cap |\Sigma|$ and $\pi'(\sigma_i) = \tau_j$. But $\pi' = \pi_0 \circ \pi$, where $\pi_0 : E(S)/E(T) \to E(S)/E(T')$ is the quotient map. Hence

$$\sigma_i = (\pi')^{-1}(\pi'(\sigma_i)) \cap |\Sigma| = \pi^{-1}(\pi_0^{-1}(\pi'(\sigma_i)) \cap |\Sigma|.$$

From this it follows easily that condition (6) is satisfied.

Assume now that the assumptions of Theorem 4.1 are satisfied. Then for any $\sigma \in \Sigma$,

$$\sigma \cap E(T) \subset \bigcap_{\sigma_i \in \Sigma_{\max}} \sigma_i =: \sigma_0. \tag{7}$$

Then $E(T') = E_{T,\Sigma}$ is the vector space generated by $E(T)$ and the face σ_0' of σ_0 of minimal dimension containing $E(T) \cap |\Sigma|$. It follows that for any cone $\sigma_i \in \Sigma$, $\pi'(\sigma_i)$ is a strictly convex cone in $E(S)/E(T')$.

We show that

$$\forall i, j : \quad \pi(\sigma_i \cap \sigma_j) = \pi(\sigma_i) \cap \pi(\sigma_j) \prec \pi(\sigma_i). \tag{8}$$

Let $\alpha \in |\Sigma|$ be such that $\pi(\alpha) \in \pi(\sigma_i) \cap \pi(\sigma_j)$. It follows from (6) that $\alpha \in \sigma_i \cap \sigma_j$. This proves that $\pi(\sigma_i) \cap \pi(\sigma_j) = \pi(\sigma_i \cap \sigma_j)$. Assume now that $\tau \prec \pi(\sigma_i)$ is the face of minimal dimension containing $\pi(\sigma_i) \cap \pi(\sigma_j)$. Let $\sigma' := \pi^{-1}(\tau) \cap \sigma_i \prec \sigma_i$. Since $\pi((\sigma')^\circ) = \tau^\circ$ we have $\pi((\sigma')^\circ) \cap \pi(\sigma_i) \cap \pi(\sigma_j) \neq \emptyset$. It follows that $(\sigma')^\circ \cap \sigma_j \neq \emptyset$ and hence $\sigma' \prec \sigma_j$. This shows that $\tau \subset \pi(\sigma_i) \cap \pi(\sigma_j)$ and hence $\tau = \pi(\sigma_i) \cap \pi(\sigma_j)$. This proves (8).

It follows that there exists a fan Υ in $E(S)/E(T')$ such that $\{\pi(\sigma_i) : \sigma_i \in \Sigma_{\max}\} = \Upsilon_{\max}$. Let $Y = U(\Upsilon)$. The corresponding morphism $Q : X \to Y$ of toric varieties is affine (because condition (3) of Lemma 1.3 is satisfied). For any $\sigma \in \Sigma_{\max}$, the open subvariety $U(\sigma)$ is saturated in X with respect to the action of T'. This follows from (6) because $U(\sigma) = q^{-1}(U(\tau))$, where $\tau = \pi(\sigma)$. Then by Proposition 3.1, $q|U(\sigma) : U(\sigma) \to U(\tau)$ is a good quotient with respect to the action of T, which proves that $q : X \to U(\Upsilon)$ is a good quotient: $U(\Upsilon) = X//T'$. This ends the proof of Theorem 4.1.

COROLLARY 4.2. *Let x, S, Σ be as in Theorem 4.1. Let E' be a linear rational subspace in $E(S)$ and let T be the subtorus of S generated by all one-parameter subgroups $\alpha \in E'$. Assume that for any $\sigma', \sigma'' \in \Sigma$,*

$$\{\sigma' + E'\} \cap (\sigma'')^\circ \neq \emptyset \Rightarrow \exists \sigma_i \in \Sigma_{\max} : \sigma', \sigma'' \prec \sigma_i. \tag{9}$$

Then there exists a good quotient $X \to X//T$.

Proof. Assume that (9) is satisfied. Let $\pi : E(S) \to E(S)/E'$ be the quotient map. For any $\sigma_i \in \Sigma_{\max}$ and any $\sigma \in \Sigma$ we have

$$\pi(\sigma^\circ) \cap \pi(\sigma_i) \neq \emptyset \Rightarrow \sigma \prec \sigma_i.$$

Therefore for any $\sigma_i \in \Sigma_{\max}$, $\pi^{-1}(\pi(\sigma_i)) \cap |\Sigma| \subset \sigma_i$. Hence $\pi^{-1}(\pi(\sigma_i)) \cap |\Sigma| = \sigma_i$ and condition (6) of Theorem 4.1 is satisfied. Hence there exists a good quotient $X \to X//T$.

THEOREM 4.3. *Let S be an n-dimensional torus, T a subtorus of S and X a toric variety defined by a fan Σ in $E(S)$. Assume that there exists a good quotient $X \to X//T$. Then $X//T$ is complete if and only if $E(S) = \bigcup_{\sigma \in \Sigma} \{\sigma + E(T)\}$.*

Proof. Notice that $E(S) = \bigcup_{\sigma \in \Sigma} \{\sigma + E(T)\}$ is equivalent to $E(S)/E(T) = \bigcup_{\sigma \in \Sigma} \pi(\sigma)$. Let, as before, T' be the torus generated by all one-parameter subgroups in $E_{T,\Sigma}$ and let $\pi' : E(S) \to E(S)/E(T')$ be the quotient morphism. Then by Corollary 2.5, $X//T$ is a toric variety with respect to the action of T' and is defined in $E(S)/E(T')$ by a fan Υ such that $\Upsilon_{\max} = \{\pi'(\sigma_i) : \sigma_i \in \Sigma_{\max}\}$. A toric variety corresponding to a fan Υ in the vector space $E(S/T')$ is complete if and only if $\bigcup_{\tau \in \Upsilon} \tau = E(S/T')$. Obviously if

$$\bigcup_{\sigma \in \Sigma} \pi(\sigma) = E(S)/E(T)$$

then

$$\bigcup_{\sigma \in \Sigma} \pi'(\sigma) = E(S)/E(T').$$

Since $\bigcup_{\sigma \in \Sigma} \pi'(\sigma) = \bigcup_{\tau \in \Upsilon} \tau$ it follows that $X//T' = X//T$ is complete.

On the other hand, assume that

$$\bigcup_{\sigma \in \Sigma} \pi'(\sigma) = E(S)/E(T'). \tag{10}$$

We have to prove that

$$\bigcup_{\sigma \in \Sigma} \pi(\sigma) = E(S)/E(T).$$

We have assumed that there exists a good quotient $U(\Sigma) \to U(\Sigma)//T$, hence according to Theorem 4.1 the condition (6) is satisfied for any $\sigma \in \Sigma_{\max}$. Let σ_0' be a cone of minimal dimension containing $E(T) \cap |\Sigma|$. Then by (10), $E(T') = \text{lin}(\sigma_0') + E(T)$ and $\text{lin}(\sigma_0') + E(T) = \sigma_0' + E(T)$. Let $\alpha \in E(S)$.

Then there exists $\sigma \in \Sigma$ such that $\alpha \in \sigma + E(T') = \sigma + \sigma'_0 + E(T)$. Since $\sigma'_0 \prec \sigma_i$ for any $\sigma_i \in \Sigma_{\max}$ (see (7)), we get $\alpha \in |\Sigma| + E(T)$. This shows that $\bigcup_{\sigma \in \Sigma} \pi(\sigma) = E(S)/E(T)$, and completes the proof.

THEOREM 4.4. *Assume that X is a toric variety with respect to an action of a torus S and T is a subtorus of S. There exists a good quotient $q : X \to X//T$ if and only if for any one-parameter group $\alpha \in N(T)$ there exists a good quotient $q_\alpha : X \to X//T_\alpha$ with respect to the action of $T_\alpha = \alpha(\mathbb{C}^*)$.*

Proof. Assume first that there exists a good quotient $q : X \to X//T$, $\alpha : \mathbb{C}^* \to T$ is a one-parameter subgroup of T and T_α is the corresponding subtorus in T. Consider the line $E(T_\alpha)$, the subspace $E_\alpha = E_{T_\alpha,\Sigma}$ and the linear maps $\pi_\alpha : E(S) \to E(S)/E_\alpha$, $\pi'_\alpha : E(S)/E_\alpha \to E(S)/E(T')$, where as before $T' \subset S$ is the subtorus generated by all one-parameter subgroups contained in $E_{T,\Sigma}$. By Theorem 4.1, the homomorphism $\pi : E(S) \to E(S)/E(T)$ satisfies condition (6). But $\pi = \pi'_\alpha \circ \pi_\alpha$, hence π_α also satisfies (6). Again by Theorem 4.1 we infer that there exists a good quotient $q : X \to X//T_\alpha$.

Assume now that for any one-parameter subgroup α of T there exists a good quotient $q_\alpha : X \to X//T_\alpha$. It follows from Theorem 4.1 that the quotient morphism π_α satisfies condition (6), i.e. for any $\sigma_i \in \Sigma_{\max}$, and $\sigma \in \Sigma$,

$$\pi_\alpha^{-1}(\pi_\alpha(\sigma_i)) \cap |\Sigma| = \sigma_i$$

or equivalently

$$\sigma \subset \pi_\alpha^{-1}(\pi_\alpha(\sigma_i)) \Rightarrow \sigma \prec \sigma_i. \tag{11}$$

Consider now $\sigma_i \in \Sigma_{\max}$ and let $\sigma \subset \pi^{-1}(\pi(\sigma_i))$ for some $\sigma \in \Sigma$. Then $\sigma \subset \{\sigma_i + E(T)\}$. There exists a one-parameter subgroup α of T such that $\sigma^\circ \cap \{\sigma_i + \mathrm{lin}(\alpha)\} \neq 0$. Consider, as before, the morphism $\pi_\alpha : E(S) \to E(S)/E(T_\alpha)$. Since $q_\alpha : X \to X//T_\alpha$ is a good quotient, it follows that $\sigma \prec \sigma_i$, and this ends the proof.

REMARK 4.5. Theorem 4.4 is also a special case of the Reduction Theorem [3], but the proof in the general situation (the action of a reductive group on a normal algebraic variety) uses much stronger methods.

5. T-maximal subsets of toric varieties. In the previous section we have described the fans Σ in $E(S)$ such that there exists a good quotient $X \to X//T$ where X is the toric variety corresponding to Σ and T is a subtorus of S. Now for a given toric variety Y corresponding to a fan Σ_0 we shall describe all T-maximal subsets of Y. It follows from Corollary 2.4 that any T-maximal subset of Y is a toric subvariety and therefore corresponds to a subfan $\Sigma \subset \Sigma_0$. Let, as before, $E(T) \subset E(S)$ be the subspace generated by the one-parameter subgroups of T, and let $\pi : E(S) \to E(S/T)$ denote the linear map induced by the quotient morphism of tori. We shall need

LEMMA 5.1. *Let Σ, Σ_1 be fans in $E(S)$ and $\Sigma \subset \Sigma_1$. Then $U(\Sigma)$ is T-saturated in $U(\Sigma_1)$ if and only if for any $\sigma \in \Sigma$,*

$$\sigma \prec \tau \in \Sigma_1 \Rightarrow \pi^{-1}\pi(\sigma) \cap \tau = \sigma. \tag{12}$$

Proof. The proof is an immediate consequence of Remark 3.3.

THEOREM 5.2. *Let X be an S-toric variety corresponding to the fan Σ_1 and let T be a subtorus of S. An open, T-invariant subvariety U is T-maximal if and only if $U = U(\Sigma)$ for a subfan Σ of Σ_1 such that for any $\sigma \in \Sigma_{\max}$,*

$$\pi^{-1}\pi(\sigma) \cap |\Sigma| = \sigma \tag{13}$$

and for any $\tau \in \Sigma_1 - \Sigma$ there exists $\sigma \in \Sigma_{\max}$ such that either

$$\pi^{-1}\pi(\sigma) \cap \tau \not\subset \sigma \tag{14}$$

or

$$\pi^{-1}\pi(\tau) \cap \sigma \not\subset \tau. \tag{15}$$

Proof. Assume first that $\Sigma \subset \Sigma_1$, $U = U(\Sigma)$ and Σ satisfies conditions (13)–(15). Then according to Theorem 4.1 there exists a good quotient $U(\Sigma) \to U(\Sigma)//T$. Consider any $\Sigma_0 \subset \Sigma_1$ which satisfies (13) and such that $\Sigma \subset \Sigma_0$. We have to prove that if $\Sigma \neq \Sigma_0$ then $U(\Sigma)$ is not saturated in $U(\Sigma_0)$. Assume that $\tau \in \Sigma_0 - \Sigma$ and $\tau \in (\Sigma_0)_{\max}$. For this τ there exists $\sigma \in \Sigma_{\max}$ satisfying (14) or (15). By the assumption we have $\pi^{-1}\pi(\tau) \cap |\Sigma_0| = \tau$. It follows that σ satisfies (14). The condition (13) for σ and τ respectively implies that $\sigma \prec \tau$. We now use Lemma 5.1 to see that $U(\Sigma)$ is not saturated in $U(\Sigma_0)$.

Assume now that $U \subset X$ is T-maximal. According to 2.4 and 4.1 there exists a subfan $\Sigma \subset \Sigma_1$ such that $U = U(\Sigma)$ and Σ satisfies (13). Suppose that there exists a cone $\tau \in \Sigma_1 - \Sigma$ such that for any $\sigma \in \Sigma_{\max}$,

$$\pi^{-1}\pi(\sigma) \cap \tau \subset \sigma \quad \text{and} \quad \pi^{-1}\pi(\tau) \cap \sigma \subset \tau.$$

Then it is easy to see that a fan $\Sigma_0 = \Sigma \cup \{\tau_i : \tau_i \prec \tau\}$ satisfies (13) and $U(\Sigma_0)$ is saturated in $U(\Sigma_0)$. But this contradicts the assumption that $U = U(\Sigma)$ is T-maximal in $U(\Sigma_1)$. This ends the proof.

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Institute of Mathematics
University of Warsaw
Banacha 2
02-097 Warszawa, Poland
E-mail: jswiec@mimuw.edu.pl

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