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ADDITIVE FUNCTIONS FOR QUIVERS WITH RELATIONS

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Abstract. Additive functions for quivers with relations extend the classical concept of additive functions for graphs. It is shown that the concept, recently introduced by T. Hübner in a special context, can be defined for different homological levels. The existence of such functions for level 2 resp. ∞ relates to a nonzero radical of the Tits resp. Euler form. We derive the existence of nonnegative additive functions from a family of stable tubes which stay tubes in the derived category, we investigate when this situation does appear and we study the restrictions imposed by the existence of a positive additive function.

Introduction. It is classical that among connected quivers exactly the extended Dynkin quivers $\vec{\Delta}$ admit a positive additive function (see [3, 6, 17]); actually, the existence of a nonnegative additive function ensures extended Dynkin type. But also a wild quiver may admit a nonzero additive function. Such an additive function λ attaches to every vertex p of $\vec{\Delta}$ an integer $\lambda(p)$ and satisfies for each p the additivity condition

$$2\lambda(p) = \sum_{p\to q} \lambda(q) + \sum_{q\to p} \lambda(q),$$

where the sum is taken over all arrows starting or ending at p. Additivity is thus a concept not concerning the quiver itself but its underlying unoriented graph.

The concept of an additive function has recently been extended by Hübner [9] to quivers with relations, therefore to finite-dimensional algebras, interpreting relations as arrows with a negative sign. His main result states that the rank function on a weighted projective line yields a positive (resp. nonnegative) additive function for the endomorphism ring of each tilting bundle (resp. tilting sheaf). Hence each concealed canonical algebra, in particular each tame concealed or tubular algebra, admits a positive additive function. This provides another explanation for the existence of additive functions on extended Dynkin quivers.

We stress in the present paper that the concept of an additive function attached to a finite-dimensional algebra Σ is homological in nature. Actually,

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we need to account for various levels of additivity, depending on the level l up to which the extension spaces $\mathrm{Ext}^i_\Sigma(S_p, S_q)$, $i \leq l$, between simple Σ-modules are taken into account. These spaces are to be interpreted as some kind of higher relations from q to p. In particular, we show that the 2-additive (resp. the ∞-additive or simply additive) functions correspond naturally to the members of the radical of the Tits form (resp. the Euler form).

The requirement of a positive additive function seems to be quite strong. We show in Theorem 2.5 that an algebra Σ, derived equivalent to a canonical algebra, has a positive additive function if and only if it is concealed canonical, i.e. the endomorphism algebra of a tilting bundle on a weighted projective line. We show, moreover, that each endomorphism ring of what we call a narrow tilting complex on a weighted projective line has a nonnegative function which is additive for each level $l \geq 2$. In particular, each quasitilted algebra of canonical type [14] has this property, which extends the results of [9].

The existence of an additive function for Σ is often derived from the existence of a family of stable tubes in the module category which stay tubes in the derived category (Proposition 2.1). We further provide a useful criterion ensuring this property for a large class of one-point extensions (Theorem 3.5).

We discuss existence and uniqueness of additive functions for various classes of algebras, including one-point extension algebras of concealed canonical algebras with modules of arbitrary regular length, thus including the pg-critical algebras of [16]. We also provide examples and counterexamples to various effects and conjectures.

Conventions. All algebras are finite-dimensional over an algebraically closed field k. We further assume these algebras to be basic and connected and to have finite global dimension, hence not to have loops in their ordinary quiver. The term module always refers to finitely generated right modules. As basic references for undefined terms we recommend [2] and [19].

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1. Basic facts on additivity. Let $\Sigma = k[\vec{\Delta}]/I$ be a basic finite-dimensional k-algebra given in terms of a quiver $\vec{\Delta}$ with an admissible ideal I of relations. We assume that the set Δ_0 of vertices of $\vec{\Delta}$ consists of the integers $1, \ldots, n$ and denote by S_p the simple Σ-module corresponding to the vertex p. We fix an integer $l \geq 2$ (resp. $l = \infty$) and introduce a new *coloured quiver* $\vec{Q}^{(l)}\Sigma$ with the same vertex set Δ_0 and whose arrows are coloured by the natural numbers i from 0 to l (resp. by all natural numbers):

from vertex p to vertex q we draw

$$a_{pq}^{(i)} = \dim_k \operatorname{Ext}_{\Sigma}^{i}(S_q, S_p)$$

arrows of colour i. In particular, $a_{pq}^{(1)}$ (resp. $a_{pq}^{(2)}$) denotes the number of arrows (resp. relations) from p to q in the ordinary quiver of Σ. The cases $l = 2$ and $l = \infty$ will be the most interesting for us. Note that the coloured quiver $\vec{Q}^{(\infty)}\Sigma$ determines the global dimension of Σ.

For each $l \geq 2$ we consider the (usually nonsymmetric) bilinear form on the Grothendieck group $\mathrm{K}_0(\Sigma)$ given on classes of simple Σ-modules by the expression

$$\langle [S_p], [S_q] \rangle^{(l)} = \sum_{i=0}^{l} (-1)^i a_{qp}^{(i)},$$

where for $l = \infty$ we remove the superscript (l). For $l = 2$ we thus recover the *Tits form*, while for $l = \infty$ we recover the *Euler form* given on classes of Σ-modules by

$$\langle [X], [Y] \rangle = \sum_{i=0}^{\infty} (-1)^i \dim_k \operatorname{Ext}_{\Sigma}^{i}(X, Y).$$

We also consider the associated symmetric bilinear form

$$(x \mid y)^{(l)} = \langle x, y \rangle^{(l)} + \langle y, x \rangle^{(l)}$$

and the corresponding *radical* of $\langle -, - \rangle^{(l)}$ or $(- \mid -)^{(l)}$ consisting of all y with $(- \mid y)^{(l)} = 0$.

Definition 1.1. An integral-valued function λ on the set Δ_0 of vertices of Σ is called *l-additive* if for each vertex p we have

$$2\lambda(p) = \sum_{i=1}^{l} (-1)^{i-1} \sum_{q \in \Delta_0} (a_{pq}^{(i)} + a_{qp}^{(i)})\lambda(q).$$

For $l = \infty$ we say that λ is *additive*. If λ is l-additive for each $l \geq 2$, we call λ a *strongly additive function*.

Also here l-additivity only depends on the unoriented (coloured) graph underlying the quiver $\vec{Q}^{(l)}\Sigma$. This can be made more precise as follows:

For any $l \geq 2$ the coloured quiver $\vec{Q}^{(l)}\Sigma$ keeps the full information on the form $\langle -, - \rangle^{(l)}$. On the other hand, the concept of an l-additive function only depends on the symmetric form $(-, -)^{(l)}$. The full information on the symmetric form is thus kept by the *nonoriented digraph* $Q^{(l)}\Sigma$, again with vertex set Δ_0, and which has $\sum_{i=0,\, i\,\mathrm{odd}}^{l} (a_{pq}^{(i)} + a_{qp}^{(i)})$ *solid* edges and $\sum_{j=2,\, j\,\mathrm{even}}^{l} (a_{pq}^{(j)} + a_{qp}^{(j)})$ *dotted* edges between p and q. For $l = \infty$ we remove the superscript (l) for $\vec{Q}^{(l)}\Sigma$ and $Q^{(l)}\Sigma$.

Since Σ is assumed to have finite global dimension, the classes $[P_1],\ldots$ $\ldots,[P_n]$ of indecomposable projective modules form a $\mathbb{Z}$-basis of $\mathrm{K}_0(\Sigma)$. Here, P_i denotes the projective cover of the simple module S_i.

In this setting we will view an additive function λ as the linear form on $\mathrm{K}_0(\Sigma)$ with $\lambda([P_i])=\lambda(i)$. We say that λ is *positive* if $\lambda(i)>0$ for each i; we call λ *nonnegative* if λ is nonzero and $\lambda(i)\geq 0$ for each i. Finally λ is called *negative* if $-\lambda$ is positive.

With each linear form λ on $\mathrm{K}_0(\Sigma)$ we further associate its *characteristic class*

$$[\lambda]=\sum_{p=1}^{n}\lambda(p)[S_p]$$

so that $\langle [P_i],[\lambda]\rangle=\lambda(i)$ for all i, and therefore $\lambda=\langle -,[\lambda]\rangle$.

PROPOSITION 1.2. *A linear form $\lambda:\mathrm{K}_0(\Sigma)\to\mathbb{Z}$ is l-additive if and only if its characteristic class $[\lambda]$ belongs to the radical of the bilinear form $\langle -,-\rangle^{(l)}$. Moreover, if λ is additive then $\lambda([\lambda])=0$.*

Proof. Identifying members of $\mathrm{K}_0(\Sigma)$ with column vectors from $\mathbb{Z}^n$ with respect to the basis of classes of simple Σ-modules, we may express $(x\,|\,y)^{(l)}$ in terms of the matrices $A^{(i)}=(a_{pq}^{(i)})$ as the matrix product

$$(x\,|\,y)^{(l)}=x^{\mathrm{t}}\Big(\sum_{i=0}^{l}(-1)^i(A^{(i)}+(A^{(i)})^{\mathrm{t}})\Big)y.$$

Now $[\lambda]$ belongs to the radical of $\langle -,-\rangle^{(l)}$ if and only if $(-\mid[\lambda])^{(l)}=0$, equivalently if

$$\sum_{i=0}^{l}(-1)^i(A^{(i)}+(A^{(i)})^{\mathrm{t}})[\lambda]=0,$$

which just expresses the additivity of λ.

Assume now that λ is additive. Then $\lambda([\lambda])=\langle[\lambda],[\lambda]\rangle=0$ since $[\lambda]$ belongs to the radical of the Euler form. ∎

The existence of a 2-additive (resp. additive) function thus relates to a nonzero radical of the Tits (resp. the Euler) form, called further on the *Tits radical* (resp. *Euler radical*).

Assume that Σ is given in terms of a quiver with relations. It is then very easy to check whether a function is 2-additive; also determining the complete Tits radical is an easy exercise in linear algebra. We note that for the calculation of the Euler radical we do not need explicit information on the higher extension spaces $\mathrm{Ext}^i_{\Sigma}(S_p,S_q)$: If $C=(\langle[P_i],[P_j]\rangle)$ denotes the *Cartan matrix* of Σ, expressed in the basis of classes of indecomposable projective modules, then the Euler radical, expressed in the same basis, is the subgroup of all x satisfying $(C + C^{\mathrm{t}})x = 0$. Note that the (i, j)-entry of C equals the dimension of $\mathrm{Hom}(P_i, P_j)$, so also the Euler radical is very easy to compute.

The Euler radical equals also the fixed point set of the *Coxeter transformation* $\Phi = \Phi_\Sigma$, the automorphism of $\mathrm{K}_0(\Sigma)$ which is uniquely determined by the validity of the formula

$$\langle y, x\rangle = -\langle x, \Phi y\rangle \quad \text{for all } x, y \in \mathrm{K}_0(\Sigma).$$

The existence of a nonzero Euler radical can thus conveniently be read off from the *Coxeter polynomial* of Σ, the characteristic polynomial of the Coxeter transformation. We note that the Coxeter polynomial is preserved under derived equivalence, in particular under tilting. The next proposition underlines the importance of additive functions.

PROPOSITION 1.3. *If Σ and Σ' are derived equivalent algebras of finite global dimension, then they have isomorphic groups of additive functions.*

Proof. Each equivalence of triangulated categories between the bounded derived categories $\mathrm{D}^{\mathrm{b}}(\mathrm{mod}(\Sigma))$ and $\mathrm{D}^{\mathrm{b}}(\mathrm{mod}(\Sigma'))$ induces an isomorphism of Grothendieck groups $\mathrm{K}_0(\Sigma) \cong \mathrm{K}_0(\Sigma')$ preserving the Euler forms, thus inducing an isomorphism from the Euler radical of Σ to the Euler radical of Σ'. ■

Passing to a derived equivalent algebra may, however, cause the loss of the positivity of an additive function: Each representation-finite algebra Σ which is tilted from a tame hereditary algebra Λ inherits from Λ a nonzero additive function $\lambda : \mathrm{K}_0(\Sigma) \to \mathbb{Z}$, which is unique up to multiplication with a nonzero integer. Since, as a tilted algebra, Σ has global dimension ≤ 2, it further follows that λ is l-additive for each $l \geq 2$. Let T be a tilting module over Λ such that $\Sigma = \mathrm{End}_\Lambda(T)$. Because Σ is representation-finite, T needs to contain an indecomposable preprojective and an indecomposable preinjective summand, where the additive rank function for Λ takes values of opposite sign.

The existence of a 2-additive function may be lost when passing to a derived equivalent algebra. Expressed in different terms this means that an algebra Σ may have a nonzero Tits radical, but the Tits radical of an algebra Σ', derived equivalent to it, may be zero (see Example 4.4).

2. Positive additive functions. A main reason for an algebra Σ to have an additive function is the existence of τ-periodic objects, in particular tubes, in the module category or in the derived category, where τ refers to the Auslander–Reiten translation of $\mathrm{mod}(\Sigma)$ and $\mathrm{D}^{\mathrm{b}}(\Sigma)$, respectively. It moreover often suffices to exhibit τ-periodic elements in $\mathrm{K}_0(\Sigma)$, where now τ stands for the Coxeter transformation Φ_Σ, the automorphism induced on the K-theoretic level by the Auslander–Reiten translation of $\mathrm{D}^{\mathrm{b}}(\Sigma)$. Recall that we assume that Σ has finite global dimension.

Of particular importance is the case of a family of stable tubes $\mathcal{T}$ in $\mathrm{mod}(\Sigma)$ which stay tubes in $\mathrm{D}^{\mathrm{b}}(\Sigma)$, because this produces a nonnegative additive function for Σ. If moreover $\mathcal{T}$ is *sincere*, that is, if for each indecomposable projective Σ-module P_i there is a nonzero homomorphism from P_i to a member of $\mathcal{T}$, then Σ even admits a positive additive function λ.

Let y be an element of $\mathrm{K}_0(\Sigma)$ of τ-period p. Then $x = \sum_{i=1}^{p} \tau^i(y)$ is stable under the Coxeter transformation τ, so it is a member of the Euler radical of $\mathrm{K}_0(\Sigma)$. Note that x may be zero even if y is nonzero. Similarly, if Y is an object of $\mathrm{D}^{\mathrm{b}}(\Sigma)$ of τ-period p, then $X = \bigoplus_{i=1}^{p} \tau^i Y$ is a nonzero τ-stable object of $\mathrm{D}^{\mathrm{b}}(\Sigma)$, yielding a τ-stable class $x = [X]$ in $\mathrm{K}_0(\Sigma)$; also here it may happen that x is zero. On the other hand, for an object Y in $\mathrm{mod}(\Sigma)$ of τ-period p, the direct sum $X = \bigoplus_{i=1}^{p} \tau^i Y$ always has a nonzero class in $\mathrm{K}_0(\Sigma)$.

PROPOSITION 2.1. (i) *If* $\mathrm{K}_0(\Sigma)$ *has a nonzero* τ*-stable element* x, *then* $\lambda = \langle -, x\rangle$ *is a nonzero additive function. This happens*, *in particular*, *if* $\mathrm{D}^{\mathrm{b}}(\Sigma)$ *has a* τ*-periodic object* X *of period* p *yielding a nonzero class* $x = \sum_{i=1}^{p} [\tau^i X]$.

(ii) *If* $\mathrm{mod}(\Sigma)$ *has a family* $\mathcal{T}$ *of stable tubes which stay tubes in* $\mathrm{D}^{\mathrm{b}}(\Sigma)$, *then* Σ *has a nonnegative additive function. If moreover* $\mathcal{T}$ *is sincere*, *then* Σ *has a positive additive function.*

Proof. Assertion (i) is a direct consequence of Proposition 1.2, and assertion (ii) follows from (i) by the remarks preceding Proposition 2.1. ∎

COROLLARY 2.2. *Assume that* Σ *has a family* $\mathcal{T}$ *of stable tubes whose members have both projective and injective dimension at most one. Then* Σ *admits a nonnegative additive function and*, *further*, *a positive additive function if* $\mathcal{T}$ *is additionally sincere.*

Proof. Since each almost split sequence $\eta : 0 \to X \to Y \to Z \to 0$ of Σ-modules with injective dimension $\mathrm{id}_{\Sigma} X \leq 1$ and projective dimension $\mathrm{pd}_{\Sigma} Z \leq 1$ yields an Auslander–Reiten triangle $X \to Y \to Y \xrightarrow{\eta} X[1]$ in $\mathrm{D}^{\mathrm{b}}(\Sigma)$ (see [5, I.4.7]), the family $\mathcal{T}$ stays a family of stable tubes in the derived category. ∎

A prominent class of algebras having tubes in the derived category is built by the *derived canonical* algebras, which, by definition, are derived equivalent to a canonical algebra. A special case of derived canonical algebras are the *concealed canonical* (resp. *almost concealed canonical*) algebras defined as the endomorphism algebras of tilting bundles (resp. tilting sheaves) on a weighted projective line $\mathbb{X}$, or equivalently as the endomorphism algebras of tilting modules T over a canonical algebra Λ, where T is built from indecomposable Λ-modules of positive (resp. nonnegative) rank; see [12]. In [13] and [14] the concealed canonical (resp. almost concealed canonical) algebras are further characterized as the connected algebras whose module category admits a sincere separating family of standard stable tubes (resp. of standard tubes not containing injectives). According to [18] the concealed canonical algebras can, moreover, be characterized as those having a sincere family of stable tubes whose modules do not lie on external short cycles.

The concealed canonical and almost concealed canonical algebras are quasitilted algebras. Recall from [7] that an algebra Σ is *quasitilted* if it is isomorphic to the endomorphism ring of a tilting object for a hereditary abelian k-category. This is equivalent to stating that $\mathrm{gl.dim}\,\Sigma \leq 2$ and further that each indecomposable module X has projective dimension or injective dimension at most one. If a quasitilted algebra Σ is derived equivalent to a canonical algebra, we say that Σ is *quasitilted of canonical type*. For further information on this class of algebras, including a characterization as the algebras having a sincere separating family of semiregular tubes, we refer to [14].

COROLLARY 2.3. (i) *For each derived canonical algebra* Σ *the rank function on* $\mathrm{D^b}(\Sigma) \cong \mathrm{D^b}(\mathrm{coh}\,\mathbb{X})$ *yields a nonzero additive function for* Σ.

(ii) *For each almost concealed canonical algebra* Σ, *more generally each quasitilted algebra of canonical type*, *the rank function yields a nonnegative strongly additive function* λ.

(iii) *For each concealed canonical algebra* Σ *the rank function yields a positive strongly additive function. Moreover*, *if* Σ *is not tubular*, *then each nonzero additive function is positive or negative.*

Proof. The corollary mostly summarizes the discussion preceding it. Note that the classes of indecomposable summands of a tilting complex form a basis of $\mathrm{K}_0(\Sigma)$. Because the rank function for $\mathrm{coh}\,\mathbb{X}$ is nonzero, each tilting complex in $\mathrm{D^b}(\mathrm{coh}\,\mathbb{X})$ contains an indecomposable direct factor of nonzero rank, which proves (i). Under the assumptions of (ii) the rank function is nonnegative. This function is strongly additive since, as a quasitilted algebra, Σ has global dimension ≤ 2. If Σ is moreover concealed canonical, the rank function yields a positive strongly additive function. The second assertion of (iii) now follows from the fact that the Euler radical has rank one if Σ is not tubular; see for instance [11]. ∎

Note that each tubular algebra has a nonzero additive function which takes the value zero, so (iii) cannot be improved.

As the following example of a pg-critical algebra shows (see [16] for a definition), there are further algebras besides the concealed canonical ones which have a positive additive function. (See Section 3 for further information.)

EXAMPLE 2.4. The following algebra has global dimension two:

$$\begin{tikzcd}
 & & & {\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]} \arrow[ddrrr] & & & \\
 & & & {\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]} \arrow[drrr] & & & \\
{\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]} \arrow[uurrr] \arrow[urrr] \arrow[dr] & & & {\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]} \arrow[dr] & & & {\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} \\
 & {\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]} \arrow[r] & {\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]} \arrow[ur] \arrow[dr] & & {\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} \arrow[r] & {\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} \arrow[ur] & \\
 & & & {\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]} \arrow[ur] & & &
\end{tikzcd}$$

where the lower square is commutative and the sum of the three paths from the source to the sink equals zero. The Tits and Euler radical agree and, moreover, form a group of rank two. The upper and the lower row, respectively, of the displayed values give a basis for the group of additive functions.

For a derived canonical algebra the requirement of a positive additive function is quite restrictive, as can be seen from our next result. We first recall some background. Since the abelian category $\mathcal{C} = \operatorname{coh}\mathbb{X}$ of coherent sheaves on a weighted projective line is hereditary [4], its derived category $\mathrm{D}^{\mathrm{b}}(\mathcal{C})$ is the additive closure of $\bigcup_{n\in\mathbb{Z}} \mathcal{C}[n]$, where each $\mathcal{C}[n]$ is a copy of $\mathcal{C}$ with objects written $X[n]$, $X \in \mathcal{C}$, $n \in \mathbb{Z}$, and where

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathcal{C})}(X[n], Y[m]) = \mathrm{Ext}_{\mathcal{C}}^{m-n}(X, Y).$$

If Λ denotes the canonical algebra attached to $\mathbb{X}$, then Σ is derived equivalent to Λ if and only if Σ is isomorphic to the endomorphism ring of a tilting complex T in $\mathrm{D}^{\mathrm{b}}(\mathcal{C})$, where a *tilting complex* T has no self-extensions, that is, $\mathrm{Hom}(T, T[n]) = 0$ for each $0 \neq n \in \mathbb{Z}$, and moreover, T generates $\mathrm{D}^{\mathrm{b}}(\mathcal{C})$ as a triangulated category. The minimal number of copies of $\mathcal{C}$ needed to contain all indecomposable summands of T is called the *width* of the tilting complex T, a concept we will need later.

THEOREM 2.5. *Let Σ be a derived canonical algebra. Then Σ admits a positive additive function λ if and only if Σ is concealed canonical, and hence is quasitilted.*

Proof. If $\Sigma = \mathrm{End}(T)$ for a tilting bundle in $\operatorname{coh}\mathbb{X}$, we have already seen that the rank function yields a positive additive function for Σ. Conversely assume that Σ is derived canonical and therefore isomorphic to the endomorphism ring of a tilting complex T in the derived category $\mathrm{D}^{\mathrm{b}}(\operatorname{coh}\mathbb{X})$ for some weighted projective line $\mathbb{X}$. If the genus of $\mathbb{X}$ is different from one, the radical of the Euler form for $\mathbb{X}$, hence for Σ, has rank one and is generated by the rank function for $\operatorname{coh}\mathbb{X}$; see [11]. If the genus of $\mathbb{X}$ equals one, the radical of the Euler form has rank two. Since the automorphism group of $\mathrm{D}^{\mathrm{b}}(\operatorname{coh}\mathbb{X})$ acts transitively on the rank one direct summands of the radical (see [11]), we can also in this case assume—without loss of generality—that a given nonzero additive function is a positive integer multiple of the rank function.

We thus derive from the assumption that the rank function is positive on each indecomposable direct summand of T. This implies—invoking $\operatorname{Hom}(\mathcal{C}[n],\mathcal{C}[m]) = 0$ for $m-n \geq 2$ and connectedness of Σ—that T lies in just one copy of $\operatorname{coh}\mathbb{X}$ and moreover is a vector bundle. Thus Σ, as the endomorphism algebra of a tilting bundle, is concealed canonical. ∎

3. Nonnegative additive functions. A major class of algebras with a nonzero additive function is obtained by forming one-point extensions, where again we distinguish the three levels of K-groups, module categories and derived categories, respectively. Often additive functions for an algebra Σ or a one-point source extension $\Sigma[M]$ (resp. sink extension $[M]\Sigma$) arise in connection with a Σ-module M whose class is periodic under the Coxeter transformation.

PROPOSITION 3.1. *Assume that Σ has finite global dimension and that S is a nonzero Σ-module whose class is p-periodic under the Coxeter transformation Φ_Σ. Let $\overline{\Sigma}$ denote the one-point sink extension $[S]\Sigma = \left(\begin{smallmatrix}\Sigma & 0\\ S & k\end{smallmatrix}\right)$. Then the following assertions hold*:

(i) *The additive function $\lambda : \mathrm{K}_0(\Sigma) \to \mathbb{Z}$ with characteristic class $[\lambda] = \sum_{j=1}^{p} \Phi_\Sigma^j[S]$ annihilates $[S]$. If moreover $[\lambda] \neq 0$, then λ extends to a nonzero additive function $\overline{\lambda}$ on $\overline{\Sigma}$ taking value zero on the extension vertex.*

(ii) *If each $\Phi_\Sigma^j[S]$ is the class of some Σ-module, then $[\lambda] \neq 0$, and the additive functions λ and $\overline{\lambda}$ are nonnegative.*

Proof. We view (right) Σ-modules as $\overline{\Sigma}$-modules with support in the subalgebra Σ. In particular, the indecomposable projectives $P_1,\dots,P_n$ over Σ become indecomposable projectives over $\overline{\Sigma}$, and there is just one additional indecomposable projective $\overline{\Sigma}$-module P_{n+1}, corresponding to the extension vertex $n+1$.

We introduce the following notations: $V = \mathrm{K}_0(\Sigma)$, equipped with the Euler form $\langle -,-\rangle_V$, further $\overline{V} = \mathrm{K}_0(\overline{\Sigma})$, equipped with the Euler form $\langle -,-\rangle_{\overline{V}}$, finally τ (resp. $\overline{\tau}$) denotes the Coxeter transformation for Σ (resp. $\overline{\Sigma}$) and $s = [S]$.

By means of the basis $[P_1],\dots,[P_{n+1}]$ of $\mathrm{K}_0(\overline{\Sigma})$ it is straightforward to verify the following assertions:

1. $\overline{V} = V \oplus \mathbb{Z}s^*$, where $s^* = [P_{n+1}]$.
2. $\langle v, s^* \rangle_{\overline{V}} = \langle v, s \rangle_V$ and $\langle s^*, v \rangle_{\overline{V}} = 0$ for all $v \in V$.
3. $\langle s^*, s^* \rangle_{\overline{V}} = 1$.

Since s is p-periodic under Φ, the orbit sum $[\lambda] = \sum_{j=1}^{p} \tau^j s$ is a fixed point for τ, hence belongs to the Euler radical of $\langle -, - \rangle_V$. First we show that $\lambda(s) = 0$. Indeed, since τ preserves the Euler form we get $\langle s, \tau^j s \rangle_V = \langle \tau^{-j} s, s \rangle_V$, hence summation over the τ-orbit of s yields $\langle s, [\lambda] \rangle_V = \langle [\lambda], s \rangle_V$. Therefore $2\lambda(s) = 2\langle s, [\lambda] \rangle_V = (s \,|\, [\lambda]) = 0$ since $[\lambda]$ belongs to the Euler radical of V.

Since $\overline{\lambda} = \langle -, [\lambda] \rangle_{\overline{V}}$, additivity of $\overline{\lambda}$ amounts to showing that $[\lambda]$ also belongs to the Euler radical of $\langle -, - \rangle_{\overline{V}}$. This follows directly from

$$\langle v, [\lambda] \rangle_{\overline{V}} + \langle [\lambda], v \rangle_{\overline{V}} = 0 \quad \text{for } v \in V, \qquad \langle s^*, [\lambda] \rangle_{\overline{V}} + \langle [\lambda], s^* \rangle_{\overline{V}} = -\lambda(s) = 0$$

and proves assertion (i).

For assertion (ii) let M_j be a module with $[M_j] = \Phi_{\Sigma}^{j}[S]$ and put $M = \bigoplus_{j=1}^{p} M_j$. Then for each $p = 1, \ldots, n$ we get $\lambda(p) = \dim_k \operatorname{Hom}(P_p, M)$, which is ≥ 0. ∎

By the definition, additive functions incorporate information about the relations for an algebra.

Corollary 3.2. *In addition to the assumptions in the proposition assume that $\overline{\lambda}$ is* 2*-additive, for instance that* gl.dim $\overline{\Sigma} \leq 2$, *and that λ is positive. Then there is at least one relation ending in the extension vertex.*

Proof. Since $\overline{\lambda}$ vanishes at the extension vertex $n+1$ we get

$$\sum_{p=1}^{n} \lambda(p) \dim_k \operatorname{Ext}^2_{\overline{\Sigma}}(S_{n+1}, S_p) = \sum_{p=1}^{n} \lambda(p) \dim_k \operatorname{Ext}^1_{\overline{\Sigma}}(S_{n+1}, S_p),$$

which by assumption is > 0. ∎

We illustrate this by our next example; see also Example 3.9.

Example 3.3. Let Σ be the Kronecker quiver $1 \rightrightarrows 2$, and let λ be the additive function given by $\lambda(1) = \lambda(2) = 2$. If M is any regular Σ-module with dimension vector (2 2) then the one-point sink extension $[M]\Sigma$ is given by the quiver

$$1 \underset{x_2}{\overset{x_1}{\rightrightarrows}} 2 \underset{y_2}{\overset{y_1}{\rightrightarrows}} 3$$

with two relations from vertex 1 to vertex 3 depending on the choice of M and which are, up to a change of bases, given as follows:

1. if $M = S \oplus S$, where S is simple regular, then $y_1 x_1 = 0 = y_2 x_1$;
2. if $M = S_1 \oplus S_2$, where S_1 and S_2 are nonisomorphic simple regular, then $y_1 x_1 = 0 = y_2 x_2$;

3. if M is indecomposable regular of quasi-length two, then $y_2x_1 = 0$ and $y_1x_1 = y_2x_2$.

Of course, the actual form of the relations does not matter for the additivity of the function $\overline{\lambda}$ extending λ with $\overline{\lambda}(3) = 0$.

In the situation of Proposition 3.1 we next show that passage to the one-point sink extension $[M]\Sigma$ will change the rank of the Euler radical, hence of the group of all additive functions, by at most one.

PROPOSITION 3.4. *Let Σ and $\overline{\Sigma}$ be as in Proposition* 3.1. *Let r* (*resp.* $\overline{r}$) *denote the rank of the Euler radical R of Σ* (*resp. the Euler radical $\overline{R}$ of $\overline{\Sigma}$*). *Then $|\overline{r} - r| \leq 1$.*

Proof. Consider the map $\pi : \overline{R} \to \mathbb{Z}$, $x \mapsto \langle s^*, x\rangle_{\overline{V}}$. Then the kernel of π is a subgroup of the Euler radical R of Σ. Hence $\overline{r} \leq r + 1$.

Further, the subgroup R' of R consisting of all x with $\langle x, s\rangle = 0$ yields a subgroup of $\overline{R}$. Hence $r - 1 \leq \overline{r}$. ■

The situation of Proposition 3.1 is frequently encountered when dealing with one-point extensions with a module from a family of stable tubes that stay tubes in the derived category. To iterate the procedure it is important to know that one again gets such tubes with the same properties over the one-point extension. We write X^* for the Σ-dual of a Σ-module X.

THEOREM 3.5. *Assume that Σ satisfies the following conditions*:

(i) $\mathrm{gl.dim}\,\Sigma \leq 2$.

(ii) *Σ has an infinite family $\mathcal{T}$ of pairwise orthogonal stable tubes whose members have* $\mathrm{pd}_\Sigma N \leq 1$ *and satisfy* $N^* = 0$.

If M is a finite direct sum of objects in $\mathcal{T}$, then the one-point sink extension $\overline{\Sigma} = [M]\Sigma$ also satisfies conditions (i) *and* (ii) *with the new family of tubes $\overline{\mathcal{T}}$ obtained from $\mathcal{T}$ by removing the tubes containing a summand from M.*

Proof. We continue to use the conventions from the proof of Proposition 3.1, in particular we identify $\mathrm{mod}(\Sigma)$ with the full exact subcategory of right $\overline{\Sigma}$-modules taking value zero at the extension vertex.

The assumption $\mathrm{pd}_\Sigma M \leq 1$ implies that $\mathrm{gl.dim}\,\overline{\Sigma} \leq 2$. Since, moreover, projective Σ-modules stay projective over $\overline{\Sigma}$, it follows that each module X from $\overline{\mathcal{T}}$ has $\mathrm{pd}_{\overline{\Sigma}} X \leq 1$ and also $\mathrm{Hom}_{\overline{\Sigma}}(X, P) = 0$ if P is projective in $\mathrm{mod}(\Sigma)$. Denote by P_{n+1} the indecomposable projective corresponding to the extension vertex. Since $\mathrm{Hom}_{\overline{\Sigma}}(X, P_{n+1}) = \mathrm{Hom}_\Sigma(X, M) = 0$ by assumption, the modules X from $\overline{\mathcal{T}}$ satisfy $\mathrm{Hom}_{\overline{\Sigma}}(X, \overline{\Sigma}) = 0$.

Next we are going to show that an almost split sequence $0 \to \tau_\Sigma X \to E \to X \to 0$, where X as before is taken from $\overline{\mathcal{T}}$, stays almost split in $\mathrm{mod}(\overline{\Sigma})$. We view the $\overline{\Sigma}$-modules as triples (k^i, Y_Σ, h) where $h : M^i \to Y_\Sigma$

is a Σ-linear map. Let $f : \tau_\Sigma X \to \overline{Y}$ be a nonisomorphism, where $\overline{Y} = (k^i, Y_\Sigma, h)$ is indecomposable over $\overline{\Sigma}$. If $i = 0$ or if $\tau_\Sigma X \to Y$ is not a split monomorphism, it is clear that f extends to E. If $i \neq 0$ and $\tau_\Sigma X \to Y$ is a split monomorphism, then $\tau_\Sigma X$ becomes a direct summand of Y. Since $\tau_\Sigma X$ is not a direct summand of $\overline{Y}$ there must be a nonzero map $M \to \tau_\Sigma X$, which is impossible. ∎

COROLLARY 3.6. *Let Σ be concealed canonical, for instance tame concealed. Then Σ has a separating tubular family $\mathcal{T}$ of standard stable tubes which stay tubes in* $\mathrm{D}^{\mathrm{b}}(\Sigma)$. *If M is any module from $\mathcal{T}$, not necessarily indecomposable, then the one-point sink extension $[M]\Sigma$ has a tubular family in the derived category and also a nonnegative additive function.*

Proof. This follows from Theorem 3.5 and Proposition 3.1. ∎

EXAMPLE 3.7. This class covers those pg-critical algebras from [16] which arise as a one-point extension from a tame concealed algebra of type $(2, 2, n)$, i.e. $\widetilde{\mathbb{D}}_{n+2}$, with a regular module of regular length two from the tube of rank n. These algebras have global dimension ≤ 2 and all have a radical of rank two. This is put into proper context by our next result.

PROPOSITION 3.8. *Let Σ be a concealed canonical algebra and M a nonzero* (*and possibly decomposable*) *module taken from a separating family of stable tubes. Then the following hold for $\overline{\Sigma} = [M]\Sigma$:*

(i) *The Euler radical for $\overline{\Sigma}$ has rank one or two.*

(ii) *Always $\overline{\Sigma}$ has a nonnegative additive function. It has a positive additive function if and only if the rank of the Euler radical for $\overline{\Sigma}$ is two.*

Proof. If Σ is nontubular then the Euler radical of Σ has rank one, and assertion (i) follows from Propositions 3.1 and 3.4. For a tubular Σ we are going to show that the group of additive functions for Σ extending to $\overline{\Sigma}$ with value zero at the extension vertex has rank one, and then by Proposition 3.4 assertion (i) also follows in this case. Each additive function for Σ has the form $\lambda = \langle -, x\rangle$ where x belongs to the Euler radical of Σ. Moreover, λ extends to an additive function $\overline{\lambda}$ on $\overline{\Sigma}$ taking value zero at the extension vertex if and only if $\langle s, x\rangle = 0$. Since x is fixed under Φ_Σ this yields $\langle v, x\rangle = 0$, where $v = \sum_{i=1}^{p} \Phi_\Sigma^j(s)$. Note that by assumption v is nonzero. Since for a tubular algebra the restriction of the Euler form to the Euler radical R is nondegenerate [11], it follows that the subgroup of R formed by all x satisfying $\langle s, x\rangle = 0$ has rank one, which concludes this part of the proof.

Proposition 3.1 yields an additive function λ_1 for $\overline{\Sigma}$ which is positive on each vertex of Σ and takes value zero at the extension vertex. If rather $\overline{\Sigma}$ has an Euler radical of rank one, it cannot have a positive additive function. On the other hand, if the rank of the Euler radical for $\overline{\Sigma}$ is two, there is another additive function λ_2 with a positive value at the extension vertex. A suitable linear combination of λ_1 and λ_2 satisfies the requirements. ∎

As shown by the passage from tame concealed to tubular (resp. from tubular to wild canonical) through one-point extensions by a quasi-simple module, the rank of the Euler radical may actually increase (resp. drop) by one. In contrast to what happens in Example 3.7, we get an Euler radical of rank one if we form the one-point extension of $\widetilde{\mathbb{A}}_{p,q}$ by a regular module of quasi-length two from one of the exceptional regular components.

Whereas the requirement of a positive additive function forces a derived canonical algebra to be concealed canonical, in particular quasitilted, the consequences of the existence of a nonnegative additive function for a derived canonical algebra Σ are much weaker. Indeed, such an algebra Σ may have arbitrarily large global dimension and also arbitrarily large width—for the realization by a tilting complex—as can be seen from Example 4.3(a). In particular, such an algebra does not need to be quasitilted. Even the additional requirement $\operatorname{gl.dim}\Sigma \leq 2$ will not enforce quasitiltedness, as is shown by the following example.

Example 3.9. The endomorphism algebra Σ of a narrow tilting complex of canonical type always has global dimension ≤ 3. Even if Σ is not quasitilted it may happen that $\operatorname{gl.dim}\Sigma = 2$. For an explicit example consider the algebra given by the quiver

$$\begin{tikzcd}
 & & 1 \arrow[dr, "y"] & & \\
0 \arrow[r, "x"] & 1 \arrow[ur, "x"] \arrow[rr] & & 1 \arrow[r, "y"] & 0
\end{tikzcd}$$

with the two zero relations $x^2 = 0$ and $y^2 = 0$, which is derived canonical of weight type (4). We have also marked the values of the rank function.

No characterization of derived canonical algebras having a nonnegative additive function is known. A natural subclass of such algebras, however, is provided by the endomorphism algebras of compact and, more specifically, of narrow tilting complexes. By definition a tilting complex T in $\mathrm{D^b}(\operatorname{coh}\mathbb{X})$ is called *compact* if its indecomposable direct summands of nonzero rank all lie in a single copy of $\operatorname{coh}\mathbb{X}$ in the derived category. Call T_+ the direct sum of these summands. Then—up to translation in the derived category—we may assume that T_+ lies in $\operatorname{coh}\mathbb{X}$ and hence is a vector bundle. It is easy to see that $\Sigma_+ = \operatorname{End}(T_+)$ is concealed canonical and that $\Sigma = \operatorname{End}(T)$ is obtained from Σ_+ by branch enlargement in the sense of [1]. Conversely, each branch enlargement of a concealed canonical algebra is isomorphic to the endomorphism ring of a compact tilting complex (see [15]). Clearly, the rank function induces a nonnegative additive function for Σ. It is interesting to note in this context that according to [1] each representation-infinite algebra which is derived tame canonical is of the above type; see [15] for a different proof.

A compact tilting complex T is called *narrow* if its indecomposable summands lie in $\operatorname{coh}\mathbb{X}$ or in $\operatorname{coh}\mathbb{X}[-1]$. It is easily checked that in this case $\Sigma = \operatorname{End}(T)$ has global dimension ≤ 3 and that the rank function induces a nonnegative function for Σ which is l-additive for each $l \geq 2$. Each quasitilted algebra of canonical type is of this type, the converse not being true; see Example 4.3(b).

There are many further instances of algebras having families of stable tubes in $\operatorname{mod}(\Sigma)$ which stay tubes in the derived category. Besides the pg-critical algebras this concerns (Skowroński, unpublished) all coil algebras, generalized multicoil algebras, and combinations of those. A complete characterization of such algebras, however, seems to be difficult.

4. Examples. In this section we illustrate the limitations of what can be true through examples. We organize the examples according to the features we want to treat.

4.1. *The classical case.* We first consider the classical case, coming from quivers without relations. For a connected graph Σ it is well known that there is a positive additive function λ for Σ if and only if Σ is an extended Dynkin diagram. Actually, the answer is the same when considering a nonnegative additive function. To see this, assume λ is a nonzero additive function on Σ with $\lambda(x) = 0$, having a neighbour y with $\lambda(y) > 0$. By additivity of λ there must also be a neighbour z of x with $\lambda(z) < 0$. This contradiction shows that any nonnegative additive function λ on Σ must be positive.

There are however many other connected graphs Σ than the extended Dynkin diagrams which admit some nonzero additive function, and no complete description is known for such graphs. One simple procedure for obtaining graphs with some nonzero additive function is putting together positive additive functions for extended Dynkin diagrams, some with opposite sign. For example we have the following:

$$\begin{tikzcd}
1 \arrow[dr] & & 1 \arrow[dl] & & -1 \arrow[dr] & & -1 \arrow[dl] \\
& 2 & & & & -2 & \\
1 \arrow[ur] & & 1 \arrow[ul] & 0 \arrow[l] \arrow[r] & -1 \arrow[ur] & & -1 \arrow[ul]
\end{tikzcd}$$

4.2. *Additive functions and tubes.* We have seen that the existence of positive or nonnegative additive functions is closely connected with the existence of tubes for the module category or for the derived category.

We first show that an algebra can admit a positive additive function without having any tube for the module category or for the derived category.

EXAMPLE 4.2(a). The algebra Σ given by the quiver

$$1 \rightrightarrows 1 \xrightarrow{x} 1 \xrightarrow{y} 1 \rightrightarrows 1$$

with the relation $y \circ x = 0$ is derived equivalent to the wild hereditary algebra Δ given by the same quiver (where we remove the relation). The function λ on the vertices of Σ whose values are depicted above is additive and 2-additive. Since Δ is wild, there is however no nonzero object in the module category or in the derived category which is stable under τ.

The next example shows that there may be a tube in the derived category giving rise to a positive additive function, but no tube in the module category.

EXAMPLE 4.2(b). Let Σ be the algebra given by the linear quiver

$$\circ \xrightarrow{x} \circ \xrightarrow{x} \circ \xrightarrow{x} \circ \xrightarrow{x} \circ \xrightarrow{x} \circ \xrightarrow{x} \circ \xrightarrow{x} \circ \xrightarrow{x} \circ$$

of 9 points with the 6 zero relations $x^3 = 0$. It is easily checked that Σ is isomorphic to the endomorphism algebra of the tilting complex on the weighted projective line $\mathbb{X} = \mathbb{X}(2,3,5)$ given, in the above order, by the objects

$$\mathcal{O},\ S,\ \tau\mathcal{O}[1],\ \tau S[1],\ \tau^2\mathcal{O}[2],\ \tau^2 S[2],\ \tau^3\mathcal{O}[3],\ \tau^3 S[3],\ \tau^4\mathcal{O}[4],$$

where $\mathcal{O}$ is the structure sheaf, τ is the Auslander–Reiten translation in $\mathrm{coh}(\mathbb{X})$, and S is the unique simple sheaf of τ-period five with $\mathrm{Hom}(\mathcal{O}, S) \neq 0$. Hence Σ is derived equivalent to any tame hereditary algebra of type $\widetilde{\mathbb{E}}_8$. In view of [8] an equivalent statement is also that Σ is iterated tilted of type $\widetilde{\mathbb{E}}_8$.

Let Σ' be a hereditary algebra of type $\widetilde{\mathbb{E}}_8$. Then $\mathrm{D^b}(\Sigma) \cong \mathrm{D^b}(\Sigma')$ has a tube giving rise to an additive function for Σ. But since Σ is of finite type, there is no tube for Σ.

Next we describe the digraph $Q\Sigma$ which has the same vertices $1, \ldots, 9$ as Σ. If $|p-q| = 1$ or 4 or 7 then there is one solid edge between p and q, and there is one dotted edge between p and q if and only if $|p-q| = 3$ or 6.

Finally, we point out that even if there are tubes in the module category, there may not be any additive functions.

EXAMPLE 4.2(c). In [10] there are examples of tame algebras, tilted from wild hereditary algebras. The Euler radical is zero, so that there is no nonzero additive function. However, the algebras have tubes. Here one can see that for any tube either the condition "projective dimension at most one" or the condition "$X^* = 0$" is violated.

4.3. *Two-additive and additive functions.* For global dimension at most two the notions of 2-additive and additive functions coincide. For higher global dimension the concepts are usually different. As we have seen, it is the question of the relationship between the Tits and the Euler radical. We give various examples to show that they do not necessarily coincide.

Consider first Example 4.2(b). The rank function yields an additive function, unique up to multiplication by an integer, and given on the above quiver by the sequence $[1,0,-1,0,1,0,-1,0,1]$. It is clear that this function is not 2-additive and further easy to check that Σ does not admit any nonzero 2-additive function.

In the next examples there are nonnegative additive functions which are not 2-additive. This happens for arbitrarily high global dimension.

EXAMPLE 4.3(a). The algebra Σ given by the quiver

$$\circ \overset{x}{\underset{y}{\rightrightarrows}} \circ \xrightarrow{x} \circ \xrightarrow{x} \circ \cdots \circ \xrightarrow{x} \circ$$

with $n+1$ vertices and the $n-1$ relations $x^2 = 0$ is derived equivalent to the canonical algebra of type $(1,n)$, that is, to a tame hereditary algebra of type $\widetilde{\mathbb{A}}_n$, and actually realizable as the endomorphism algebra of the compact tilting complex on the weighted projective line $\mathbb{X}(n)$ of width $n-1$ consisting of

$$\mathcal{O},\ \mathcal{O}(\vec{c}),\ S,\ \tau S[1],\ \tau^2 S[2], \ldots, \tau^{n-2} S[n-2],$$

where S is the simple sheaf of τ-period n with $\mathrm{Hom}(\mathcal{O}, S) \neq 0$. In particular, Σ admits an additive function $[1,1,0,0,\ldots,0,0]$, which is however not 2-additive. Moreover, $\mathrm{gl.dim}\,\Sigma = n$, so that we can have arbitrarily large global dimension.

Next we give an example where the Tits radical is properly contained in the Euler radical.

EXAMPLE 4.3(b). The poset algebra Σ given by the quiver

$$\begin{tikzcd}
 & \circ \arrow[r] \arrow[ddr] & \circ \arrow[dr] & \\
\circ \arrow[ur] \arrow[r] \arrow[dr] & \circ \arrow[ur] \arrow[r] & \circ \arrow[r] & \circ \\
 & \circ \arrow[ur] \arrow[r] & \circ \arrow[ur] &
\end{tikzcd}$$

with all 6 possible commutativity relations is derived equivalent to the canonical algebra of tubular type $(3,3,3)$. Actually, Σ is the endomorphism algebra of a narrow tilting complex on the weighted projective line $\mathbb{X}(3,3,3)$. The algebra Σ has global dimension three and has an Euler radical of rank two.

The two rows of the scheme below constitute a basis of the group of additive functions:

$$\begin{array}{cccc} & \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] & \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right] & \\ \left[\begin{smallmatrix}-1\\0\end{smallmatrix}\right] & \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] & \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right] & \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] \\ & \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] & \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right] & \end{array}$$

The lower row defines a generator for the Tits radical which, in this case, is thus properly contained in the Euler radical. Moreover, and in contrast to Example 2.4, which also has an Euler radical of rank two, here there does not exist any positive additive function.

The next example, due to the referee, shows that positive additive and 2-additive functions do not always coincide.

EXAMPLE 4.3(c). The string algebra Σ given by the quiver

$$\circ \overset{x}{\underset{y}{\rightrightarrows}} \circ \overset{x}{\underset{y}{\rightrightarrows}} \circ \overset{x}{\underset{y}{\rightrightarrows}} \circ$$

with all possible relations $xy=0=yx$ has global dimension three. The Euler radical has rank three, and is generated by $[1,1,0,0]$, $[0,0,1,1]$ and $[0,1,1,0]$, whereas the Tits radical has rank one and is generated by $[0,1,1,0]$. Hence, there is a positive additive function but no positive 2-additive function.

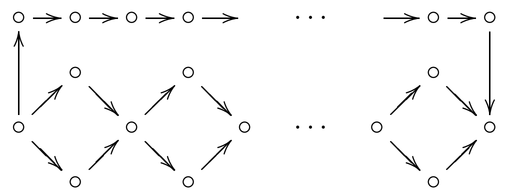
4.4. *Change under derived equivalence.* We have seen that the existence of a nonzero additive function is preserved under derived equivalence. This is however not the case for 2-additive functions.

Consider again Example 4.2(b). The algebras Σ and Σ' are derived equivalent, and Σ' has a 2-additive function. It is however easy to see that Σ does not have such a function.

This example also shows that for two derived equivalent algebras one may have a positive additive function for one, but for the other one not even a nonnegative additive function.

4.5. *Radical rank and nonnegative additive functions.* The main classes of algebras admitting a positive additive function are on the one hand the concealed canonical algebras and on the other the pg-critical algebras, which have Euler radical rank one or two. The next example shows that there are examples of algebras having arbitrarily high rank of the Euler radical, and having a positive additive function.

EXAMPLE 4.5. The algebra



with n commutative squares has global dimension two and a radical of rank n. The additive function λ taking value one on each vertex is positive; moreover, for each of the $n-1$ convex subquivers of shape

$$\begin{tikzcd} 1 \arrow[dr] & & 1 \\ & 2 \arrow[ur] \arrow[dr] & \\ 1 \arrow[ur] & & 1 \end{tikzcd}$$

we get an additive function, extending the above function by 0 for the remaining vertices. This yields a basis $\lambda, \lambda_1, \dots, \lambda_{n-1}$ for the group of additive functions, equivalently for the radical, expressed in the basis of classes of simple modules.

REFERENCES

[1] I. Assem and A. Skowroński, *Algebras with cycle-finite derived categories*, Math. Ann. 280 (1988), 441–463.

[2] M. Auslander, I. Reiten and S. O. Smalø, *Representation Theory of Artin Algebras*, Cambridge Univ. Press, 1995.

[3] H. S. M. Coxeter, *Regular Polytopes*, 3rd ed., Dover, 1973.

[4] W. Geigle and H. Lenzing, *A class of weighted projective curves arising in representation theory of finite dimensional algebras*, in: Singularities, Representations of Algebras, and Vector Bundles, Lecture Notes in Math. 1273, Springer, 1987, 265–297.

[5] D. Happel, *Triangulated Categories in the Representation Theory of Finite Dimensional Algebras*, London Math. Soc. Lecture Note Ser. 119, Cambridge Univ. Press, 1988.

[6] D. Happel, U. Preiser and C. M. Ringel, *Binary polyhedral groups and Euclidean diagrams*, Manuscripta Math. 31 (1980), 317–329.

[7] D. Happel, I. Reiten and S. O. Smalø, *Tilting in abelian categories and quasitilted algebras*, Mem. Amer. Math. Soc. 575 (1996).

[8] D. Happel, J. Rickard and A. Schofield, *Piecewise hereditary algebra*, Bull. London Math. Soc. 20 (1988), 23–28.

[9] T. Hübner, *Rank additivity for quasitilted algebras of canonical type*, Colloq. Math. 75 (1998), 183–193.

[10] O. Kerner, *Tilting wild algebras*, J. London Math. Soc. 39 (1989), 29–47.

[11] H. Lenzing and H. Meltzer, *Sheaves on a weighted projective line of genus one, and representations of a tubular algebra*, in: Representations of Algebras (Ottawa, 1992), CMS Conf. Proc. 14, Amer. Math. Soc., 1993, 313–337.

[12] —, —, *Tilting sheaves and concealed-canonical algebras*, in: Representations of Algebras (Cocoyoc, 1994), CMS Conf. Proc. 18, Amer. Math. Soc., 1996, 455–473.

[13] H. Lenzing and J. A. de la Peña, *Concealed-canonical algebras and separating tubular families*, Proc. London. Math. Soc. 78 (1999), 513–540.

[14] H. Lenzing and A. Skowroński, *Quasi-tilted algebras of canonical type*, Colloq. Math. 71 (1996), 161–181.

[15] H. Meltzer, *Exceptional vector bundles, tilting sheaves and tilting complexes on weighted projective lines*, Habilitationsschrift, TU Chemnitz, 1997.

[16] R. Nörenberg and A. Skowroński, *Tame minimal non-polynomial growth strongly simply connected algebras*, in: Representations of Algebras (Cocoyoc, 1994), CMS Conf. Proc. 18, Amer. Math. Soc., 1996, 519–538.

[17] I. Reiten, *Dynkin diagrams and the representation theory of algebras*, Notices Amer. Math. Soc. 44 (1997), 546–566.

[18] I. Reiten and A. Skowroński, *Sincere stable tubes*, preprint, 1999.

[19] C. M. Ringel, *Tame Algebras and Integral Quadratic Forms*, Lecture Notes in Math. 1099, Springer, 1984.

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