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FINITE GROUPS WITH GLOBALLY PERMUTABLE LATTICE OF SUBGROUPS

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Abstract. The notions of permutable and globally permutable lattices were first introduced and studied by J. Krempa and B. Terlikowska-Osłowska [4]. These are lattices preserving many interesting properties of modular lattices. In this paper all finite groups with globally permutable lattices of subgroups are described. It is shown that such finite p-groups are exactly the p-groups with modular lattices of subgroups, and that the non-nilpotent groups form an essentially larger class though they have a description very similar to that of non-nilpotent modular groups.

1. Preliminaries. Let L be a lattice with the least element 0 and the greatest element 1. Furthermore for all $a \le b \in L$ let $[a,b] = \{x \in L : a \le x \le b\}$ be an interval of L.

Let $X = \{x_1, \ldots, x_n\} \subset L \setminus \{0\}$. Then we will say that X is (or the elements $x_1, \ldots, x_n$ are):

- *independent* if for every $1 \le i \le n$ we have $x_i \wedge (\bigvee_{k \ne i} x_k) = 0$;
- *sequentially independent* if for every $k < n$ we have $(\bigvee_{j=1}^{k} x_j) \wedge x_{k+1} = 0$.

As in [4] a lattice L will be called *permutable* if any 3-element sequentially independent subset of L is in fact independent. In other words, L is permutable if for all $x, y, z \in L$,

$$x \wedge y = 0 \ \&\ (x \vee y) \wedge z = 0 \Rightarrow (y \vee z) \wedge x = 0 \ \&\ (z \vee x) \wedge y = 0.$$

A lattice L will be called *globally permutable* if all non-empty intervals of L are permutable lattices. It is clear that every globally permutable lattice is permutable but not conversely.

Permutable lattices were first studied in [4] where their basic properties were described and some known results concerning uniform dimension of modular lattices were extended to this broader class of lattices. The description of finite permutable lattices (see Theorem 1.1 below) suggests that they should be very similar to modular lattices. On the other hand the modularity of subgroup lattices is an interesting property of groups. It is therefore of some interest and importance to study the role which the class

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of permutable lattices plays in investigation of groups. The theory of subgroup lattices has been studied for many years. For a survey of known results see for instance the book [6] of Schmidt.

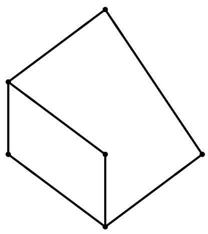
In this paper we give a description of all finite groups with globally permutable lattice of subgroups. We are indebted to J. Krempa for encouraging us to study this topic and for many useful conversations.

We begin with some known results. Our notation is standard and follows [2], [3] and [6]. In particular, $L(G)$ denotes the lattice of subgroups of a group G.

THEOREM 1.1 ([4]). *Let L be a finite lattice. Then*:

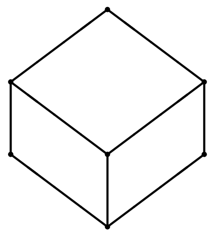
(a) *L is globally permutable if and only if none of the sublattices of L is isomorphic to the lattice L_1 or L_2 (see figures below).*

(b) *L is permutable if and only if none of the sublattices of L containing 0 of L is isomorphic to L_1 or L_2.*



L_1

Fig. 1



L_2

Fig. 2

COROLLARY 1.2. *Any finite modular lattice is globally permutable.*

THEOREM 1.3 ([6]). *Let G be a finite p-group. The following conditions are equivalent*:

(a) *$L(G)$ is modular.*

(b) *Either G is a Hamiltonian 2-group, or G contains an abelian normal subgroup A with cyclic factor group G/A; further there exists an element $b \in G$ with $G = A\langle b\rangle$ and a positive integer s such that $b^{-1}ab = a^{1+p^s}$ for all $a \in A$, with $s \geq 2$ in case $p = 2$.*

(c) *Each section of G of order p^3 is modular.*

We say that G is a P^*-*group* if G is a semidirect product of an elementary abelian normal subgroup A by a cyclic group $\langle t\rangle$ of prime power order such that t induces a power automorphism of prime order on A.

THEOREM 1.4 ([6]). *A finite group has a modular subgroup lattice if and only if it is a direct product of P^*-groups and modular p-groups with relatively prime orders.*

For subgroups $H_1, \dots, H_n$ of a group G we denote by $L(H_1, \dots, H_n)$ the sublattice of $L(G)$ generated by these subgroups.

LEMMA 1.5 ([5]). *Let G be an arbitrary group. If A and B are normal subgroups of G and C is a subgroup of G then $L(A, B, C)$ is modular.*

We begin with a simple observation concerning finite p-groups and nilpotent groups.

PROPOSITION 1.6. *Let G be a finite p-group. The lattice $L(G)$ is globally permutable if and only if $L(G)$ is modular.*

Proof. By Theorem 1.3 we need only show that the dihedral group of order 8 and the non-abelian group of order p^3 and exponent p, $p > 2$, are not globally permutable.

First let $G = \langle a, b \mid a^4 = b^2 = 1,\ b^{-1}ab = a^3 \rangle$ be dihedral of order 8. For $A = \langle ab \rangle$, $B = \langle a^2 \rangle$, $C = \langle b \rangle$, we have $|A| = |B| = |C| = 2$, $|A \vee B| = 4 = |B \vee C|$, $A \vee C = A \vee B \vee C = G$ and obviously $A \wedge B = A \wedge C = B \wedge C = \{e\}$. Hence the lattice $L(A, B, C)$ is isomorphic to L_2.

Now let $G = \langle a, b, c \mid a^p = b^p = c^p = 1,\ ab = bac,\ ac = ca,\ cb = bc \rangle$. Again it is easily seen that for the subgroups $A = \langle a \rangle$, $B = \langle b \rangle$, $C = \langle c \rangle$ the lattice $L(A, B, C)$ is isomorphic to L_2. ■

Since the direct product of globally permutable lattices is globally permutable, we have:

COROLLARY 1.7. *Let G be a finite nilpotent group. Then $L(G)$ is globally permutable if and only if $L(G)$ is modular.* ■

The following example shows that this is not the case in general.

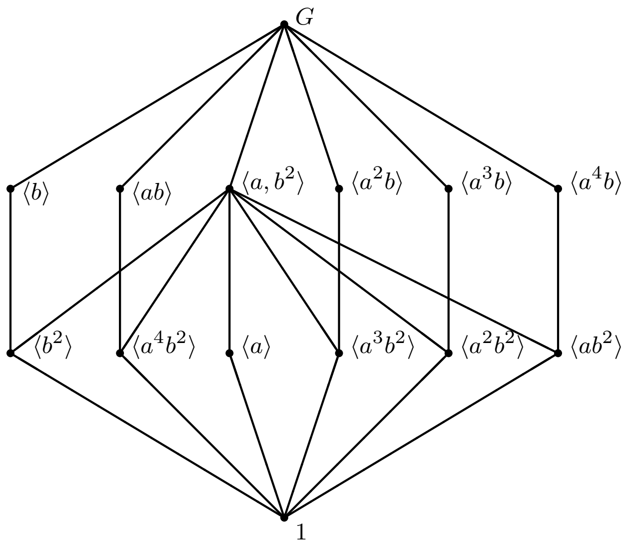


Fig. 3

EXAMPLE 1. Let

$$G = \langle a, b \mid a^5 = b^4 = 1,\ b^{-1}ab = a^2 \rangle,$$

that is, G is a group of order 20 which is a semidirect product of a cyclic group of order 5 and its group of automorphisms. It is clear that $L(G)$ is not modular. Moreover, as is seen from Fig. 3, $L(G)$ does not contain a sublattice isomorphic to L_1 or L_2. Hence $L(G)$ is globally permutable. This is the smallest group whose lattice is globally permutable but not modular, as will be seen from Theorem 2.9. Note also that in $L(G)$ a normal subgroup and two arbitrary subgroups generate a modular lattice.

2. Main results. Now we turn to the general case. We begin with recalling Thompson's classification of minimal simple groups.

THEOREM 2.1 ([8]). *Every minimal simple group is isomorphic to one of the following groups*:

(a) $\mathrm{PSL}(2, 2^p)$, *p any prime*,
(b) $\mathrm{PSL}(2, 3^p)$, *p any odd prime*,
(c) $\mathrm{PSL}(2, p)$, *p any prime exceeding 3 such that* $p^2 + 1 \equiv 0 \pmod 5$,
(d) $\mathrm{Sz}(2^p)$, *p any odd prime*,
(e) $\mathrm{PSL}(3, 3)$.

LEMMA 2.2. *If G is soluble and the lattice $L(G)$ is globally permutable then G is supersoluble.*

Proof. Suppose that G is not supersoluble with smallest possible order. Let

$$1 = G_0 \le G_1 \le \ldots \le G_n = G$$

be a chain of normal subgroups of G, which cannot be ramified. If G_1 is cyclic then G/G_1 is again globally permutable and not supersoluble, which contradicts the minimality of G. So G_1 is an elementary abelian p-group which is not cyclic and for each $g \in G_1$, $\langle g^G \rangle = G_1$.

Fix $x \in G_1$. We show that there exists a p'-element y in G such that $x^y \notin \langle x \rangle$. Suppose not, that is, $g^y \in \langle g \rangle$ for every p'-element y of G and every $g \in G_1$. Let q be the largest prime dividing $|G|$ and let Q be a Sylow q-subgroup of G. By ([3], VI, 9.1), G_1Q is a normal subgroup of G. If $q > p$ then elements of Q act trivially on G_1. Otherwise there exist $y \in Q$ and $x_1, x_2 \in G_1$ such that $\langle x_1 \rangle \cap \langle x_2 \rangle = 1$ and $x_1^y = x_2$. It is easily seen that $L(\langle x_1 \rangle, \langle x_2 \rangle, \langle y \rangle) \simeq L_1$.

So $G_1Q = G_1 \times Q$ and then Q is a characteristic subgroup of G_1Q, that is, Q is normal in G. Moreover G/Q is not supersoluble, which is not possible by minimality of G. Hence $q = p$ and then $G_1 \le Q$. But by ([3], III, 2.6), G_1 must be contained in the center $Z(Q)$ of Q, so all cyclic subgroups of G_1

must be invariant under conjugation by all elements of G. This contradicts the fact that G_1 is a minimal normal subgroup of G and is not cyclic.

Now let $A = \langle x\rangle$, $B = \langle x^y\rangle$, $C = \langle y\rangle$, where $x \in G_1$ and y is a p'-element such that $x^y \notin \langle x\rangle$. Standard considerations show that the sublattice $L(A,B,C)$ of $L(G)$ is isomorphic to L_1. ■

THEOREM 2.3. *Let G be a finite group. If $L(G)$ is globally permutable then G is supersoluble.*

Proof. We use Thompson's classification of minimal simple groups (Theorem 2.1). First we show that none of the groups listed in Theorem 2.1 is globally permutable.

Let G be the Suzuki group $\mathrm{Sz}(2^p)$ with p prime. Then $G = \langle A, C\rangle$ where

$$A = \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ a^\theta + b & a^\theta & 1 & 0 \\ a^{2+\theta} + ab + b^\theta & b & a & 1 \end{pmatrix} \,\middle|\, a, b \in \mathrm{GF}(2^p) \right\rangle,$$

$$C = \left\langle \begin{pmatrix} 1 & a & a^{1+\theta} + b & a^{2+\theta} + ab + b^\theta \\ 0 & 1 & a^\theta & b \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix} \,\middle|\, a, b \in \mathrm{GF}(2^p) \right\rangle$$

and θ is the automorphism of $\mathrm{GF}(2^p)$ such that $\theta^2 = 2$ ([2]). Let

$$B = \left\langle \begin{pmatrix} c^{1+\theta} & 0 & 0 & 0 \\ 0 & c^\theta & 0 & 0 \\ 0 & 0 & c^\theta & 0 \\ 0 & 0 & 0 & c^{-1-\theta} \end{pmatrix} \right\rangle$$

with $0 \neq c \in GF(2^p)$. Then $B \not\leq \langle A, C\rangle$ and $L(A,B,C) \simeq L_2$.

Now let $G = \mathrm{PSL}(2,p)$ with p prime such that $p^2 + 1 \equiv 0 \pmod 5$. Let $F = \mathrm{GF}(p)$ and $\zeta \in F$ be a generator of the multiplicative group F^*. It is easily seen that as in the previous case the subgroups

$$A = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle, \quad B = \left\langle \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle, \quad C = \left\langle \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \right\rangle$$

generate a sublattice of $L(\mathrm{SL}(2,p))$ isomorphic to L_2. Moreover the image of $L(A,B,C)$ in $L(\mathrm{PSL}(2,p))$ under the epimorphism induced by the natural epimorphism $\mathrm{SL}(2,p) \to \mathrm{PSL}(2,p)$ is isomorphic to $L(A,B,C)$.

Now let $G = \mathrm{PSL}(3,3)$. Let $F = Z_3$. Consider three elements $a, b \in \mathrm{UT}(3,3)$ and $c \in D(3,3)$ such that

$$a = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad c = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $A = \langle a \rangle$, $B = \langle b \rangle$, $C = \langle c \rangle$. Then $|A| = |B| = 3$, $|C| = 2$ and $A \wedge B = A \wedge T = B \wedge T = E$. Since $b = cac$, we have $A \vee C = A \vee B \vee C = B \vee C$ and $(B \vee A) \wedge C = E$. Hence $L(A,B,C)$ is isomorphic to L_1.

Finally, let G be one of the groups $\mathrm{SL}(2,2^p)$, $\mathrm{SL}(2,3^p)$ where p is a prime as in Theorem 2.1. It is well known that the upper triangular subgroups of these groups are soluble but not supersoluble. So by Lemma 2.2 they are not globally permutable and their images in $\mathrm{PSL}(*,*)$ are not either. ∎

LEMMA 2.4. *Let $G = PH$ be a semidirect product of a normal elementary abelian p-group P and a cyclic p'-group $H = \langle y \rangle$. If $L(G)$ is globally permutable, then there exists an integer k such that $y^{-1}xy = x^k$ for all $x \in P$. Moreover there exists a prime $q < p$ such that $|G/C_G(P)| = q^n$.*

Proof. Let $h \in H$ and $x \in P$. If $x^h \notin \langle x \rangle$ then one can easily check that $L(\langle x^h \rangle, \langle x \rangle, \langle h \rangle) \simeq L_1$. Hence all cyclic subgroups of P are normal in G. Let $\langle x_1, x_2 \rangle$ be a subgroup of P of order p^2. If $x_1^y = x_1^{k_1}$, $x_2^y = x_2^{k_2}$ where $k_1 \not\equiv k_2 \pmod{p}$ then $(x_1x_2)^y = x_1^{k_1}x_2^{k_2} \notin \langle x_1x_2 \rangle$. That is, the cyclic subgroup $\langle x_1x_2 \rangle$ is not normal in G. Therefore $k_1 \equiv k_2 \pmod{p}$ and we can replace all k_i by a fixed k for all $x \in P$. Now suppose that there are distinct primes q, r such that qr divides $|G/C_G(P)|$. In other words there exist $y_1, y_2 \in \langle y \rangle$ and integers m_1, m_2 such that $o(y_1) = q$, $o(y_2) = r$, $m_1 \not\equiv 1 \pmod{p}$, $m_2 \not\equiv 1 \pmod{p}$ and $x^{y_1} = x^{m_1}$, $x^{y_2} = x^{m_2}$ for all $x \in G$. Let $A = \langle y_1 \rangle$, $B = \langle x \rangle$, $C = \langle xy_2 \rangle$, where x is a fixed element of P, $x \neq 1$. We have $|A \vee B| = pq$, $|B \vee C| = pr$, $A \vee C = A \vee B \vee C$ and $A \wedge B = A \wedge C = B \wedge C = \{e\}$. Hence the lattice $L(A,B,C)$ is isomorphic to L_2. Therefore $|G/C_G(P)|$ is a power of a prime q and of course $q \mid p - 1$. ∎

COROLLARY 2.5. *Let p be the largest prime dividing $|G|$ and let P be an elementary abelian Sylow p-subgroup of G. If $L(G)$ is globally permutable then $G/C_G(P)$ is cyclic.* ∎

LEMMA 2.6. *Let p be the largest prime dividing $|G|$ and let P be a Sylow p-subgroup of G. If $L(G)$ is globally permutable and $C_G(P) \neq G$ then P is elementary abelian.*

Proof. Let G be a minimal counter-example to the lemma. By Corollary 2.3 and ([3], VI, 9.1), P is a normal subgroup of G. By Lemma 2.4 and by the choice of G there exists an element y of order q^n, q prime, $q < p$, such that $y \in G \setminus C_G(P)$ and $G = \langle P, y \rangle = P\langle y \rangle$. Since $\Phi(P) \lhd G$ and

$\langle \Phi(P), y\rangle < G$, $\Phi(P)$ is elementary abelian and again by Lemma 2.4 there exist integers k and m such that for all $z \in \Phi(P)$,

$$z^y = z^k, \tag{1}$$

and for every $g \in P$ there exists $z \in \Phi(P)$ such that

$$g^y = g^m z. \tag{2}$$

As P is modular, by Proposition 1.6, but not elementary abelian, there exists an element x in P of order p^2 (Theorem 1.3) and $x^y = x^k z$ for a certain element $z \in \Phi(P)$. Now it follows from (1) and (2) that $\langle x, z\rangle \lhd G$, $p^2 \le |\langle x, z\rangle| \le p^3$ and by minimality of G, $\langle x, y, z\rangle = G$. If $|\langle x, z\rangle| = p^2$ then $z \in \langle x\rangle$ and $\langle x\rangle \lhd G$. For the subgroups $A = \langle y\rangle$, $B = \langle x^p\rangle$, $C = \langle xy\rangle$ we have $L(A, B, C) \simeq L_2$, a contradiction. Therefore $|\langle x, z\rangle| = p^3$ and $\Phi(\langle x, z\rangle) = \langle x^p\rangle$. Now in the factor group $\langle x', y', z'\rangle = \langle x, y, z\rangle/\langle x^p\rangle$ the subgroups $A = \langle x'\rangle$, $B = \langle z'\rangle$, $C = \langle y'\rangle$ again generate a sublattice isomorphic to L_2. This contradiction ends the proof. ■

LEMMA 2.7. *Let p be the largest prime dividing $|G|$ and let P be a Sylow p-subgroup of G. If $L(G)$ is globally permutable and P is not a direct factor of G, then there exists a prime q and a Sylow q-subgroup Q of G such that PQ is a direct factor of G.*

Proof. Let $\{p_1, \ldots, p_t\}$ be the set of all primes smaller than p dividing $|G|$, $p_1 > \ldots > p_t$. Since, by Theorem 2.3, G is supersoluble, it has a Sylow system $\{P, P_1, \ldots, P_t\}$ of p_i-subgroups of G ([3], VI, 2.3). Hence P is normal in G and there exists a p'-subgroup H of G such that $G = PH$, $H = P_1 \ldots P_t$ and $P_1 \ldots P_i \lhd H$ for $i = 1, \ldots, t$. Let $q = p_s$ be the largest prime among $p_1, \ldots, p_t$ such that certain q-elements act non-trivially by conjugations on P. We denote by K the subgroup $P_1 \ldots P_{s-1}$. By the choice of q, $K \le C_G(P)$ and then $PK = P \times K \lhd G$.

Let $Q = P_s$ be the Sylow q-subgroup of G. If $Q \not\le C_G(K)$ then there exists $i, 1 \le i \le s-1$, such that $Q \not\le C_G(P_i)$. Since $L(PQ)$ and $L(P_iQ)$ are globally permutable, P and P_i are elementary abelian by Lemma 2.6. Let $y \in Q \setminus (C_G(P) \cup C_G(K))$ and let $x_1 \in P$, $x_2 \in P_i$, $x_1, x_2 \ne 1$. By Lemma 2.4, $y^{-1}x_1y = x_1^{k_1}$ $y^{-1}x_2y = x_2^{k_2}$. It is easy to see that $L(\langle y\rangle, \langle x_1\rangle, \langle yx_1x_2\rangle) \simeq L_2$, a contradiction.

Therefore $Q \le C_G(K)$ and $PKQ = PQ \times K$. Let $T = P_{s+1} \ldots P_t$. Since $H = (K \times Q)T$, we have $G = ((PQ) \times K)T$. Now suppose that there exists a prime $r = p_j$, $r < q$, such that the Sylow r-subgroup $R = P_j$ is not contained in $C_G(PQ)$. Let $z \in R \setminus C_G(PQ)$. If $z \in C_G(Q)$ then $z \notin C_G(P)$ and for $y \in Q \setminus C_G(P)$ and $1 \ne x \in P$ we have $L(\langle y\rangle, \langle x\rangle, \langle xz\rangle) \simeq L_2$. So $z \notin C_G(Q)$. Since $L(QR)$ is globally permutable, Q is elementary abelian and there exists an integer k such that for every $a \in Q$, $z^{-1}az = a^k$. Now suppose $y \in Q$ does not centralize P. Then in the factor group $G/C_G(P)$, the images of y, z are non-trivial and $\langle zC_G(P), yC_G(P)\rangle \simeq \langle y, z\rangle$ is non-abelian. This contradicts $G/C_G(P)$ being cyclic by Corollary 2.5. Thus all elements of T commute with all elements of PQ and then $G = (PQ) \times (KT)$. ∎

LEMMA 2.8. *Let p and q be primes with $p > q$ and let $G = PQ$, where $P \lhd G$ is a Sylow p-subgroup and Q is a Sylow q-subgroup of G. If $L(G)$ is globally permutable and $Q \not\leq C_G(P)$ then Q is cyclic.*

Proof. Suppose by contradiction that Q is not cyclic. Since by Corollary 2.5 the commutator subgroup Q' of Q is contained in $C_Q(P)$ it has to be a normal subgroup of G. So we may assume that $Q' = 1$ (otherwise replace G by G/Q'). Now let $1 \neq x \in P$, $y \in Q \setminus C_Q(P)$ and $1 \neq z \in C_Q(P) \setminus \langle y\rangle$. We may also assume that y and z are of order q. It is easily seen that for $A = \langle y\rangle$, $B = \langle y^x\rangle$, $C = \langle y^x z\rangle$ we have $L(A, B, C) \simeq L_2$. ∎

As an immediate consequence of Lemmas 2.4, 2.6, 2.7 and 2.8 we get the following.

PROPOSITION 2.9. *Let G be a finite group which is not a direct product of its nontrivial subgroups. Then $L(G)$ is globally permutable if and only if there exist primes p and q, $p > q$, such that*:

(a) $|G| = p^n q^m$;

(b) *A Sylow p-subgroup P of G is normal in G and elementary abelian*;

(c) *Sylow q-subgroups of G are cyclic*;

(d) *If $Q = \langle y \mid y^{q^m} = 1\rangle$ is a Sylow q-subgroup of G, then there exists an integer k such that $g^y = g^k$ for all $g \in P$, and $k^{q^m} \equiv 1 \pmod{p}$.*

Let p and q be primes and let k, m, n, r be positive integers such that $m \geq k$ and $q^r \mid p - 1$, $k^{q^r} \equiv 1 \pmod{p}$, $k^{q^{r-1}} \not\equiv 1 \pmod{p}$. The groups described in Proposition 2.9 have the following presentation in terms of generators and relations:

$$(3)\quad G = \langle y, x_1, \ldots, x_n \mid y^{q^m} = x_i^p = 1,\ [x_i, x_j] = 1,\ y^{-1}x_i y = x_i^k,\ i, j = 1, \ldots, n\rangle.$$

We will complete the classification of finite groups with globally permutable lattice of subgroups by proving that $L(G)$ is globally permutable. We begin with listing some elementary properties of G.

Let $P = \langle x_1, \ldots, x_n\rangle$. It is clear that P is a Sylow p-subgroup of G and $P \lhd G$. For a subgroup H of G we denote by P_H a Sylow p-subgroup of H. One can easily see that $P_H = P \cap H$.

LEMMA 2.10. (i) *If $H \leq G$ then there exists $x \in P$ and an integer i, $0 \leq i \leq m$, such that $H = \langle y^{q^i} x, P_H\rangle$.*

(ii) $Z(G) = \langle y^{q^r}\rangle$.

(iii) *A subgroup H of G is normal in G if and only if $H \leq PZ(G)$ or $P \leq H$.*

(iv) *A subgroup H is not normal in G if and only if $P_H \neq P$ and Sylow q-subgroups of H contain non-trivial central elements.*

LEMMA 2.11. (i) *If H and K are q-subgroups of G such that $H \not\leq K$ and $K \not\leq H$ then $H \cap K = Z(G)$.*

(ii) *If H and K are not normal subgroups of G then $Z(G) \leq H \cap K$.*

(iii) *If H and K are subgroups of G such that $H \cap K \ntriangleleft G$ then some Sylow q-subgroup of H is contained in K or some Sylow q-subgroup of K is contained in H.*

(iv) *If $H, K \leq G$ then $P_H \cap P_K = P_{H \cap K}$.*

PROPOSITION 2.12. *If a group G has presentation* (3) *then $L(G)$ is globally permutable.*

Proof. We show that neither L_1 nor L_2 can be embedded into $L(G)$. Let A, B, C be subgroups of G, at most one of them being normal in G (Lemma 1.5). Suppose by way of contradiction that $L(A, B, C) \simeq L_1$ or $L(A, B, C) \simeq L_2$, that is, A, B and C are situated as in Figure 4 or Figure 5.

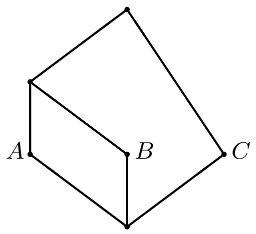


Fig. 4

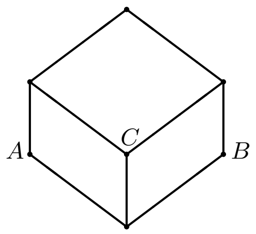


Fig. 5

Since by Lemma 2.11(ii), $Z(G) \leq A \wedge B \wedge C$, we can consider $G/Z(G)$ in place of G and assume that $Z(G) = 1$. By Lemma 2.10(i) we can assume that

$$A = \langle y^{q^i} x, P_A \rangle, \quad B = \langle y^{q^j} x', P_B \rangle, \quad C = \langle y^{q^l}, P_C \rangle$$

where $x, x' \in P$. We consider several special cases:

I. $A \wedge B \wedge C \triangleleft G$. It is obvious that $P \not\leq A \wedge B \wedge C$ because all the subgroups of G containing P form a chain. Hence by Lemma 2.10(iv) we need only show that a non-trivial q-element belongs to $A \wedge B \wedge C$, which contradicts the normality of $A \wedge B \wedge C$.

I.1. $L(A, B, C) \simeq L_1$. Suppose first that exactly one of the subgroups A, B, C is normal in G. Note that, by Lemma 2.10(iii), a normal subgroup of G either contains P or is contained in P.

I.1.a. Let $A \lhd G$. Assume first that $A \leq P$, i.e. $A = P_A$. Since $B, C \ntriangleleft G$ we have $y^{q^j} \neq 1 \neq y^{q^l}$. Moreover

$$\langle y^{q^l}, P_A, P_C\rangle = A \vee C = B \vee C = \langle y^{q^l}, y^{q^j}, x', P_B, P_C\rangle.$$

Hence there are $a \in P_A$ and $c \in P_C$ such that $x' = ac$. Therefore $y^{q^j}c = (y^{q^j}x')a^{-1} \in A \vee B$. Observe that $l \leq j$; otherwise Sylow q-subgroups of $A \vee C$ and $B \vee C$ would have different orders. Thus $y^{q^j}c \in C$ and then $1 \neq y^{q^j}c \in (A \vee B) \wedge C = A \wedge B \wedge C$. This contradicts the fact that $A \wedge B \wedge C \lhd G$.

Now assume that $P \leq A$. Then $P_A = P$ and $P_C = P_C \wedge P = P_B \wedge B = P_B$. Of course $P_C < P$—otherwise either $A \leq C$ or $C \leq A$. In particular we have $P \nleq A \wedge C = A \wedge B \wedge C$. Since by assumption $C \ntriangleleft G$ again we obtain $y^{q^l} \neq 1$. So if $y^{q^i} \neq 1$ there exists an element $1 \neq y' \in \langle y^{q^i}\rangle \cap \langle y^{q^l}\rangle \leq A \wedge C = A \wedge B \wedge C$ (as $A = \langle y^{q^i}, P\rangle$). This again contradicts the fact that $A \wedge B \wedge C \lhd G$. Hence $y^{q^i} = 1$ and then $A = P$. This is a special case of the situation $A \leq P$ which was already considered.

I.1.b. $C \lhd G$ and $A, B \ntriangleleft G$. Assume that $C \leq P$. Since $A \vee C = B \vee C$, Sylow q-subgroups of A and B must have the same order. This means that $i = j$. Now

$$A \vee B = \langle y^{q^i}x, x^{-1}x', P_A, P_B\rangle \leq B \vee C = \langle y^{q^i}x', P_B, P_C\rangle.$$

So there exist $b \in P_B$ and $c \in P_C$ such that $x^{-1}x' = bc$ and then $x^{-1}x'b^{-1} = c \in (A \vee B) \wedge C = B \wedge C \leq B$. Hence $x^{-1}x' \in B$ and $y^{q^i}x = y^{q^i}x'(x^{-1}x')^{-1} \in B$. Therefore $y^{q^i}x \in A \wedge B$. But $y^{q^i}x \neq 1$, which contradicts the normality of $A \wedge B \wedge C$ in G. Now assume that $P < C$, that is, $P_C = P$ and $y^{q^l} \neq 1$. Then $\langle y^{q^i}x\rangle$ has non-trivial intersection with C. As in the previous case this means that $A \wedge B \wedge C = A \wedge C \ntriangleleft G$.

I.1.c. Now let $A, B, C \ntriangleleft G$, that is, $y^{q^i}, y^{q^j}, y^{q^l} \neq 1$. Since A and B play symmetric roles in $L(A, B, C)$ we may assume that $i \leq j$. Let $z \in P$ be such that $(y^{q^i}x)^{q^{j-i}}z = y^{q^j}x'$, that is, $z = x^t x'$ for some $t \in \mathbb{N}$. Similarly to the previous case we have $A \vee B = \langle y^{q^i}x, z, P_A, P_B\rangle$. Moreover

$$\langle y^{q^{\min\{i,l\}}}, x, P_A, P_C\rangle = A \vee C = B \vee C = \langle y^{q^{\min\{j,l\}}}, x', P_B, P_C\rangle.$$

It is clear that $x \neq 1 \neq x'$ and even more, $x \notin P_A \vee P_C$; otherwise there are $a \in P_A$ and $c \in P_C$ such that $x = ac$ and then $xa^{-1} = c$ and $y^{q^i}xa^{-1} = y^{q^i}c$. Let $1 \neq y' \in \langle y^{q^i}\rangle \cap \langle y^{q^j}\rangle \cap \langle y^{q^l}\rangle$ and $d \in \mathbb{N}$ be such that $(y^{q^i})^d = y'$. Then $(y^{q^i}xa^{-1})^d = (y^{q^i}c)^d = y'c' \in A \wedge C = A \wedge B \wedge C$ and $y'c' \notin P$, a contradiction. Analogously $x' \notin P_B \vee P_C$. We now have $z = x^t x' \in A \vee C = A \vee B \vee C$, i.e. there are $a \in A$, $c \in C$ and an integer v such that $x^t x' = x^v ac$.

If $v \equiv 0 \pmod p$ then $za^{-1} = c \in (A \vee B) \wedge C = A \wedge B \wedge C$. Hence $z = (za^{-1})a \in A$ and then $y^{q^j}x' = (y^{q^i}x)^{q^{j-i}}z \in A \wedge B$, a contradiction. If $v \equiv t \pmod p$ then $x'a^{-1} = c$, that is, $y^{q^j}c = y^{q^j}x'a^{-1} \in A \vee B$. Again let $1 \neq y' \in$

$\langle y^{q^j}\rangle \cap \langle y^{q^l}\rangle$ and $(y^{q^j})^d = y'$. Thus $(y^{q^j}x'a^{-1})^d = (y^{q^j}c)^d = y'c' \in (A\vee B)\wedge C$. So we may assume that $v \not\equiv 0 \pmod p$ and $v \not\equiv t \pmod p$. Take an integer w such that $uw \equiv -1 \pmod p$. Therefore $c^w = (x^{-v}za^{-1})^w = xz^wa^{-w}$ and then $A \vee B \ni ((y^{q^i}x)z^wa^{-w})^d = (y^{q^i}c^w)^d = y'c' \in C$ for a suitable d such that $(y^{q^i})^d = y' \neq 1$. This is the last contradiction which ends part I.1.

I.2. $L(A,B,C) \simeq L_2$. The proof is similar to that in I.1.

I.2.a. Let $A \lhd G$ and $A \leq P$ (i.e. $A = P_A$). We also have $y^{q^j} \neq 1 \neq y^{q^l}$ and

$$A \vee C = \langle y^{q^l}, P_A, P_C\rangle, \qquad A \vee B = \langle y^{q^j}x', P_A, P_B\rangle = A \vee B \vee C.$$

Since Sylow q-subgroups of $A\vee B$ have order $|y^{q^j}x'|$, we obtain $|y^{q^l}| \leq |y^{q^j}x'|$, that is, $l \geq j$. This means that $B \vee C = \langle y^{q^j}, x', P_B, P_C\rangle$. Observe now that $x' \in \langle P_A, P_B\rangle$, that is, $x' = ab$ where $a \in P_A$, $b \in P_B$. Hence $x'b^{-1} = a \in (B \vee C) \wedge A = A \wedge B \wedge C$ and then $x' = (x'b^{-1})b \in B$. Thus $y^{q^j} \in B$, which implies $1 \neq y^{q^l} \in B \wedge C$, a contradiction.

The case $P \leq A$ can be easily reduced to $P = A$, which belongs to the case just considered.

I.2.b. $C \lhd G$. Again we only consider the situation $C \leq P$, because the case $P \leq C$ easily reduces to $P = C$. So let $C \leq P$, that is, $C = P_C$. Since A and B play symmetric roles in $L(A,B,C)$ we may assume that $i \leq j$. We now have

$$A \vee C = \langle y^{q^i}x, P_A, P_C\rangle, \qquad A \vee B = \langle y^{q^i}x, y^{q^j}x', P_A, P_B\rangle.$$

Let z be an element of P such that $(y^{q^i}xz)^{q^{j-i}} = y^{q^j}x'$. Hence

$$A \vee B = \langle y^{q^i}x, z, P_A, P_B\rangle.$$

Since $P_C \not\leq P_A$ (otherwise $C \leq A$) we can find $c \in P_C \setminus P_A$. Take $a \in P_A$ and $b \in P_B$ such that $c = z^d ab$ as $c \in A \vee B$. If $d \equiv 0 \pmod p$ then $ca^{-1} = b \in (A \vee C) \wedge B = A \wedge B \wedge C$. Hence $c = (ca^{-1})a \in A$. So $d \not\equiv 0 \pmod p$ and we may assume that $d = 1$ (if it is not the case we can take a suitable power of c in place of c). Thus $(y^{q^i}x)ca^{-1} = (y^{q^i}xz)b$ and

$$(y^{q^i}xca^{-1})^{q^{j-i}} = y^{q^j}x'b' \in (A \vee C) \wedge B = A \wedge B \wedge C.$$

I.2.c. Let $A, B, C \ntriangleleft G$, that is, $y^{q^i}, y^{q^j}, y^{q^l} \neq 1$. Let $1 \neq y' \in \langle y^{q^i}\rangle \cap \langle y^{q^j}\rangle \cap \langle y^{q^l}\rangle$. We have

$$A \vee B = \langle y^{q^i}x, y^{q^j}x', P_A, P_B\rangle, \qquad A \vee C = \langle y^{q^{\min\{i,l\}}}, x, P_A, P_C\rangle,$$
$$C \vee B = \langle y^{q^{\min\{j,l\}}}, x', P_B, P_C\rangle.$$

Since $B \vee C, A \vee C < A \vee B$, we have

$$A \vee B = \langle y^{q^i}, y^{q^j}, x, x', P_A, P_B\rangle.$$

By the argument similar to that used in I.1.c we have $x, x' \notin P_A \vee P_B$. In fact $x \in P_A \vee P_B$ implies $x = ab$ with $a \in P_A$ and $b \in P_B$, and then $y^{q^i} x a^{-1} = y^{q^i} b \in A$; hence for $d \in \mathbb{N}$ such that $1 \neq (y^{q^i})^d \in \langle y' \rangle$ we have $1 \neq (y^{q^i} x a^{-1})^d = (y^{q^i} b)^d \in (B \vee C) \wedge A$. Hence $P_{A \vee B} = \langle x, x', P_A, P_B \rangle$. Let $c \in P_C$ be arbitrary. Then $c = x^w x'^v ab$ for suitable $a \in P_A$, $b \in P_B$ and $w, v \in \mathbb{N}$. Thus $x^{-w} c a^{-1} = x'^v b \in (A \vee C) \wedge (B \vee C) = C$, that is, there exist $c' \in P_C$ and a power b' of b such that $x' b' = c'$. Therefore $y^{q^j} x' b' = y^{q^j} c'$. Now for $t \in \mathbb{N}$ such that $1 \neq (y^{q^j})^t \in \langle y' \rangle$ we obtain $1 \neq (y^{q^j} x' b')^t = (y^{q^j} c')^t \in C \wedge B = A \wedge B \wedge C$.

II. $A \wedge B \wedge C \ntrianglelefteq G$. Since $A \wedge B \ntrianglelefteq G$, by Lemma 2.11(iii) there exists a q-element y_1 such that $A = \langle y_1^{q^j}, P_A \rangle$, $B = \langle y_1^{q^j}, P_B \rangle$. Hence $A \wedge B = \langle y_1^{q^{\max\{i,j\}}}, P_A \wedge P_B \rangle$. Repeating the argument for the subgroups $A \vee B$ and C we deduce that there exists a q-element y_2 such that

$$A \vee B = \langle y_2^{q^{\min\{i,j\}}}, P_A \vee P_B \rangle, \qquad C = \langle y_2^{q^l}, P_C \rangle.$$

We may then assume that

$$A = \langle y^{q^i}, P_A \rangle, \qquad B = \langle y^{q^j}, P_B \rangle, \qquad A = \langle y^{q^l}, P_C \rangle$$

and because A and B play similar roles we may assume that $i \leq j$. Suppose now that $P(A, B, C) \simeq L_1$. Since P has a modular lattice of subgroups and $P_{A \vee C} = P_A \vee P_C$ we have

$$P_{A \vee B} = P_{A \vee B} \wedge (P_A \vee P_C) = P_A \vee (P_{A \vee B} \wedge P_C) = P_A \vee P_{(A \vee B) \wedge C} = P_A.$$

Hence $P_A < P_B$ and $B < A$, a contradiction.

If $L(A, B, C) \simeq L_2$ then analogously

$$P_{A \vee C} = P_{A \vee C} \wedge (P_A \vee P_B) = P_A \vee (P_{A \vee C} \wedge P_B) = P_A \vee P_{(A \vee C) \wedge B} = P_A.$$

Note further that the subgroups $A \wedge B$, $A \wedge C$ and $B \wedge C$ are equal, so their Sylow q-subgroups have the same order, which by Lemma 2.11(iii) means that two of the numbers i, j, l are equal and the third one is not greater than the first two. So if $l < i$ then $i = j$ and Sylow q-subgroups of $A \vee B$ have smaller order than Sylow q-subgroups of $A \vee C$, which is impossible. Thus $l \geq i$ and then $C \leq A$. This is the last contradiction which ends the proof of the proposition. ∎

Now we are able to summarize the results.

THEOREM 2.13. *Let G be a finite group. Then $L(G)$ is globally permutable if and only if G is one of the following groups*:

(a) *a finite modular p-group*;

(b) *a group described in* (3);

(c) *a direct product of groups given in* (a) *and* (b), *with pairwise co-prime orders.*

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