C O L L O Q U I U M   M A T H E M A T I C U M

VOL. 82 1999 NO. 1

DISSIDENT ALGEBRAS

BY

ERNST DIETERICH (UPPSALA)

Abstract. Given a euclidean vector space $V = (V, \langle\ \rangle)$ and a linear map $\eta : V \wedge V \to V$, the anti-commutative algebra (V, η) is called *dissident* in case $\eta(v \wedge w) \notin \mathbb{R}v \oplus \mathbb{R}w$ for each pair of non-proportional vectors $(v, w) \in V^2$. For any dissident algebra (V, η) and any linear form $\xi : V \wedge V \to \mathbb{R}$, the vector space $\mathbb{R} \times V$, endowed with the multiplication

$$(\alpha, v)(\beta, w) = (\alpha\beta - \langle v, w\rangle + \xi(v \wedge w), \alpha w + \beta v + \eta(v \wedge w)),$$

is a quadratic division algebra. Up to isomorphism, each real quadratic division algebra arises in this way.

Vector product algebras are classical special cases of dissident algebras. Via composition with definite endomorphisms they produce new dissident algebras, thus initiating a construction of dissident algebras in all possible dimensions $m \in \{0, 1, 3, 7\}$ and of real quadratic division algebras in all possible dimensions $n \in \{1, 2, 4, 8\}$. For $m \leq 3$ and $n \leq 4$, this construction yields complete classifications. For $m = 7$ it produces a 28-parameter family of pairwise non-isomorphic dissident algebras. For $n = 8$ it produces a 49-parameter family of pairwise non-isomorphic real quadratic division algebras.

0. Terminology. Let k be a field of characteristic not two. By an *algebra* we mean any finite-dimensional vector space A over k endowed with a k-bilinear multiplication $A \times A \to A$, $(x, y) \mapsto xy$. *Quadratic algebras*, *division algebras* and *weak division algebras* are non-zero algebras with the following respective properties. The "quadratic property" states that A has an identity element 1 and each $x \in A$ satisfies the equation $x^2 = \alpha x + \beta 1$ with $\alpha, \beta \in k$; the "division property" states that $xy = 0$ always implies $x = 0$ or $y = 0$; the "weak division property" states that for each $x \in A \setminus \{0\}$, the subalgebra $k\langle x\rangle$ generated by x is a division algebra.

1. Quadratic algebras. Given any quadratic algebra A, we denote by $V = \{v \in A \mid v^2 \in k1\} \setminus (k1 \setminus \{0\})$ its set of *purely imaginary* elements. Frobenius' lemma (cf. [7]) states that V is a linear subspace of A which is supplementary to $k1$. Hence Frobenius' decomposition $A = k1 \oplus V$ determines projections $\lambda : A \to k$ and $\iota : A \to V$ such that $x = \lambda(x)1 + \iota(x)$ for all $x \in A$. Further, λ determines the symmetric bilinear form $\langle\ \rangle : A \times A \to k$, $\langle x, y\rangle =$

1991 *Mathematics Subject Classification*: 15A63, 17A35, 17A45, 57S25.

$2\lambda(x)\lambda(y) - \frac{1}{2}\lambda(xy+yx)$, with associated quadratic forms $q, \omega : A \to k$, $q(x) = \langle x, x\rangle = 2\lambda(x)^2 - \lambda(x^2)$ and $\omega(x) = q(x) - \lambda(x)^2 = \lambda(x)^2 - \lambda(x^2)$.

LEMMA 1. *In each quadratic algebra A, the following identities hold for all $x, y \in A$:*

(i) $\langle x, 1\rangle = \lambda(x)$,
(ii) $\iota(x)^2 = -\omega(x)1$,
(iii) $x^2 = 2\lambda(x)x - q(x)1$,
(iv) $xy + yx = 2\lambda(x)y + 2\lambda(y)x - 2\langle x, y\rangle 1$.

Proof (see also [8]). (i) follows directly from the definition of $\langle\ \rangle$. The identity

$$\begin{aligned}\lambda(\iota(x)^2) &= \lambda((x-\lambda(x)1)^2)\\ &= \lambda(x^2 - 2\lambda(x)x + \lambda(x)^2 1) = \lambda(x^2) - \lambda(x)^2 = -\omega(x)\end{aligned}$$

implies (ii). Therefore $-\omega(x)1 = \iota(x)^2 = x^2 - 2\lambda(x)x + \lambda(x)^2 1$, whence (iii) follows. Application of (iii) to each of $x+y$, x and y in $xy + yx = (x+y)^2 - x^2 - y^2$ yields (iv). ∎

We denote by $\mathcal{Q}$ the category of all quadratic algebras, where morphisms $\varphi : A \to A'$ in $\mathcal{Q}$ are algebra morphisms such that $\varphi(1) = 1'$. We also consider the category $\mathcal{E}$ whose objects are *exterior triples* (V, ξ, η) consisting of a finite-dimensional vector space V over k endowed with a symmetric bilinear form $\langle\ \rangle$, and of linear maps $\xi : V \wedge V \to k$ and $\eta : V \wedge V \to V$. Morphisms $\sigma : (V, \xi, \eta) \to (V', \xi', \eta')$ in $\mathcal{E}$ are orthogonal linear maps $\sigma : V \to V'$ satisfying $\xi = \xi'(\sigma \wedge \sigma)$ and $\sigma\eta = \eta'(\sigma \wedge \sigma)$. The categories $\mathcal{Q}$ and $\mathcal{E}$ are related by the functor $\mathcal{G} : \mathcal{E} \to \mathcal{Q}$ which is defined on objects by $\mathcal{G}(V, \xi, \eta) = k \times V$, with multiplication

$$(\alpha, v)(\beta, w) = (\alpha\beta - \langle v, w\rangle + \xi(v \wedge w), \alpha w + \beta v + \eta(v \wedge w)),$$

and on morphisms by $\mathcal{G}(\sigma) = \mathbb{I}_k \times \sigma$.

Consider the following example. The exterior triple $E = (k, o, o)$, with symmetric bilinear form $\langle v, w\rangle = -vw$, determines the quadratic algebra $A = \mathcal{G}E = k \times k \xrightarrow{\sim} k[X]/(X^2 - 1)$. The endomorphism monoids of E and A are $\mathcal{E}(E, E) = \{\mathbb{I}_k, -\mathbb{I}_k\}$ and

$$\mathcal{Q}(A, A) = \left\{ \begin{pmatrix} \mathbb{I}_k & 0 \\ 0 & \mathbb{I}_k \end{pmatrix}, \begin{pmatrix} \mathbb{I}_k & 0 \\ 0 & -\mathbb{I}_k \end{pmatrix}, \begin{pmatrix} \mathbb{I}_k & \mathbb{I}_k \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \mathbb{I}_k & -\mathbb{I}_k \\ 0 & 0 \end{pmatrix} \right\}$$

respectively. So the map of endomorphisms $\mathcal{G}(E, E) : \mathcal{E}(E, E) \to \mathcal{Q}(A, A)$ is not surjective, and hence the functor $\mathcal{G} : \mathcal{E} \to \mathcal{Q}$ is not an equivalence of categories. But still, it is "nearly an equivalence" in the following sense.

PROPOSITION 2. *The functor* $\mathcal{G}:\mathcal{E}\to\mathcal{Q}$ *has the following properties*:

(i) *It is faithful and dense. It detects and exhausts isomorphisms* ([1]).

(ii) *It induces a bijection* $\overline{\mathcal{G}}:\mathrm{Ob}(\mathcal{E})/{\simeq}\xrightarrow{\sim}\mathrm{Ob}(\mathcal{Q})/{\simeq}$ *between the sets of isoclasses of the respective categories.*

(iii) *It induces an equivalence of full subcategories* $\mathcal{G}^{\mathrm{a}}:\mathcal{E}^{\mathrm{a}}\xrightarrow{\sim}\mathcal{Q}^{\mathrm{a}}$, *where* $\mathcal{E}^{\mathrm{a}}$ *is formed by all exterior triples* (V,ξ,η) *such that the quadratic form* $q_V:k\times V\to k$, $q_V(\alpha,v)=\alpha^2+\langle v,v\rangle$, *is anisotropic*, *and* $\mathcal{Q}^{\mathrm{a}}$ *is formed by all quadratic algebras* A *whose quadratic form* $q:A\to k$, $q(x)=2\lambda(x)^2-\lambda(x^2)$, *is anisotropic.*

P r o o f. (i) The functor $\mathcal{G}$ is faithful and detects isomorphisms, by construction. Given $A\in\mathcal{Q}$ with Frobenius decomposition $A=k1\oplus V$, define the exterior triple $\Gamma(A)=((V,\langle\ \rangle),\xi,\eta)$ by

$$\langle v,w\rangle=-\frac{1}{2}\lambda(vw+wv),\qquad \xi(v\wedge w)=\frac{1}{2}\lambda(vw-wv),\qquad \eta(v\wedge w)=\iota(vw).$$

Then the canonical linear isomorphism $k\times V\xrightarrow{\sim}k1\oplus V$ is in fact an isomorphism of quadratic algebras $\mathcal{G}\Gamma(A)\xrightarrow{\sim}A$. Thus $\mathcal{G}$ is dense. Let $E=(V,\xi,\eta)$ and $E'=(V',\xi',\eta')$ be exterior triples, and let $\varphi\in\mathcal{Q}(\mathcal{G}E,\mathcal{G}E')$ be an isomorphism. Then

$$\varphi=\begin{pmatrix}\mathbb{I}_k & \varepsilon\\ 0 & \sigma\end{pmatrix},$$

where $\varepsilon:V\to k$ is a linear form and $\sigma:V\xrightarrow{\sim}V'$ is a linear isomorphism. For each $v\in V$, the identities

$$\begin{gathered}\varphi(0,v)=(\varepsilon(v),\sigma(v)),\\ (\varphi(0,v))^2=\varphi((0,v)^2)=\varphi(-\langle v,v\rangle,0)=(-\langle v,v\rangle,0)\end{gathered}$$

imply that either $\varepsilon(v)=0$ or $\sigma(v)=0$. Since $\ker\sigma=0$ we conclude that $\varepsilon=0$. Now the algebra morphism property of φ implies that σ is orthogonal and satisfies both $\xi=\xi'(\sigma\wedge\sigma)$ and $\sigma\eta=\eta'(\sigma\wedge\sigma)$. Hence $\sigma\in\mathcal{E}(E,E')$ such that $\mathcal{G}(\sigma)=\varphi$. Thus $\mathcal{G}$ exhausts isomorphisms.

(ii) follows immediately from (i).

(iii) By construction, if $E\in\mathcal{E}^{\mathrm{a}}$ then $\mathcal{G}E\in\mathcal{Q}^{\mathrm{a}}$, and if $A\in\mathcal{Q}^{\mathrm{a}}$ then $\Gamma(A)\in\mathcal{E}^{\mathrm{a}}$. So $\mathcal{G}$ induces a functor $\mathcal{G}^{\mathrm{a}}:\mathcal{E}^{\mathrm{a}}\to\mathcal{Q}^{\mathrm{a}}$ which is faithful and dense. To prove that $\mathcal{G}^{\mathrm{a}}$ is full, let $E=(V,\xi,\eta)$ and $E'=(V',\xi',\eta')$ be exterior triples in $\mathcal{E}^{\mathrm{a}}$, and let

$$\varphi=\begin{pmatrix}\mathbb{I}_k & \varepsilon\\ 0 & \sigma\end{pmatrix}\in\mathcal{Q}(\mathcal{G}E,\mathcal{G}E').$$

([1]) The functor $\mathcal{G}:\mathcal{E}\to\mathcal{Q}$ is said to *detect isomorphisms* if for all $E,E'\in\mathcal{E}$ and for all $\sigma\in\mathcal{E}(E,E')$, if $\mathcal{G}(\sigma)$ is an isomorphism then so is σ. We say that $\mathcal{G}:\mathcal{E}\to\mathcal{Q}$ *exhausts isomorphisms* if for all $E,E'\in\mathcal{E}$ and for each isomorphism $\varphi\in\mathcal{Q}(\mathcal{G}E,\mathcal{G}E')$ there exists a morphism $\sigma\in\mathcal{E}(E,E')$ such that $\mathcal{G}(\sigma)=\varphi$.

As above, we find that for each $v \in V$ either $\varepsilon(v)=0$ or $\sigma(v)=0$. If $\sigma(v)=0$, then $(\varepsilon(v)^2,0)=(\varphi(0,v))^2=(-\langle v,v\rangle,0)$ implies $\varepsilon(v)^2+\langle v,v\rangle=0$, hence $\varepsilon(v)=0$ and $v=0$ since q_V is anisotropic. We conclude that $\varepsilon=0$, hence $\sigma \in \mathcal{E}(E,E')$ and $\mathcal{G}(\sigma)=\varphi$. So $\mathcal{G}^{\mathrm{a}}:\mathcal{E}^{\mathrm{a}} \to \mathcal{Q}^{\mathrm{a}}$ is full, faithful and dense, i.e. an equivalence of categories. ∎

2. Quadratic weak division algebras. Henceforth let $k=\mathbb{R}$. Then $\mathcal{Q}^{\mathrm{a}}$ coincides both with the category of all quadratic algebras A whose associated quadratic form q is positive-definite and with the category of all quadratic weak division algebras. The first of these statements follows from Sylvester's inertia theorem, while the second is a consequence of the following proposition.

PROPOSITION 3. *For each quadratic algebra A, the following assertions are equivalent*:

(i) *A is a weak division algebra.*
(ii) *ω is positive-semidefinite with null space $\mathbb{R}1$.*
(iii) *$q|V$ is positive-definite.*
(iv) *q is positive-definite.*

Proof. Since each quadratic algebra A is power-associative, the minimal polynomial $\mu_x \in \mathbb{R}[X]$ is well defined for all $x \in A$. Lemma 1 shows that $\mu_x = X^2 - 2\lambda(x)X + q(x)$ for all $x \in A \setminus \mathbb{R}1$, in particular $\mu_x = X^2+\omega(x)$ for all $x \in V\setminus\{0\}$. Substitution $X \mapsto x$ induces the canonical isomorphism $\mathbb{R}[X]/\mu_x \xrightarrow{\sim} \mathbb{R}\langle x,1\rangle$ (2).

Assuming (i), we conclude for any $x \in A\setminus\mathbb{R}1$ that $\omega(x)\neq 0$, by Lemma 1(ii). So $\mathbb{R}\langle\iota(x)\rangle = \mathbb{R}\langle\iota(x),1\rangle \xrightarrow{\sim} \mathbb{R}[X]/\mu_{\iota(x)}$ is a division algebra. Thus $\mu_{\iota(x)}$ is irreducible and therefore $\omega(x)>0$. Since $\omega(x)=0$ for all $x\in\mathbb{R}1$, by definition of ω, this proves (ii).

The equivalence of (ii), (iii) and (iv) follows immediately from the identities $q(x)=\lambda(x)^2+\omega(x)$ and $\omega(x)=q(\iota(x))$, valid for all $x\in A$ by definition of ω resp. by Lemma 1, (ii) and (iii).

Assuming (ii), we conclude for any $x\in A\setminus\mathbb{R}1$ that μ_x is irreducible. So $\mathbb{R}\langle x\rangle=\mathbb{R}\langle x,1\rangle \xrightarrow{\sim} \mathbb{R}[X]/\mu_x$ is a division algebra. Since $\mathbb{R}\langle x\rangle \xrightarrow{\sim} \mathbb{R}$ is a division algebra for all $x\in\mathbb{R}1\setminus\{0\}$, this proves (i). ∎

Note that Proposition 3 generalizes the well known fact that q is positive-definite for each quadratic division algebra (cf. [8]).

A basis $b=(b_0,b_1,\ldots,b_m)$ of an algebra A with identity element 1 is

(2) By $\mathbb{R}\langle a,b\rangle$ we denote the subalgebra of A generated by $a,b\in A$.

called *unital* if it satisfies the following system of equations:

$$\begin{cases} b_0 = 1, \\ b_i^2 = -1 & \text{for all } i \in \{1, \dots, m\}, \\ b_i b_j + b_j b_i = 0 & \text{for all } (i,j) \in \{1, \dots, m\}^2 \text{ with } i \neq j. \end{cases}$$

PROPOSITION 4. *For each algebra A with identity element* 1, *the following assertions are equivalent*:

(i) *A is a quadratic weak division algebra.*
(ii) *A admits a unital basis.*

Proof. If A is a quadratic weak division algebra, then $A=(A, \langle\ \rangle)$ is a euclidean vector space (Proposition 3), thus admitting an orthonormal basis with leading vector 1. But this is the same thing as a unital basis (Lemma 1).

Conversely, let $b = (b_0, b_1, \dots, b_m)$ be a unital basis of A. Then each $x \in A$, with coordinate column ϱ in b, satisfies the equation

$$(*) \qquad x^2 = 2\varrho_0 x - (\varrho^{\mathrm{T}}\varrho)1.$$

So A is a quadratic algebra. Accordingly, a linear form $\lambda : A \to \mathbb{R}$ and quadratic forms $q, \omega : A \to \mathbb{R}$ are associated with A. Comparing $(*)$ with Lemma 1(iii) we infer that $\lambda(x) = \varrho_0$ and $q(x) = \varrho^{\mathrm{T}}\varrho$. Hence q is positive-definite. Equivalently, A is a quadratic weak division algebra (Proposition 3). ■

3. Quadratic division algebras. Given any finite-dimensional real vector space V, we denote by $\mathcal{P}(V)$ the set of all 2-dimensional subspaces of V. Each linear map $\eta : V \wedge V \to V$ induces a map $\overline{\eta} : \mathcal{P}(V) \to \mathbb{P}(V) \cup \{0\}$, $\mathbb{R}v \oplus \mathbb{R}w \mapsto \mathbb{R}\eta(v \wedge w)$. We call η *dissident* if $\overline{\eta}(P) \not\subset P$ for all $P \in \mathcal{P}(V)$, and *incident* otherwise.

PROPOSITION 5. *Let A be a real quadratic algebra. Then A is a division algebra if and only if the following two conditions are satisfied*:

(a) *The quadratic form $q|V : V \to \mathbb{R},\ v \mapsto -\lambda(v^2)$, is positive-definite.*
(b) *The linear map $\eta : V \wedge V \to V,\ v \wedge w \mapsto \iota(vw)$ is dissident.*

Proof. *The condition* (a)$\wedge$(b) *is sufficient.* Let $x = \alpha 1 + v$ and $y = \beta 1 + w$ be elements of $A \setminus \{0\}$, with $\alpha, \beta \in \mathbb{R}$ and $v, w \in V$. We first consider the case where v and w are proportional, say $w = \gamma v$ with $\gamma \in \mathbb{R}$. If $w = 0$, then $xy \neq 0$ follows immediately. If $w \neq 0$, then we obtain

$$\frac{1}{\gamma} xy = (\alpha 1 + v)\left(\frac{\beta}{\gamma} 1 + v\right) = \left(\frac{\alpha\beta}{\gamma} - q(v)\right)1 + \left(\alpha + \frac{\beta}{\gamma}\right)v,$$

and $q(v) > 0$ due to (a). Hence Frobenius' lemma implies $xy \neq 0$. In case v and w are not proportional, (b) implies

$$xy = (\alpha\beta - \langle v, w\rangle + \xi(v \wedge w))1 + \beta v + \alpha w + \eta(v \wedge w) \neq 0.$$

The condition (a)$\wedge$(b) *is necessary.* Condition (a) is necessary by Proposition 3. If η is incident, then we may choose $\alpha, \beta \in \mathbb{R}$ and orthonormal vectors $v, w \in V$ such that $\beta v + \alpha w + \eta(v \wedge w) = 0$. Set $x = \alpha 1 + v$ and $y_t = \beta_t 1 + w_t$ where, for any parameter $t \in [-\pi/2, \pi/2]$,

$$\beta_t = \alpha \sin t + \beta \cos t \quad \text{and} \quad w_t = -(\sin t)v + (\cos t)w.$$

Then

$$xy_t = (\alpha\beta_t - \langle v, w_t\rangle + \xi(v \wedge w_t))1 + \beta_t v + \alpha w_t + \eta(v \wedge w_t) = f(t)1,$$

where $f(t) = (1 + \alpha^2)\sin t + (\alpha\beta + \xi(v \wedge w))\cos t$. Choosing t to be a zero for the continuous function $f : [-\pi/2, \pi/2] \to \mathbb{R}$ we obtain $xy_t = 0$, whence A is not a division algebra. ∎

4. Dissident algebras. Anti-commutative euclidean algebras V are identified with pairs (V, η) consisting of a euclidean vector space $V = (V, \langle\ \rangle)$ and a linear map $\eta : V \wedge V \to V$, via $vw = \eta(v \wedge w)$. Accordingly, morphisms $\sigma : (V, \eta) \to (V', \eta')$ between anti-commutative euclidean algebras are orthogonal linear maps $\sigma : V \to V'$ satisfying $\sigma\eta = \eta'(\sigma \wedge \sigma)$. We define a *dissident* (resp. *incident*) *algebra* to be any anti-commutative euclidean algebra (V, η) such that η is dissident (resp. incident).

Dissident algebras generalize the classical notion of a vector product algebra ([3]). Vector product algebras are known to constitute 4 isoclasses which are represented by those (V, π) which arise from one of the alternative division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. Conversely, each alternative division algebra is isomorphic to $\mathcal{G}(V, o, \pi)$, for some vector product algebra (V, π) ([5], [9]). More generally, Proposition 2(iii) and Proposition 5 explain the importance of dissident algebras for the investigation of quadratic division algebras: the functor $(V, \eta) \mapsto \mathcal{G}(V, o, \eta)$ is an equivalence between the category of dissident algebras and the category formed by all quadratic division algebras satisfying $\xi = o$.

This raises the interesting question of how to construct dissident algebras. In fact, it turns out that vector product morphisms not only provide the simplest examples of dissident algebras, but also produce new dissident algebras when composed with definite endomorphisms ([4]). In low dimensions we can prove that this construction of dissident algebras is complete.

LEMMA 6. *Let $\pi : V \wedge V \to V$ be a vector product.*

([3]) Recall that a *vector product algebra* is an anti-commutative euclidean algebra (V, π) satisfying $\langle \pi(u \wedge v), w\rangle = \langle u, \pi(v \wedge w)\rangle$ and $|\pi(u \wedge v)|^2 = |u|^2|v|^2 - \langle u, v\rangle^2$ for all $u, v, w \in V$ (cf. [9]).

([4]) A linear endomorphism $\varepsilon : V \to V$ of a euclidean vector space V is said to be *definite* if the quadratic form $q_\varepsilon : V \to \mathbb{R},\ v \mapsto \langle v, \varepsilon(v)\rangle$, is positive-definite or negative-definite.

(i) *If* $\varepsilon : V \to V$ *is a definite endomorphism*, *then* $\varepsilon\pi : V \wedge V \to V$ *is a dissident linear map.*

(ii) *If* $\eta : V \wedge V \to V$ *is a dissident linear map and* $\dim V \leq 3$, *then there exists a definite endomorphism* $\varepsilon : V \to V$ *such that* $\varepsilon\pi = \eta$.

(iii) *If* $\dim V = 3$, *then the assignment* $\varepsilon \mapsto \varepsilon\pi$ *is a bijection between the set of all definite endomorphisms of* V *and the set of all dissident algebra structures on* V.

Proof. (i) Let $v, w \in V$ be orthonormal vectors. The vector product property of π implies $\mathbb{R}v \oplus \mathbb{R}w \subset \pi(v \wedge w)^\perp$ and $|\pi(v \wedge w)| = 1$. Definiteness of ε further implies $\varepsilon\pi(v \wedge w) \notin \pi(v \wedge w)^\perp$. Hence $\varepsilon\pi(v \wedge w) \notin \mathbb{R}v \oplus \mathbb{R}w$.

(ii) In case $\dim V \leq 2$, the assertion is true for trivial reasons. If $\dim V = 3$, then $\pi : V \wedge V \to V$ is an isomorphism. Given $u \in V \setminus \{0\}$, choose $v, w \in u^\perp$ such that $u = \pi(v \wedge w)$. Then $\eta(v \wedge w) \notin \mathbb{R}v \oplus \mathbb{R}w = u^\perp$, and therefore $\langle u, \eta\pi^{-1}(u)\rangle = \langle u, \eta(v \wedge w)\rangle \neq 0$. Hence $\varepsilon = \eta\pi^{-1}$ is definite and $\varepsilon\pi = \eta$.

(iii) The assignment $\varepsilon \mapsto \varepsilon\pi$ defines a map from the set of all definite endomorphisms of V to the set of all dissident algebra structures on V, by (i). This map is surjective by (ii), and injective as π is an isomorphism. ∎

Proposition 7. *Dissident algebras exist in dimensions* 0, 1, 3 *and* 7 *only.*

Proof. Given any dissident algebra (V, η), the quadratic algebra $A = \mathcal{G}(V, o, \eta)$ is a division algebra, by Proposition 5. Application of the celebrated "$(1, 2, 4, 8)$-theorem" ([6], [10], [2], [1]) to A yields the assertion. ∎

Let (V, η) be a dissident algebra. For each $v \in V \setminus \{0\}$, the endomorphism $v\cdot : V \to V$ induces an epimorphism $v^\perp \to vv^\perp$ which (by dissidence) is in fact an isomorphism. Thus $vv^\perp$ is a hyperplane in V. Hence η determines a selfmap $\eta_{\mathbb{P}} : \mathbb{P}(V) \to \mathbb{P}(V)$, defined by $\eta_{\mathbb{P}}(\mathbb{R}v) = (vv^\perp)^\perp$. Observe that $\pi_{\mathbb{P}} = \mathbb{I}_{\mathbb{P}(V)}$ for each vector product $\pi : V \wedge V \to V$.

The adjoint of an endomorphism $\varepsilon : V \to V$ will be denoted by $\varepsilon^* : V \to V$. If ε is invertible then so is ε^*, and $(\varepsilon^*)^{-1} = (\varepsilon^{-1})^*$ will be denoted by ε^{-*}.

Lemma 8. *Let* $\eta = \varepsilon\pi$, *where* $\varepsilon : V \to V$ *is a definite endomorphism and* $\pi : V \wedge V \to V$ *is a vector product. Then* $\eta_{\mathbb{P}} = \mathbb{P}(\varepsilon^{-*})$.

Proof. Let $v, \overline{v} \in V \setminus \{0\}$ be such that $\eta_{\mathbb{P}}(\mathbb{R}v) = \mathbb{R}\overline{v}$. Then $\langle\varepsilon^*(\overline{v}), \pi(v \wedge x)\rangle = \langle\overline{v}, \varepsilon\pi(v \wedge x)\rangle = \langle\overline{v}, vx\rangle = 0$ for all $x \in v^\perp$. Thus $\mathbb{R}\varepsilon^*(\overline{v}) = \pi_{\mathbb{P}}(\mathbb{R}v) = \mathbb{R}v$, whence $\eta_{\mathbb{P}}(\mathbb{R}v) = \mathbb{R}\overline{v} = \mathbb{P}(\varepsilon^{-*})(\mathbb{R}v)$. ∎

Proposition 9. *Let* $\eta = \varepsilon\pi = \varepsilon'\pi'$, *where* $\varepsilon, \varepsilon' : V \to V$ *are definite endomorphisms and* $\pi, \pi' : V \wedge V \to V$ *are vector products. Then either* $(\varepsilon, \pi) = (\varepsilon', \pi')$ *or* $(\varepsilon, \pi) = (-\varepsilon', -\pi')$.

Proof. The hypothesis implies by Lemma 8 that $\eta_{\mathbb{P}} = \mathbb{P}(\varepsilon^{-*}) = \mathbb{P}(\varepsilon'^{-*})$. In view of the short exact sequence

$$1 \to \mathrm{GL}(\mathbb{R}) \to \mathrm{GL}(V) \to \mathrm{PGL}(V) \to 1$$

we conclude that $\varepsilon = r\varepsilon'$ for some $r \in \mathbb{R} \setminus \{0\}$. Hence we obtain $r\varepsilon'\pi = \varepsilon'\pi'$, whence $r\pi = \pi'$. Since both π and π' are vector products, we infer that $r = 1$ or $r = -1$. ∎

5. Summary. Let $m \in \{0,1,3,7\}$ and $n = m+1$. Choose a vector product $\pi : \mathbb{R}^m \wedge \mathbb{R}^m \to \mathbb{R}^m$ with respect to the natural scalar product $v \bullet w = v^{\mathrm{T}}w$. The orthogonal group $O(\mathbb{R}^m)$ acts canonically on the set of all vector products in $\mathbb{R}^m$ via $\sigma \cdot \pi' = \sigma\pi'(\sigma^{-1} \wedge \sigma^{-1})$. Denote by $O_\pi(\mathbb{R}^m) = \{\sigma \in O(\mathbb{R}^m) \mid \sigma \cdot \pi = \pi\}$ the isotropy group of π. Moreover, denote by $\mathbb{R}^{m\times m}_{\mathrm{ant}}$ (resp. $\mathbb{R}^{m\times m}_{\mathrm{pos}}$) the set of all anti-symmetric (resp. positive-definite) matrices in $\mathbb{R}^{m\times m}$, and set $\mathcal{P}_m = \mathbb{R}^{m\times m}_{\mathrm{ant}} \times \mathbb{R}^{m\times m}_{\mathrm{pos}}$. Then the map

$$\Phi_\pi : \mathcal{P}_m \to \mathcal{Q}, \qquad (X,Y) \mapsto \mathcal{G}(\mathbb{R}^m, \xi, \eta),$$

where $\xi(v \wedge w) = v^{\mathrm{T}}Xw$ and $\eta(v \wedge w) = Y\pi(v \wedge w)$, has the following properties.

THEOREM 10. (i) *For each matrix pair $(X,Y) \in \mathcal{P}_m$ the quadratic algebra $\Phi_\pi(X,Y)$ is a division algebra of dimension n.*

(ii) *For each quadratic division algebra D of dimension $n \le 4$ there exists a matrix pair $(X,Y) \in \mathcal{P}_m$ such that $\Phi_\pi(X,Y) \xrightarrow{\sim} D$.*

(iii) *For all matrix pairs $(X,Y),(X',Y') \in \mathcal{P}_m$, the quadratic division algebras $\Phi_\pi(X,Y)$ and $\Phi_\pi(X',Y')$ are isomorphic if and only if $(X',Y') = (SXS^{\mathrm{T}}, SYS^{\mathrm{T}})$ for some $S \in O_\pi(\mathbb{R}^m)$.*

Proof. (i) follows from Proposition 5 and Lemma 6(i).

(ii) follows from Propositions 2 and 5, Lemma 6(ii) and the fact that $-\mathbb{I}_V : (V,\xi,\eta) \xrightarrow{\sim} (V,\xi,-\eta)$ is an isomorphism for each exterior triple (V,ξ,η).

(iii) Let $(X,Y),(X',Y') \in \mathcal{P}_m$ be given. If $(SXS^{\mathrm{T}}, SYS^{\mathrm{T}}) = (X',Y')$ for some $S \in O_\pi(\mathbb{R}^m)$, then the linear automorphism $\sigma : \mathbb{R}^m \xrightarrow{\sim} \mathbb{R}^m$ corresponding to S is an isomorphism $\sigma : (\mathbb{R}^m,\xi,\eta) \xrightarrow{\sim} (\mathbb{R}^m,\xi',\eta')$ of exterior triples. Accordingly, $\mathbb{I}_{\mathbb{R}} \times \sigma : \Phi_\pi(X,Y) \xrightarrow{\sim} \Phi_\pi(X',Y')$ is an isomorphism of quadratic algebras. Conversely, if $\varphi : \Phi_\pi(X,Y) \xrightarrow{\sim} \Phi_\pi(X',Y')$ is an isomorphism of quadratic algebras, then $\varphi = \mathbb{I}_{\mathbb{R}} \times \sigma$ for some isomorphism $\sigma : (\mathbb{R}^m,\xi,\eta) \xrightarrow{\sim} (\mathbb{R}^m,\xi',\eta')$ of exterior triples. Hence the matrix $S \in O(\mathbb{R}^m)$ corresponding to σ satisfies $SXS^{\mathrm{T}} = X'$ and, due to Proposition 9, both $SYS^{\mathrm{T}} = Y'$ and $S \in O_\pi(\mathbb{R}^m)$. ∎

Keeping the above setting, denote by $\mathcal{D}^{\mathrm{q}}_n$ the category of all quadratic division algebras of dimension n. If $n \le 2$, then $\mathcal{D}^{\mathrm{q}}_n$ constitutes one isomorphism class, represented by $\mathbb{R}$ and $\mathbb{C}$ respectively.

In case $n = 4$, the chosen vector product $\pi : V \wedge V \to V$ is bijective. Identifying $V \wedge V$ with V via π, definite endomorphisms $\varepsilon : V \to V$ are identified with dissident linear maps $\eta : V \wedge V \to V$, by Lemma 6(iii). Thus Proposition 5 specializes to [4], Proposition 2. From Theorem 10 we infer that $\Phi_\pi : \mathcal{P}_3 \to \mathcal{Q}$ induces a bijection $\overline{\Phi}_\pi : \mathcal{P}_3/\mathrm{SO}_3(\mathbb{R}) \xrightarrow{\sim} \mathcal{D}_4^{\mathrm{q}}/\simeq$ ([5]). Thus the classification of all 4-dimensional quadratic division algebras up to isomorphism is equivalent to the classification of all matrix pairs $(X, Y) \in \mathcal{P}_3$ up to simultaneous conjugation by matrices $S \in \mathrm{SO}_3(\mathbb{R})$.

The latter problem admits a complete solution by application of classical theory! Namely, set $\mathcal{K}_3 = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathcal{T}_3$ where $\mathcal{T}_3 = \{\delta \in \mathbb{R}^3 \mid 0 < \delta_1 \le \delta_2 \le \delta_3\}$, and define $\Psi : \mathcal{K}_3 \to \mathcal{P}_3$ by $\Psi(x, y, \delta) = (A_x, A_y + \Delta_\delta)$ where

$$A_x = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix} \quad \text{and} \quad \Delta_\delta = \begin{pmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_2 & 0 \\ 0 & 0 & \delta_3 \end{pmatrix}.$$

Given $(X, Y) \in \mathcal{P}_3$, decompose $Y = Y^{\mathrm{a}} + Y^{\mathrm{s}}$ into its anti-symmetric summand Y^{a} and its symmetric summand Y^{s} and transform Y^{s} to the diagonal form Δ_δ, applying Jacobi's spectral theorem.

This reasoning shows that $\Psi : \mathcal{K}_3 \to \mathcal{P}_3$ induces a bijection $\overline{\Psi} : \mathcal{K}_3/\sim \xrightarrow{\sim} \mathcal{P}_3/\mathrm{SO}_3(\mathbb{R})$, where $(x, y, \delta) \sim (x', y', \delta')$ if and only if $\delta = \delta'$ and $(Sx, Sy) = (x', y')$ for some $S \in \mathrm{SO}_3(\mathbb{R})$ such that $S^{\mathrm{T}} \Delta_\delta S = \Delta_\delta$. Interpreting $(x, y, \delta) \in \mathcal{K}_3$ as a configuration in $\mathbb{R}^3$, formed by a pair of points (x, y) and an ellipsoid $E_\delta = \{z \in \mathbb{R}^3 \mid z^{\mathrm{T}} \Delta_\delta z = 1\}$, the equivalence $(x, y, \delta) \sim (x', y', \delta')$ means geometrically that $E_\delta = E_{\delta'}$ and $(x, y), (x', y')$ lie in the same orbit under the action of the special orthogonal symmetry group of E_δ. Thus we recover the classification of $\mathcal{D}_4^{\mathrm{q}}$ in terms of configurations in $\mathbb{R}^3$ consisting of an ellipsoid and a pair of points, as presented in [3], [4]. Explicitly, the composed map $\Phi_\pi \Psi : \mathcal{K} \to \mathcal{D}_4^{\mathrm{q}}$ is given by $\Phi_\pi \Psi(x, y, \delta) = \mathbb{R} \times \mathbb{R}^3$, with multiplication

$$(\alpha, v)(\beta, w) = (\alpha\beta - v^{\mathrm{T}} w + v^{\mathrm{T}} A_x w, \alpha w + \beta v + A_y \pi(v \wedge w) + \Delta_\delta \pi(v \wedge w)).$$

Forgetting about X in $(X, Y) \in \mathcal{P}_3$ and about x in $(x, y, \delta) \in \mathcal{K}_3$, we clearly obtain a classification of all 3-dimensional dissident algebras in terms of configurations in $\mathbb{R}^3$ consisting of an ellipsoid and a single point, with equivalence relation analogous to the above. Here, the dissident algebra $(\mathbb{R}^3, \eta)$ constructed from (y, δ) is given by $\eta(v \wedge w) = A_y \pi(v \wedge w) + \Delta_\delta \pi(v \wedge w)$.

In case $n = 8$, the question whether the image of $\Phi_\pi : \mathcal{P}_7 \to \mathcal{D}_8^{\mathrm{q}}$ exhausts all isoclasses of $\mathcal{D}_8^{\mathrm{q}}$ is still open. It is equivalent to the interesting question whether any dissident linear map $\eta : \mathbb{R}^7 \wedge \mathbb{R}^7 \to \mathbb{R}^7$ admits a factorization $\eta = \varepsilon\pi$ into a vector product $\pi : \mathbb{R}^7 \wedge \mathbb{R}^7 \to \mathbb{R}^7$ and a definite endomorphism

([5]) Recall that the euclidean vector space $\mathbb{R}^3$ admits two vector products π_1 and $\pi_2 = -\pi_1$ only, where π_1 is characterized by the identity $\pi_1(v \wedge w) \bullet x = \det(v|w|x)$, valid for all $v, w, x \in \mathbb{R}^3$. This implies $O_\pi(\mathbb{R}^3) = \mathrm{SO}_3(\mathbb{R})$.

$\varepsilon : V \to V$. However, irrespective of the answer to this question we may use the map $\Phi_\pi : \mathcal{P}_7 \to \mathcal{D}_8^{\mathrm{q}}$ to construct large families of 8-dimensional quadratic division algebras.

COROLLARY 11. *Let* $\mathcal{K}_7^< = \{(X,Y,\delta) \in \mathbb{R}_{\mathrm{ant}}^{7\times 7} \times \mathbb{R}_{\mathrm{ant}}^{7\times 7} \times \mathcal{T}_7 \mid 0 < Y_{ij}$ *for all* $1 \le i < j \le 7$ *and* $0 < \delta_1 < \ldots < \delta_7\}$, *and define* $\Psi : \mathcal{K}_7^< \to \mathcal{P}_7$ *by* $\Psi(X,Y,\delta) = (X, Y + \Delta_\delta)$. *Then* $\Phi_\pi\Psi(\mathcal{K}_7^<)$ *is a* 49-*parameter family of pairwise non-isomorphic objects in* $\mathcal{D}_8^{\mathrm{q}}$. *The composed map* $\Phi_\pi\Psi : \mathcal{K}_7^< \to \mathcal{D}_8^{\mathrm{q}}$ *is given explicitly by* $\Phi_\pi\Psi(X,Y,\delta) = \mathbb{R} \times \mathbb{R}^7$, *with multiplication*

$$(\alpha, v)(\beta, w) = (\alpha\beta - v^{\mathrm{T}}w + v^{\mathrm{T}}Xw, \alpha w + \beta v + Y\pi(v \wedge w) + \Delta_\delta \pi(v \wedge w)).$$

Proof. Let $(X,Y,\delta), (X',Y',\delta') \in \mathcal{K}_7^<$ be such that

$$\Phi_\pi\Psi(X,Y,\delta) \xrightarrow{\sim} \Phi_\pi\Psi(X',Y',\delta').$$

Then $(SXS^{\mathrm{T}}, SYS^{\mathrm{T}}, S\Delta_\delta S^{\mathrm{T}}) = (X', Y', \Delta_{\delta'})$ for some $S \in O_\pi(\mathbb{R}^7)$, by Theorem 10(iii). From $S\Delta_\delta S^{\mathrm{T}} = \Delta_{\delta'}$ we infer that $\delta = \delta'$ and $S = \Delta_\varepsilon$ for some $\varepsilon \in \{1,-1\}^7$. Entering this diagonal form of S into the equation $SYS^{\mathrm{T}} = Y'$, we deduce from the constraints $0 < Y_{ij}$ and $0 < Y'_{ij}$ for all $i < j$ that $S = \mathbb{I}_7$ or $S = -\mathbb{I}_7$. Hence $(X,Y,\delta) = (X',Y',\delta')$. ∎

Finally, let us note that on forgetting about X in $(X,Y,\delta) \in \mathcal{K}_7^<$ we obtain a 28-parameter family of pairwise non-isomorphic 7-dimensional dissident algebras $(\mathbb{R}^7, \eta)$, where $\eta(v \wedge w) = Y\pi(v \wedge w) + \Delta_\delta\pi(v \wedge w)$.

Acknowledgements. The author expresses his gratitude for the hospitality he received during his research visits at the Mathematics Department of Nicholas Copernicus University in Toruń in May 1997 and at the "Sonderforschungsbereich diskrete Strukturen in der Mathematik" at Bielefeld University in October 1998. The present article grew out of these occasions.

REFERENCES

[1] J. F. Adams, *Vector fields on spheres*, Ann. of Math. 75 (1962), 603–632.
[2] M. F. Atiyah and F. Hirzebruch, *Bott periodicity and the parallelizability of the spheres*, Proc. Cambridge Philos. Soc. 57 (1961), 223–226.
[3] E. Dieterich, *Zur Klassifikation vierdimensionaler reeller Divisionsalgebren*, Math. Nachr. 194 (1998), 13–22.
[4] —, *Real quadratic division algebras*, Comm. Algebra, to appear.
[5] B. Eckmann, *Stetige Lösungen linearer Gleichungssysteme*, Comm. Math. Helv. 15 (1942/43), 318–339.
[6] H. Hopf, *Ein topologischer Beitrag zur reellen Algebra*, ibid. 13 (1940/41), 219–239.
[7] M. Koecher and R. Remmert, *Isomorphiesätze von Frobenius, Hopf und Gelfand–Mazur*, in: Zahlen, Springer-Lehrbuch, 3. Auflage, 1992, 182–204.
[8] —, —, *Cayley-Zahlen oder alternative Divisionsalgebren*, ibid., 205–218.
[9] —, —, *Kompositionsalgebren. Satz von Hurwitz. Vektorprodukt-Algebren*, ibid., 219–232.

[10] J. Milnor, *Some consequences of a theorem of Bott*, Ann. of Math. 68 (1958), 444–449.

Matematiska institutionen
Uppsala universitet
Box 480
S-751 06 Uppsala, Sweden
E-mail: ernstdie@math.uu.se

Received 23 November 1998;
revised 12 March 1999