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SIMPLY CONNECTED RIGHT MULTIPEAK ALGEBRAS AND THE SEPARATION PROPERTY

BY

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Abstract. Let $R = k(Q, I)$ be a finite-dimensional algebra over a field k determined by a bound quiver (Q, I). We show that if R is a simply connected right multipeak algebra which is chord-free and $\widetilde{\mathbb{A}}$-free in the sense defined below then R has the separation property and there exists a preprojective component of the Auslander–Reiten quiver of the category $\mathrm{prin}(R)$ of prinjective R-modules. As a consequence we get in 4.6 a criterion for finite representation type of $\mathrm{prin}(R)$ in terms of the prinjective Tits quadratic form of R.

1. Introduction. Let k be a field. We consider triangular simply connected right multipeak algebras $R = kQ/I$, where Q is a finite quiver and I is an admissible ideal in the path algebra kQ. Triangularity means that the ordinary quiver Q of R has no oriented cycles. Following [13] we say that R is a *right multipeak algebra* if the right socle $\mathrm{soc}(R_R)$ of R is R-projective. The main objective of the paper is a criterion for R to have the separation property [2]. We prove in Section 4 that R has the separation property when R is chord-free (see 2.5) and $\widetilde{\mathbb{A}}$-free as a right multipeak algebra. Our main result, Theorem 4.5, is analogous to [21, Theorem 4.1] and [1, 1.2]; cf. [6].

Recall from [1, 1.2] that if R is schurian, triangular, simply connected and does not contain any full subcategory (see 2.2) isomorphic to $k\widetilde{\mathbb{A}}_m$, $m \geq 1$, then R has the separation property. Our result is a version of this statement: algebras considered are right multipeak algebras and the requirement that R is $\widetilde{\mathbb{A}}$-free as a right multipeak algebra is a weaker version of $\widetilde{\mathbb{A}}$-freeness considered in [1], [3]. The condition that R is chord-free plays a similar role as the assumption that R is schurian. Note that the arguments used in [1] do not work in our situation: our assumptions on R do not imply that R is schurian. Moreover, R (viewed as a k-category) admits full subcategories isomorphic to $k\widetilde{\mathbb{A}}_m$ for some $m \geq 1$, although R is $\widetilde{\mathbb{A}}$-free as a right multipeak algebra (see the Example in 2.5). Hence the arguments used in [3, 2.3] and [5, 2.9] do not apply here.

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In Section 2 we recall and discuss the basic concepts related to the notion of the fundamental group of a bound quiver. The next section is devoted to investigating the fundamental group of the bound quiver of the reflection-dual algebra $R^\bullet$ associated with R introduced in [14, 2.6] and [15, 17.4]. The reflection duality important for socle projective modules over right multipeak algebras is a substitute of the usual duality for modules over finite-dimensional algebras.

The proof of the main result is contained in Section 4. Following the ideas of Skowroński [21, 4.1] we apply induction on the rank $\mathbf{r}_R$ of the Grothendieck group $\mathbf{K}_0(R)$ of R. In order to do it we prove in Proposition 4.1 that, under suitable assumptions, if R is a one-point extension $B[M]$ of a simply connected algebra B with $\mathbf{r}_B < \mathbf{r}_R$ then B is also simply connected.

There is another similarity to the results of [21], namely we prove in 4.5 that the algebras considered in our paper have the first Hochschild cohomology group zero.

As an application we obtain in 4.6 a criterion for the finite representation type of the category of prinjective R-modules over a triangular chord-free simply connected right multipeak algebra R. Our result can be applied to incidence algebras of posets with zero-relations investigated by Simson [18]–[20] as a tool for determining the representation theory of lattices over special orders. They also form a nice class of examples of algebras admitting only inner derivations.

2. Preliminaries. The main aim of this section is to recall the notion of the fundamental group of a bound quiver.

2.1. By a *quiver* we mean a tuple $Q = (Q_0, Q_1, s, t)$ of two sets, the set Q_0 of vertices and Q_1 of arrows, and two functions $s, t : Q_1 \to Q_0$. Instead of (Q_0, Q_1, s, t) we usually write (Q_0, Q_1). Given an arrow $\alpha \in Q_1$ we call $s(\alpha)$ and $t(\alpha)$ the *source* and the *sink* of α respectively. We denote by α^{-1} the formal inverse of α and set $s(\alpha^{-1}) = t(\alpha)$ and $t(\alpha^{-1}) = s(\alpha)$. A *sink* (resp. *source*) of Q is a vertex which is not a source (resp. sink) of any arrow in Q.

If $Q = (Q_0, Q_1, s, t)$ and $Q' = (Q_0', Q_1', s', t')$ are two quivers such that $s(\alpha) = s'(\alpha)$ and $t(\alpha) = t'(\alpha)$ for every $\alpha \in Q_1 \cap Q_1'$ then their intersection $Q \cap Q' = (Q_0 \cap Q_0', Q_1 \cap Q_1')$ is defined in the obvious way. For $A \subseteq Q_0$ we denote by $Q \setminus A$ the quiver $(Q_0 \setminus A, \overline{Q}_1)$, where $\overline{Q}_1 = \{\alpha \in Q_1 : s(\alpha) \notin A, t(\alpha) \notin A\}$.

A *walk* in Q is a sequence $u = \alpha_1 \ldots \alpha_n$ of arrows and formal inverses of arrows in Q such that $s(\alpha_{i+1}) = t(\alpha_i)$ for $i = 1, \ldots, n-1$. The trivial walk at $x \in Q_0$ is denoted by e_x. If u is as above we define the source $s(u)$ of u to be $s(\alpha_1)$ and the sink $t(u)$ to be $t(\alpha_n)$. We denote by u^{-1} the inverse walk $\alpha_n^{-1} \ldots \alpha_1^{-1}$.

If all $\alpha_1, \ldots, \alpha_r$ are arrows (not inverses of arrows) then we call u a *path*. Two paths u and v are *parallel* if $s(u) = s(v)$ and $t(u) = t(v)$.

If u and v are two walks with $s(v) = t(u)$ then we define the composition uv in the obvious way.

A walk u is a *loop* (at x) provided $s(u) = t(u) = x$. It is well known that given $x \in Q_0$ the composition of walks induces a group structure on the set of homotopy classes of loops at x. The homotopy relation is induced by the topological structure associated with the quiver Q in the usual way. The group obtained that way is called the *fundamental group* of Q at x and it is denoted by $\Pi_1(Q, x)$. If Q is connected then the isomorphism class of $\Pi_1(Q, x)$ does not depend on the choice of x. In this case we shall speak about the fundamental group of Q and denote it by $\Pi_1(Q)$.

Assume that Q is connected and T is a maximal tree in Q. If $\alpha_1, \ldots, \alpha_r$ are all arrows of Q not belonging to T then $\Pi_1(Q)$ can be identified with the free group with free generators $\alpha_1, \ldots, \alpha_r$ [22, 3.7]. Under this identification each walk u in Q can be regarded as an element of $\Pi_1(Q)$: we identify arrows that belong to T with the unit element of $\Pi_1(Q)$.

2.2. Given a field k the path algebra of Q with coefficients in k is denoted by kQ. If I is an admissible ideal in kQ then the pair (Q, I) is called a *bound quiver* and $k(Q, I)$ denotes the bound quiver algebra kQ/I of (Q, I) (cf. [4, 2.1]). We agree that the trivial paths e_x, $x \in Q_0$, form a complete set of primitive orthogonal idempotents of R.

Fix Q and I as above and let $R = k(Q, I)$. Recall that the algebra R is said to be *connected* if the quiver Q is connected. By *connected components* of R we mean the algebras determined by connected components of the quiver Q. The algebra R is *triangular* if Q has no oriented cycle, and it is *schurian* if $\dim_k e_x R e_y \le 1$ for all $x, y \in Q_0$. It is easy to check that R is a right multipeak algebra if and only if for any w in kQ not belonging to I there exists a path v terminating at a sink of Q such that $uv \notin I$.

It is often convenient to treat R as a k-category with Q_0 as objects and with morphism spaces $R(x, y) = e_x R e_y$ for $x, y \in Q_0$. The composition is induced by multiplication in R. Given two paths u, v in Q we denote by uRv the subspace of R generated by the I-cosets of paths of the form uwv in Q.

For $x \in Q_0$ we denote by S_x the simple R-module $e_x R/\mathrm{rad}(e_x R)$ associated with x and by P_x its R-projective cover $e_x R$. Here $\mathrm{rad}(X) = X\,\mathrm{rad}(R)$ is the Jacobson radical of the module X.

For $A \subseteq Q_0$ we denote by R_A the full subcategory of R with $Q_0 \setminus A$ as objects. In algebraic terms this means that $R_A \cong \mathrm{End}_R(\bigoplus_{x \in Q_0 \setminus A} P_x)$. Given a vertex x of Q we write R_x instead of $R_{\{x\}}$.

We identify in the usual way an R-module M with a k-representation $(M(x), M(\alpha))_{x \in Q_0, \alpha \in Q_1}$ of (Q, I). Given a path $u = \alpha_1 \ldots \alpha_r$ in Q we denote by $M(u) : M(s(\alpha)) \to M(t(\alpha))$ the composition $M(\alpha_r)\dots M(\alpha_1)$. By the *support* of M we mean the subset $\mathrm{supp}(M) = \{x \in Q_0 : M(x) \neq 0\}$ of Q_0.

2.3. Let I be an admissible ideal in the path algebra kQ. Following [7], [10, 1.3] we say that an element $\omega = \sum_{i=1}^n \lambda_i u_i$ of I is a *minimal relation* in I provided $u_1, \dots, u_n$ are parallel paths in Q, $\lambda_1, \dots, \lambda_r \in k$, $n \geq 2$ and for any proper subset J of $\{1, \dots, n\}$ we have $\sum_{j\in J} \lambda_j u_j \notin I$.

We say that a path u *appears* in $\omega = \sum_{i=1}^n \lambda_i u_i$ *with coefficient* μ provided $\sum_{i;u_i=u} \lambda_i = \mu$. If u appears in ω with a nonzero coefficient then we just say that u *appears* in ω. If α' and α'' are arrows such that $u_1'\alpha'u_2'$ and $u_1''\alpha''u_2''$ are different parallel paths appearing in a minimal relation for some paths u_1', u_2', u_1'', u_2'' in Q then we also say that α' and α'' *appear in a minimal relation in* I.

Let Ω be a fixed set of minimal relations generating the two-sided ideal I in kQ. Following [7] we denote by $\approx_\Omega$ the *homotopy relation defined by* Ω; it is the smallest equivalence relation on the set of walks in Q satisfying:

(a) if u and v are homotopic in Q then $u \approx_\Omega v$,
(b) if $u \approx_\Omega u'$, $v \approx_\Omega v'$ and $t(u) = s(v)$, $t(u') = s(v')$ then $uv \approx_\Omega u'v'$,
(c) if u and v appear in a minimal relation belonging to Ω then $u \approx_\Omega v$.

We denote by $\Pi_1((Q,I), x, \Omega)$ the group of homotopy classes of loops at x and call it the *fundamental group of the bound quiver* (Q,I) *at the vertex* x *with respect to the set* Ω. Again if Q is connected then this group does not depend on the choice of x and we speak about the *fundamental group of* (Q,I) *with respect to* Ω and denote it by $\Pi_1((Q,I), \Omega)$ (cf. [7], [10], [1]).

2.4. Assume that Q is connected and fix a maximal tree $T = (T_0, T_1)$ in Q. As above we identify $\Pi_1(Q)$ with the free group on the set $Q_1 \setminus T_1$ of free generators. Fix a set Ω of minimal relations generating I and denote by $N(\Omega)$ the normal subgroup of $\Pi_1(Q)$ generated by all elements of the form uv^{-1}, where u, v are parallel paths appearing in a minimal relation belonging to Ω. Then by [12] and [16, Remark 3.6] (see also [9]),

$$\Pi_1((Q,I), \Omega) \cong \Pi_1(Q)/N(\Omega).$$

The lemma below implies that $N(\Omega)$ and consequently $\Pi_1((Q,I), \Omega)$ do not depend on the choice of Ω.

LEMMA. *In the notation above assume that* Ω *and* Ω' *are two sets of generators of* I *consisting of minimal relations. Then* $N(\Omega) = N(\Omega')$.

Proof. We show that $N(\Omega) \subseteq N(\Omega')$, the remaining inclusion follows analogously. It is enough to prove that if $\omega' \in \Omega'$ is a minimal relation and u, v appear in ω' then $uv^{-1} \in N(\Omega)$. Since Ω generates I there exist elements $\omega_i \in \Omega$, paths u_i, v_i and $\lambda_i \in k$ for $i = 1, \dots, r$ such that $\omega' = \sum_{i=1}^r \lambda_i \widetilde{\omega}_i$, where $\widetilde{\omega}_i$ denotes $u_i\omega_i v_i$ for $i = 1, \dots, r$.

We introduce a relation $\smile$ in $\{1,\ldots,r\}$ by writing $i \smile j$ provided there exists a path appearing in both $\widetilde{\omega}_i$ and $\widetilde{\omega}_j$. Then:

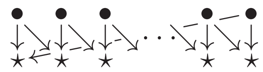
(1) If i and j belong to the same connected component of $\{1,\ldots,r\}$ with respect to the relation $\smile$, u appears in $\widetilde{\omega}_i$ and v appears in $\widetilde{\omega}_j$ then $uv^{-1} \in N(\Omega)$.

(2) If $\mathcal{C}$ is a component with respect to $\smile$ and u appears in $\omega'_{\mathcal{C}} = \sum_{i\in\mathcal{C}} \widetilde{\omega}_i$ with coefficient λ_u then u appears in ω' with the same coefficient.

By minimality of ω' it follows from (2) that for any component $\mathcal{C}$ we have $\omega'_{\mathcal{C}} = \omega'$. Now the assertion follows from (1). ■

Since $N(\Omega)$ does not depend on the choice of Ω we denote it by $N(I)$; also, we shorten the notation $\Pi_1((Q,I),\Omega)$ to $\Pi_1(Q,I)$.

2.5. From now on we assume that $R = k(Q,I)$ is a right multipeak algebra. Denote by $\max Q$ the set of all sinks in Q and put $Q^- = Q\setminus \max Q$. We say that the algebra R is $\widetilde{\mathbb{A}}_m$-*free*, $m \geq 1$, if it does not contain a full subcategory isomorphic to $k\widetilde{\mathbb{A}}_m$, where

$\widetilde{\mathbb{A}}^*_m$: 

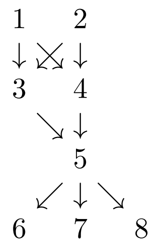
(m stars at the bottom), and $\widetilde{\mathbb{A}}_1$ is the Kronecker two-arrow quiver. If R is $\widetilde{\mathbb{A}}_m$-free for every $m = 1,2,\ldots$ then we say that R is $\widetilde{\mathbb{A}}$-*free*. Observe that R is $\widetilde{\mathbb{A}}_1$-free if and only if $\dim_k e_xRe_p \leq 1$ for any $x \in Q_0$ and $p \in \max Q$.

We say that a triangular right multipeak algebra R is *chord-free* if for any arrow α in the ordinary quiver Q of R with $t(\alpha) \notin \max Q$ there is no path u different from α and parallel to α. In particular, the only multiple arrows in Q terminate in $\max Q$.

Lemma. *Let $R \cong kQ/I \cong kQ/I'$ for admissible ideals I, I' be a chord-free $\widetilde{\mathbb{A}}_1$-free right multipeak algebra. Then a path u in Q belongs to I if and only if it belongs to I'.*

Proof. This follows from the observation that our assumptions imply that for any arrow $\alpha \in Q_1$ the space $e_{s(\alpha)}Re_{t(\alpha)}$ is 1-dimensional. ■

Example. Let Q be the quiver



Denote the arrow from i to j by α_{ij}. Let I be the ideal in kQ generated by the following elements:

$$\alpha_{13}\alpha_{35}-\alpha_{14}\alpha_{45},\quad \alpha_{23}\alpha_{35}-\alpha_{24}\alpha_{45},\quad \alpha_{35}\alpha_{57},\quad \alpha_{45}\alpha_{58}.$$

The algebra $R=kQ/I$ is a chord-free $\widetilde{\mathbb{A}}$-free right multipeak algebra but it contains a full subcategory isomorphic to $k\widetilde{\mathbb{A}}_2$.

2.6. Under the assumption that R is a chord-free $\widetilde{\mathbb{A}}_1$-free right multipeak algebra we give a new description of the homotopy relation from 2.3. Let $\sim$ be the smallest equivalence relation on the set of walks in Q satisfying the conditions (a) and (b) in 2.3 (with $\approx_\Omega$ replaced by $\sim$) and the condition

(c′) if u and w are parallel paths in Q and there exists a path v in Q ending at a sink of Q such that $uv\notin I$ and $wv\notin I$, then $u\sim w$.

LEMMA. *If R is a chord-free $\widetilde{\mathbb{A}}_1$-free right multipeak algebra then the relations $\approx_\Omega$ and $\sim$ coincide on the set of walks in Q.*

Proof. Assume first that u and w are parallel paths in Q and there exists a path v in Q ending at a sink of Q such that $uv\notin I$ and $wv\notin I$. Since R is an $\widetilde{\mathbb{A}}_1$-free it follows that $\lambda uv+wv\in I$ for some nonzero $\lambda\in k$. Then $u\approx_\Omega v$ and hence the relation $\sim$ is contained in $\approx_\Omega$.

To prove the converse inclusion let $\sum_{i=1}^r\lambda_i w_i$ be a minimal relation in I. Let x be the sink of w_i, $i=1,\ldots,r$. It is enough to prove that $w_i\sim w_j$ for any $1\le i,j\le r$.

Assume that $W_1,\ldots,W_s$ are equivalence classes of the relation $\sim$ restricted to the set $\{w_1,\ldots,w_r\}$ and let $s>1$. For $j=1,\ldots,s$ let S_j be the set of $p\in\max Q$ such that there exists a path v from x to p with $wv\notin I$ for some $w\in W_j$. For any $p\in\max Q$ such that $e_xRe_p\ne 0$ let v_p be a path in Q from x to p not belonging to I. Since R is $\widetilde{\mathbb{A}}_1$-free any two paths from x to a fixed vertex $p\in\max Q$ are equal modulo I. It follows that $S_j=\{p\in\max Q: wv_p\notin I \text{ for all } w\in W_j\}$ for any $j=1,\ldots,s$. The sets S_j are nonempty and pairwise disjoint for $j=1,\ldots,s$.

Observe that by minimality of $\sum_{i=1}^r\lambda_i w_i$ we have

$$\sum_{w_i\in W_1}\lambda_i w_i\ne 0$$

and since R is a right multipeak algebra,

$$\Big(\sum_{w_i\in W_1}\lambda_i w_i\Big)\Big(\sum_{p\in\max Q,\, e_xRe_p\ne 0} v_p\Big)\ne 0.$$

This yields a contradiction as the left hand side equals

$$\Big(\sum_{w_i\in W_1}\lambda_i w_i\Big)\Big(\sum_{p\in S_1} v_p\Big)=\Big(\sum_{i=1}^r\lambda_i w_i\Big)\Big(\sum_{p\in S_1} v_p\Big)=0. \ \blacksquare$$

Corollary. *Suppose that* $R = k(Q, I)$ *is a chord-free* $\widetilde{\mathbb{A}}_1$*-free right peak algebra and* $R \cong kQ/I \cong kQ/I'$, *where* I *and* I' *are admissible ideals in* kQ. *Then* $\Pi_1(Q, I) \cong \Pi_1(Q, I')$. *In particular*, *the algebra* R *is simply connected in the sense of* [1] *if and only if there exists a bound quiver* (Q, I) *such that* $R \cong kQ/I$ *and the group* $\Pi_1(Q, I)$ *is trivial.*

Proof. The assertion follows from the above Lemma and the fact (see 2.5) that a path u in Q belongs to I if and only if it belongs to I'. ∎

3. Right multipeak algebras and a reflection duality. Throughout this section we assume that R is a triangular $\widetilde{\mathbb{A}}_1$-free right multipeak algebra.

3.1. We represent the algebra R in the triangular matrix form

$$R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$$

where $A = k(Q^-, I^-)$, I^- is the restriction of the ideal I to kQ^- and $B = k(\max Q) \cong \prod_{p \in \max Q} k_p$, with $k_p = k$ for $p \in \max Q$. According to [14, Definition 2.6] (see also [15]) the *reflection dual algebra* $R^\bullet$ is

$$R^\bullet = \begin{pmatrix} A^{\mathrm{op}} & DM \\ 0 & B^{\mathrm{op}} \end{pmatrix}$$

where $DM = \mathrm{Hom}_k(M, k)$ is the bimodule dual to M. It follows from [15, 17.4] that $R^\bullet$ is a right multipeak algebra as well.

3.2. *A construction.* Our main aim in this section is to present the construction of a new bound quiver $(Q^\bullet, I^\bullet)$ such that $R^\bullet \cong k(Q^\bullet, I^\bullet)$ and the fundamental groups of (Q, I) and $(Q^\bullet, I^\bullet)$ coincide. We follow the idea of [14, Definition 2.16].

Let $\mathcal{B}$ be a set of paths in Q such that the I-cosets of the elements of $\mathcal{B}$ form a k-basis of the left A-socle of M. Each $u \in \mathcal{B}$ is a path terminating in $\max Q$ and such that $u \notin I$ but $\alpha u \in I$ for any arrow α. Given two vertices y, p of Q such that $p \in \max Q$ and $y \notin \max Q$ we define the set $\mathcal{B}_{y,p} = \{u \in \mathcal{B} : s(u) = y,\ t(u) = p\}$.

Observe that since R is $\widetilde{\mathbb{A}}_1$-free each path u parallel to an element b of $\mathcal{B}$ equals λb modulo I for some $\lambda \in k$.

Define the quiver $Q^\bullet = Q^\bullet_{\mathcal{B}} = (Q^\bullet_0, Q^\bullet_1)$, where the set $Q^\bullet_0$ of vertices of $Q^\bullet$ coincides with Q_0 and

$$Q^\bullet_1 = \{\alpha^{-1} : \alpha \in Q_1,\ t(\alpha) \notin \max Q\} \cup \{b^* : b \in \mathcal{B}\},$$

where b^* are new arrows. We set $s(b^*) = y$ and $t(b^*) = p$ if $b \in \mathcal{B}_{y,p}$.

The ideal $I^\bullet = I^\bullet_{\mathcal{B}}$ is generated by elements of the following types:

(1) $\sum_{i=1}^r \lambda_i u_i^{-1}$, where all u_i are paths in Q^{-1} and $\sum_{i=1}^r \lambda_i u_i \in I$,

(2) $u^{-1} b^*$ if $b \in \mathcal{B}_{y,p}$, u is a path from y to x in Q and $uRe_p = 0$,

(3) $\lambda_2 u_1^{-1} b_1^* - \lambda_1 u_2^{-1} b_2^*$ if $b_i \in \mathcal{B}_{y_i,p}$, u_i is a path from y_i to x in Q for $i = 1, 2$ and there exists a path $v \notin I$ from x to p such that $\lambda_i b_i - u_i v \in I$ for some $\lambda_i \in k$ and $i = 1, 2$.

Since R is $\widetilde{\mathbb{A}}_1$-free the element of type (3) above does not depend (up to a scalar multiplication) on the choice of v.

Example. Let Q be the quiver

$$\begin{array}{ccccc} 1 & & & & \\ \downarrow & & & & \\ 2 & & & & 3 \\ & \searrow & & \swarrow & \\ & & 4 & & \\ & \swarrow & & \searrow & \\ 5 & & & & 6 \end{array}$$

We denote by α_{ij} the arrow from i to j. Let I be the ideal generated by $\alpha_{12}\alpha_{24}\alpha_{46}$ and let $\mathcal{B} = \{b_1, b_2, b_3, b_4\}$ where $b_1 = \alpha_{12}\alpha_{24}\alpha_{45}$, $b_2 = \alpha_{34}\alpha_{45}$, $b_3 = \alpha_{24}\alpha_{46}$, $b_4 = \alpha_{34}\alpha_{46}$.

The quiver $Q_{\mathcal{B}}^{\bullet}$ has the form

$$\begin{array}{ccccc} & & 2 & \leftarrow & 4 \\ & \swarrow & \downarrow & & \downarrow \\ 1 & & 6 & \leftarrow & 3 \\ & \searrow & & \swarrow & \\ & & 5 & & \end{array}$$

If α'_{ij} denotes the arrow in $Q_{\mathcal{B}}^{\bullet}$ starting from i and ending at j then $\alpha'_{42} = \alpha_{24}^{-1}$, $\alpha'_{21} = \alpha_{12}^{-1}$, $\alpha'_{43} = \alpha_{34}^{-1}$, $\alpha'_{15} = b_1^*$, $\alpha'_{35} = b_2^*$, $\alpha'_{26} = b_3^*$, $\alpha'_{36} = b_4^*$.

Since $b_3 - \alpha_{24}\alpha_{46} = 0$ and $b_4 - \alpha_{34}\alpha_{46}$ in kQ, according to (2) we have

$$\alpha_{24}^{-1} b_3^* - \alpha_{34}^{-1} b_4^* \in I_{\mathcal{B}}^{\bullet}.$$

Analogously, $\alpha_{24}^{-1}\alpha_{12}^{-1} b_1^* - \alpha_{34}^{-1} b_2^* \in I_{\mathcal{B}}^{\bullet}$. The ideal $I_{\mathcal{B}}^{\bullet}$ is generated by commutativity relations, and $k(Q^{\bullet}, I^{\bullet})$ is the incidence algebra of a poset.

3.3. Lemma. *If $R = k(Q, I)$ is a triangular $\widetilde{\mathbb{A}}_1$-free right multipeak algebra then there exists an algebra isomorphism*

$$k(Q^{\bullet}, I^{\bullet}) \cong R^{\bullet}.$$

Proof. This follows from Proposition 2.19 and Corollary 2.22 of [14]. ∎

3.4. Proposition. *Suppose that $R = k(Q, I)$ is an $\widetilde{\mathbb{A}}_1$-free triangular connected right multipeak algebra and let $(Q^{\bullet}, I^{\bullet})$ be the reflection dual bound quiver to (Q, I) with respect to a set $\mathcal{B}$ of paths. Then there exists a group isomorphism*

$$\Pi_1(Q, I) \cong \Pi_1(Q^{\bullet}, I^{\bullet}).$$

P r o o f. Let $T = (T_0, T_1)$ be a maximal tree in Q such that the restriction $T^- = T \cap Q^-$ of T to Q^- is a maximal tree in Q^-. Let $Q_1 \setminus T_1 = \{\alpha_1, \ldots, \alpha_r, \gamma_1, \ldots, \gamma_s\}$, where $\alpha_1, \ldots, \alpha_r$ are arrows in $(Q^-)_0$ and $\gamma_1, \ldots, \gamma_s$ are arrows in $Q_1 \setminus Q_1^-$.

Construct a maximal tree T' in $Q^\bullet$ such that $Q_1^\bullet \setminus T_1' = \{\alpha_1^{-1}, \ldots, \alpha_r^{-1}, b_1^*, \ldots, b_t^*\}$, where $b_1^*, \ldots, b_t^*$ are arrows from $(Q^\bullet)^-$ to $\max Q^\bullet$.

Recall from 2.1 that we agreed to treat walks in $Q^\bullet$ (resp. in Q) as elements of $\Pi_1(Q^\bullet)$ (resp. $\Pi_1(Q)$). Denote by $[w]$ the image of $w \in \Pi_1(Q)$ in $\Pi_1(Q, I)$. Define a homomorphism

$$\Phi : \Pi_1(Q^\bullet) \to \Pi_1(Q, I)$$

by setting $\Phi(\alpha_i^{-1}) = [\alpha_i^{-1}]$ for $i = 1, \ldots, r$ and $\Phi(b_j^*) = [b_j]$ for $j = 1, \ldots, t$.

We are going to prove that Φ induces a homomorphism $\overline{\Phi} : \Pi_1(Q^\bullet, I^\bullet) \to \Pi_1(Q, I)$. By 2.4 it is enough to prove that if two paths w_1, w_2 in $Q^\bullet$ appear in a minimal relation generating $I^\bullet$ then $\Phi(w_1) = \Phi(w_2)$. This is clear if w_1 and w_2 are paths in $(Q^\bullet)^-$. It remains to consider the case when $w_1 = u_1^{-1} b_{i_1}^* \notin I^\bullet$, $w_2 = u_2^{-1} b_{i_2}^* \notin I^\bullet$ and $\lambda_2 u_1^{-1} b_{i_1}^* - \lambda_1 u_2^{-1} b_{i_2}^*$ for some $\lambda_1, \lambda_2 \in k^*$ is a relation of type (3) in 3.2. It follows that if $t(b_{i_1}) = t(b_{i_2}) = p$ and x is a sink of u_1 and of u_2 then there exists a path v from x to p in Q such that $b_{i_1} - \lambda_1 u_1 v \in I$ and $b_{i_2} - \lambda_2 u_2 v \in I$. Then $[b_{i_1}] = [u_1 v]$ and $[b_{i_2}] = [u_2 v]$ in $\Pi_1(Q, I)$, hence $\Phi(w_1) = [u_1^{-1} b_{i_1}] = [u_2^{-1} b_{i_2}] = \Phi(w_2)$.

In order to define a map

$$\Psi : \Pi_1(Q) \to \Pi_1(Q^\bullet, I^\bullet)$$

inducing the inverse to $\overline{\Phi}$ first consider an arrow γ_j in Q and let u_j be a path such that $u_j \gamma_j$ is a nonzero element of the left socle of M. Assume that $s(u_j \gamma_j) = y_j$ and $t(u_j \gamma_j) = p_j$ and let $b_j \in \mathcal{B}_{y_j, p_j}$ and $\lambda_j \in k^*$ be such that $\lambda_j b_j - u_j \gamma_j \in I$.

Now define $\Psi(\alpha_i) = [\alpha_i]$ for $i = 1, \ldots, r$ and $\Psi(\gamma_j) = [u_j^{-1} b_j]$ for $j = 1, \ldots, s$. Observe that $\Psi(\gamma_j)$ does not depend on the choice of u_j thanks to the assumption that R is $\widetilde{\mathbb{A}}_1$-free.

Next we prove that $\Psi(N(I)) = \{1\}$. Take any minimal relation $\omega \in I$ and let u and v appear in ω. If u and v are paths in Q^{-1} then it is easy to observe that $[u] = [v]$ in $\Pi_1(Q^\bullet, I^\bullet)$. Otherwise, since R is $\widetilde{\mathbb{A}}_1$-free, we can assume that ω is of the form $\lambda u + \mu v$ with $\lambda, \mu \in k^*$. Let $u = u'\gamma$, $v = v'\delta$, where γ, δ are arrows. Let w be a path in Q such that $wu'\gamma$ and $wv'\delta$ are elements of the left socle of M and let $b \in \mathcal{B}$ be the element linearly dependent on each of $wu'\gamma$ and $wv'\delta$. Then

$$\Psi(u) = \Psi(u')\Psi(\gamma) = [u'][wu']^{-1}[b] = [w^{-1}b],$$
$$\Psi(v) = \Psi(v')\Psi(\delta) = [v'][wv']^{-1}[b] = [w^{-1}b],$$

which proves that $\Psi(N(I)) = \{1\}$ and Ψ induces a homomorphism

$$\overline{\Psi} : \Pi_1(Q, I) \to \Pi_1(Q^\bullet, I^\bullet).$$

It is easy to check that $\overline{\Phi}$ and $\overline{\Psi}$ are inverse to each other. ∎

3.5. Lemma. *Assume that x is a vertex in Q^- and let $S = R_x$ be the full subcategory of R obtained by deleting the vertex x. Then*

$$S^\bullet \cong (R^\bullet)_x,$$

where $(R^\bullet)_x$ is by definition the full subcategory of $R^\bullet$ obtained by removing the vertex x.

The proof is routine and is left to the reader. ∎

3.6. Lemma. *Assume that R is a chord-free $\widetilde{\mathbb{A}}_1$-free right multipeak algebra with ordinary quiver Q and x is a source or a sink in Q^-. Then the algebras $R^\bullet$ and R_x are chord-free and $\widetilde{\mathbb{A}}_1$-free.*

Proof. The statement about $\widetilde{\mathbb{A}}_1$-freeness is clear; the remaining assertion also follows immediately from the definition of a chord-free algebra. ∎

4. Separation property. From now on we assume that R is a triangular, connected, chord-free $\widetilde{\mathbb{A}}$-free right multipeak algebra. In the proof of our main theorem the following proposition is crucial.

4.1. Proposition (cf. [21]). *Assume that $R = k(Q, I)$ is a triangular, connected, chord-free $\widetilde{\mathbb{A}}$-free right multipeak algebra which is simply connected. Let x be a sink or a source in Q^-. Then each connected component of the algebra R_x is a simply connected right multipeak algebra.*

The main tool for the proof of the proposition is the following lemma.

Lemma. *Let $R = k(Q, I)$ be a right multipeak chord-free $\widetilde{\mathbb{A}}$-free triangular algebra and let x be a source in Q. Assume that $Q_1, \ldots, Q_r$ are connected components of $Q \setminus \{x\}$ and I_j is the restriction of I to Q_j for $j = 1, \ldots, r$. Then there exists a surjective homomorphism*

$$\Pi_1(Q, I) \to \prod_{j=1}^{r} \Pi_1(Q_j, I_j).$$

Proof. Denote by $\widetilde{Q}_j$ the full subquiver of Q containing Q_j and x and by $\widetilde{I}_j$ the restriction of I to $\widetilde{Q}_j$ for $j = 1, \ldots, r$. It is easy to see that

$$\Pi_1(Q, I) \cong \Pi_1(\widetilde{Q}_1, \widetilde{I}_1) * \ldots * \Pi_1(\widetilde{Q}_r, \widetilde{I}_r)$$

(free product of groups). Thus without loss of generality we can assume that the quiver $Q \setminus \{x\} = \overline{Q}$ is connected.

Let T be a maximal tree in Q such that $\overline{T} = T \cap \overline{Q}$ is a maximal tree in $\overline{Q}$. Denote by U the set of arrows starting at x. There is exactly one belonging to T among them, say $\alpha_0 \in T_1 \cap U$.

We define a homomorphism

$$\Phi : \Pi_1(Q) \to \Pi_1(\overline{Q}, \bar{I})$$

in the following way. If β is an arrow in $\overline{Q}_1 \setminus T_1$ then we set $\Phi(\beta) = [\beta]$. To define Φ on elements of U we introduce in U a partial order $\preceq$ satisfying:

(i) If $\alpha \prec \alpha'$ is a minimal relation in $(U, \preceq)$ then there exist paths w, w' in Q with $t(w) = t(w') \in \max Q$ such that $\alpha w \notin I$ and $\alpha' w' \notin I$.

(ii) Every connected component of U with respect to $\preceq$ has a smallest element.

(iii) The arrow α_0 is minimal in U.

(iv) The poset $(U, \preceq)$ is a tree.

(v) The relation $\preceq$ is maximal among those satisfying (i)–(iv).

The existence of such an order follows easily by induction on the cardinality of U. Let $\alpha_1 \prec \ldots \prec \alpha_n$ be a sequence of minimal relations in U such that α_1 is a minimal element in U. We define $\Phi(\alpha_s)$ by induction on s. Set $\Phi(\alpha_1) = 1$. Assume that $s > 1$ and $\Phi(\alpha_{s-1})$ has already been defined. Let v_s, u_s be paths such that $t(v_s) = t(u_s) \in \max Q$ and $\alpha_{s-1} v_s \notin I$, $\alpha_s u_s \notin I$. Then we set $\Phi(\alpha_s) = \Phi(\alpha_{s-1})[v_s] \cdot [u_s]^{-1}$.

Thanks to condition (iv) this definition is correct.

It is clear that Φ is surjective; we prove that it induces a homomorphism

$$\overline{\Phi} : \Pi_1(Q, I) \to \Pi_1(\overline{Q}, \bar{I}).$$

Let u, u' be parallel paths which are homotopy equivalent. We prove that $\Phi(u) = \Phi(u')$. If u and u' do not start at x the assertion follows by the description of the homotopy relation given in 2.6 (observe that by Lemma 3.6 the algebra R_x is chord-free and $\widetilde{\mathbb{A}}_1$-free).

Assume now that u and u' start at x and let $u = \alpha v$, $u' = \alpha' v'$, where $\alpha, \alpha' \in U$. By Lemma 2.6 without loss of generality we can assume that there exists a path w ending at $\max Q$ such that $\alpha v w \notin I$ and $\alpha v' w \notin I$. We need to prove that $\Phi(\alpha)[v] = \Phi(\alpha')[v']$.

Let

$$\alpha_1 \prec \ldots \prec \alpha_n \quad \text{and} \quad \alpha'_1 \prec \ldots \prec \alpha'_{n'}$$

be sequences of minimal relations in U such that $\alpha_1 = \alpha'_1$ is the maximal common predecessor of α_n and $\alpha'_{n'}$ and $\alpha_n = \alpha$, $\alpha'_{n'} = \alpha'$. The existence of such sequences follows from the conditions (iv) and (v).

Let $\alpha_i v_{i+1} \notin I$ and $\alpha_{i+1} u_{i+1} \notin I$ be parallel paths terminating at $\max Q$ for $i = 1, \ldots, n-1$ and similarly let $\alpha'_j v'_{j+1} \notin I$ and $\alpha'_{j+1} u'_{j+1} \notin I$ be parallel paths terminating at $\max Q$ for $j = 1, \ldots, n' - 1$. Denote by x_i the sink of

α_i for $i = 1, \ldots, n$ and by x'_j the sink of α'_j for $j = 1, \ldots, n'$. Denote by p_i the sink of $\alpha_i v_{i+1}$ and by p'_j the sink of $\alpha'_{j+1} u'_{j+1}$. Moreover, let p be the sink of $\alpha v w$.

Observe that $p_2 = \ldots = p_n = p = p'_2 = \ldots = p'_{n'}$ since otherwise the full subcategory of R formed by $x_1, \ldots, x_n, x'_2, \ldots, x'_{n'}$ and $p_2, \ldots, p_n, p, p'_2, \ldots$ $\ldots, p'_{n'}$ contains a subcategory isomorphic to $k\widetilde{\mathbb{A}}_s$ for some $s \geq 2$, contrary to our assumption that R is $\widetilde{\mathbb{A}}$-free.

The following equalities hold in $\Pi_1(\overline{Q}, \bar{I})$:

$$\begin{aligned} [v_2] &= [v'_2], \\ [u_i] &= [v_{i+1}] \quad \text{for } i = 2, \ldots, n-1, \\ [u_n] &= [v][w], \\ [u'_j] &= [v'_{j+1}] \quad \text{for } j = 2, \ldots, n'-1, \\ [u'_{n'}] &= [v'][w]. \end{aligned}$$

It follows that

$$\begin{aligned} \Phi(\alpha)[v] &= \Phi(\alpha_n)[v] = \Phi(\alpha_{n-1})[v_n][u_n]^{-1}[v] = \ldots \\ &= \Phi(\alpha_1)[v_2][u_2]^{-1} \ldots [v_n][u_n]^{-1}[v] \\ &= \Phi(\alpha_1)[v_2][u_2]^{-1} \ldots [v_{n-1}][u_{n-1}]^{-1}[v_n][w]^{-1} \\ &= \Phi(\alpha_1)[v_2][u_2]^{-1} \ldots [v_{n-1}][w]^{-1} = \ldots = \Phi(\alpha_1)[v_2][w]^{-1}. \end{aligned}$$

Analogously we get $\Phi(\alpha')[v'] = \Phi(\alpha_1)[v'_2][w]^{-1}$. Thus the equality $[v_2] = [v'_2]$ yields $\Phi(\alpha)[v] = \Phi(\alpha')[v']$. ■

Proof of the Proposition. It is clear that R_x is a right peak algebra. If x is a source in Q^- the remaining assertion follows directly from the lemma above. Otherwise we use reflection duality. The vertex x is then a source in $Q^\bullet$ and the assertion follows by the above Lemma and 3.3–3.5. ■

4.2. Now we are going to prove that simply connected triangular chord-free $\widetilde{\mathbb{A}}$-free right multipeak algebras have the separation property.

Recall from [21, 2.3] (comp. [2]) that if $R = k(Q, I)$ then a vertex x of Q is called *separating* in R if the restriction of the module $\operatorname{rad}(P_x)$ to any connected component of $R_{x^\triangledown}$ is indecomposable, where $P_x = e_x R$ is the indecomposable projective R-module associated with x, and x^∇ is the set of vertices y of Q such that there exists a path from y to x in Q or $x = y$.

If $R = k(Q, I)$ and every vertex of Q is separating in R then we say that R has the *separation property*.

A special case of the general result is treated separately in the following lemma.

Lemma. *Assume that $R = k(Q, I)$ is a chord-free $\widetilde{\mathbb{A}}$-free triangular right multipeak algebra, x is the unique source in Q and each vertex of Q^- except*

x is the sink of an arrow starting at x. If $\Pi_1(Q,I)$ is trivial then the vertex x is separating.

P r o o f. Every vertex of Q apart from x is either a sink of Q or a sink of Q^-. Set $M = \mathrm{rad}(P_x)$. It is easy to see that under the assumptions of the Lemma, if x is not separating then there exist in Q parallel paths u, w such that $u \in I$. Hence we easily conclude by 2.6 that there are two paths from x to $t(\alpha)$ which are not homotopic. ∎

4.3. Lemma. *Let x, y be vertices of Q such that there is no arrow $\alpha \in Q_1$ with $s(\alpha) = x$ and $t(\alpha) = y$ and let $Q_1, \ldots, Q_r$ be connected components of the ordinary quiver Q' of $R_{\{x,y\}}$. Assume that*

(a) *for any $1 \le j \le r$ there exists a vertex z_j of Q_j and paths u_j, v_j in Q such that $s(u_j) = x$, $t(u_j) = s(v_j) = z_j$ and $t(v_j) = y$,*

(b) *for any minimal relation $\sum_{i=1}^{s} \lambda_i w_i$ there exists $1 \le j \le r$ such that all the paths $w_1, \ldots, w_s$ have vertices in the set $(Q_j)_0 \cup \{x, y\}$.*

Then there exists a surjective group homomorphism

$$h : \Pi_1(Q,I) \to \mathbf{F}_{r-1}$$

where $\mathbf{F}_{r-1}$ is the free nonabelian group with $r-1$ free generators $f_1, \ldots, f_{r-1}$.

P r o o f. Any loop at the vertex x in Q can be represented as a composition of walks $w_1, \ldots, w_m$ for some $m \ge 1$ such that $s(w_i), t(w_i) \in \{x, y\}$ for any $i = 1, \ldots, m$, and any vertex of w_i which is neither a source nor a sink of w_i is not equal to x or y. Observe that if $s(w_i) \ne t(w_i)$ then all the vertices of w_i belong to $(Q_j)_0 \cup \{x, y\}$ for exactly one $j \in \{1, \ldots, r\}$. With each w_i we associate the numbers $d(w_i)$ and $\varepsilon(w_i)$ in the following way:

$$d(w_i) = \begin{cases} 0 & \text{if } s(w_i) = t(w_i), \\ j & \text{if } s(w_i) \ne t(w_i), \text{ the vertices of } w_i \text{ belong to } (Q_j)_0 \cup \{x, y\}, \end{cases}$$

and

$$\varepsilon(w_i) = \begin{cases} 0 & \text{if } s(w_i) = t(w_i), \\ 1 & \text{if } s(w_i) = x,\ t(w_i) = y, \\ -1 & \text{if } s(w_i) = y,\ t(w_i) = x. \end{cases}$$

Let

$$\widetilde{h}(w) = f_{d(w_1)}^{\varepsilon(w_1)} \ldots f_{d(w_m)}^{\varepsilon(w_m)} \in \mathbf{F}_{r-1},$$

where $f_0 = f_r$ is the unit element of $\mathbf{F}_{r-1}$.

Condition (a) implies that $\widetilde{h}(w)$ depends only on the homotopy class of w and hence $\widetilde{h}$ induces a group homomorphism $h : \Pi_1(Q,I) \to \mathbf{F}_{r-1}$, which is surjective thanks to the assumption (b). ∎

4.4. Lemma (cf. [21]). *Suppose that $R = k(Q,I)$ is a chord-free $\widetilde{\mathbb{A}}$-free triangular right multipeak algebra and R is simply connected. Let x be a* *vertex of Q such that the algebra R_x is connected. Then* $\operatorname{End}_R(\operatorname{rad} P(x)) \cong k$ *or $P(x)$ is a simple module.*

Proof. The proof mimics that of Lemma 4.2 in [21]. We proceed by induction on $|Q_0|$. Denote by M the radical $\operatorname{rad} P_x$ of P_x. Since Q has no multiple arrows, the multiplicities of simple modules occurring in $M/\operatorname{rad} M$ are equal to 1, and thus it is enough to show that M is indecomposable. By Proposition 4.1 one can assume that x is a unique source in Q.

If x is a sink of Q^- or a sink of Q then the assertion is clear; now suppose otherwise. By Lemma 4.2 we can assume that there exists a sink y in Q^- such that there is no arrow from x to y in Q. Assume that $M \cong N_1 \oplus \ldots \oplus N_r$, $r \geq 2$, $N_i \neq 0$ for $i = 1, \ldots, r$. It follows from 4.1 that each connected component of the algebra R_y is simply connected. Denote by M', N'_j the restrictions of M and N_j to R_y for $j = 1, \ldots, r$. Since the simple R-module corresponding to y is not a direct summand of M it follows that $N'_j \neq 0$ for $j = 1, \ldots, r$. By the induction hypothesis there exist pairwise different connected components $Q_1, \ldots, Q_r$ of the quiver Q' of $R_{\{x,y\}}$ such that $\operatorname{supp}(N'_j) \subseteq (Q_j)_0$ for $j = 1, \ldots, r$.

We show that the elements x, y and components $Q_1, \ldots, Q_r$ satisfy the assumptions of Lemma 4.3. The assumption (a) follows easily.

We prove that if there is a minimal relation $\omega = \sum_{i=1}^{s} \lambda_i u_i$ in I then the vertices of all paths u_i, $i = 1, \ldots, r$, belong to $(Q_j)_0 \cup \{x, y\}$ for some j. This is clear if x is not the source of ω. So consider the case when x is the source of ω.

Suppose the contrary and let the vertices of $u_1, \ldots, u_l$ belong to $(Q_1)_0 \cup \{x, y\}$ and the vertices of $u_{l+1}, \ldots, u_s$ belong to $\bigcup_{i=2}^{r} (Q_r)_0 \cup \{x, y\}$ for some $l < s$. Denote by z the sink of ω. Since $u_1 \notin I$ it follows that $N'_1(z) \neq 0$. Minimality of ω implies $\sum_{i=1}^{l} \lambda_i u_i \notin I$.

Take $v \in P_x(x)$ such that $m_1 = \sum_{i=1}^{l} \lambda_i P_x(u_i)(a)$ is a nonzero element of $N_1(z)$ and consider the projection $p_1 : M \to N_1$. Clearly, $p_1(m_1) \neq 0$. Observe that $p_1(m_2) = 0$ where $m_2 = \sum_{l+1}^{s} P_x(u_i)(a)$ since $m_2 \in N_2 \oplus \ldots \oplus N_r$. This contradicts the assumption that $m_1 + m_2 = \sum_{i=1}^{s} \lambda_i P_x(u_i)(a) = 0$.

It follows that M is indecomposable. ■

Example (cf. [21, 2.1]). We now show the importance of the assumption that R is chord-free. Let $R = k(Q, I)$, where Q is the quiver

$$\begin{array}{ccc} 2 & \leftarrow & 1 \\ & \searrow & \downarrow \\ & & 3 \\ & \swarrow & \downarrow \\ 4 & & 5 \end{array}$$

and I is the two-sided ideal in kQ generated by the elements $\alpha_{23}\alpha_{34}$ and

$\alpha_{12}\alpha_{23}\alpha_{35} - \alpha_{13}\alpha_{35}$, with α_{ij} the arrow of Q from i to j. The algebra R is a right multipeak $\widetilde{\mathbb{A}}$-free algebra, the quiver Q has no multiple arrows, the group $\Pi_1(Q,I)$ is trivial, but the vertex 1 of Q is not separating in R. The algebra R is not chord-free: the arrow α_{13} is parallel to the path $\alpha_{12}\alpha_{23}$.

4.5. We denote by $H^1(R)$ the first Hochschild cohomology group $H^1(R,R)$ of the algebra R with coefficients in R and with the natural R-R-bimodule structure (see [21]).

THEOREM. *Assume that $R = k(Q,I)$ is a triangular simply connected chord-free $\widetilde{\mathbb{A}}$-free right multipeak algebra. Then*:

(a) *The algebra R has the separation property.*
(b) *The first Hochschild cohomology group $H^1(R)$ vanishes.*

Proof. Both assertions follow from 4.4: (a) is an immediate consequence, whereas the proof of [21, Theorem 4.1] directly applies to (b). ∎

4.6. Let $R = k(Q,I)$ be a right multipeak algebra, which we represent in the triangular matrix form

$$R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}.$$

Following [11], [17, Section 2] define the category $\mathrm{prin}(R) = \mathrm{prin}(R)^A_B$ of *prinjective R-modules* to be the full subcategory of $\mathrm{mod}(R)$ (the category of right finitely generated R-modules) consisting of modules X admitting a short exact sequence

$$0 \to P'' \to P' \to X \to 0,$$

where P' is projective and P'' is semisimple projective.

According to [11, 4.1] the *prinjective Tits quadratic form* associated with R is the integral quadratic form

$$\mathbf{q}_R : \mathbb{Z}^{Q_0} \to \mathbb{Z}$$

given by

$$\mathbf{q}_R(v) = \sum_{x\in Q_0} v_x^2 + \sum_{x,y\in Q_0^-} v_x v_y \dim_k R(x,y) - \sum_{p\in \max Q}\ \sum_{x\in Q_0^-} v_p v_x \dim_k R(x,p)$$

for any $v = (v_x)_{x\in Q_0} \in \mathbb{Z}^{Q_0}$.

The reader is referred to [11], [15] for the definitions of the Auslander–Reiten quiver of the category $\mathrm{prin}(R)$ and the preprojective components.

It is proved in [11, 4.2, 4.13] that if the category $\mathrm{prin}(R)$ is of finite representation type, that is, there are only finitely many isomorphism classes of indecomposable modules in $\mathrm{prin}(R)$, then the form $\mathbf{q}_R$ is weakly positive, which means that $\mathbf{q}_R(v) > 0$ for every nonzero element $v \in \mathbb{Z}^{Q_0}$ with nonnegative coefficients. The converse is true under the assumption that the Auslander–Reiten quiver of $\mathrm{prin}(R)$ has a preprojective component.

Recall from [13], [17] that $\mathrm{mod}_{\mathrm{sp}}(R)$ is the full subcategory of $\mathrm{mod}(R)$ formed by modules having projective socles.

THEOREM. *Assume that R is a triangular chord-free simply connected right peak algebra. Then*

(1) *If R is an $\widetilde{\mathbb{A}}$-free right multipeak algebra then the Auslander–Reiten quiver of the category* $\mathrm{prin}(R)$ *has a preprojective component.*

(2) *The following conditions are equivalent*:

(i) *the prinjective Tits quadratic form $\mathbf{q}_R$ is weakly positive*,

(ii) *the category* $\mathrm{prin}(R)$ *is of finite representation type*,

(iii) *the category* $\mathrm{mod}_{\mathrm{sp}}(R)$ *is of finite representation type.*

Proof. (1) By Theorem 4.5, R has the separation property, thus the existence of a preprojective component can be proved analogously to [3, Theorem 2.5] (cf. [8, 3.4]).

(2) The equivalence of conditions (ii) and (iii) follows from the properties of the adjustment functor $\mathbf{\Theta}$ (see [17, Lemma 2.1]). If the prinjective Tits quadratic form $\mathbf{q}_R$ is weakly positive or the category $\mathrm{prin}(R)$ is of finite representation type then R is $\widetilde{\mathbb{A}}$-free (cf. [8]). Thus, in view of (1), the equivalence (i)$\Leftrightarrow$(ii) follows again by [11, 4.13]. ∎

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